# Geometric Structures in Tensor Representations

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**Abstract**

The main goal of this paper is to study the geometric structures associated with the representation of tensors in subspace based formats. To do this we use a property of the so-called minimal subspaces which allows us to describe the tensor representation by means of a rooted tree. By using the tree structure and the dimensions of the associated minimal subspaces, we introduce, in the underlying algebraic tensor space, the set of tensors in a tree-based format with either bounded or fixed tree-based rank. This class contains the Tucker format and the Hierarchical Tucker format (including the Tensor Train format). In particular, we show that the set of tensors in the tree-based format with bounded (respectively, fixed) tree-based rank of an algebraic tensor product of normed vector spaces is an analytic Banach manifold. Indeed, the manifold geometry for the set of tensors with fixed tree-based rank is induced by a fibre bundle structure and the manifold geometry for the set of tensors with bounded tree-based rank is given by a finite union of connected components where each of them is a manifold of tensors in the tree-based format with a fixed tree-based rank. The local chart representation of these manifolds is often crucial for an algorithmic treatment of high-dimensional PDEs and minimization problems. In order to describe the relationship between these manifolds and the natural ambient space, we introduce the definition of topological tensor spaces in the tree-based format. We prove under natural conditions that any tensor of the topological tensor space under consideration admits best approximations in the manifold of tensors in the tree-based format with bounded tree-based rank. In this framework, we also show that the tangent (Banach) space at a given tensor is a complemented subspace in the natural ambient tensor Banach space and hence the set of tensors in the tree-based format with bounded (respectively, fixed) tree-based rank is an immersed submanifold. This fact allows us to extend the Dirac-Frenkel variational principle in the bodywork of topological tensor spaces.

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# 1 Introduction

Tensor approximation methods play a central role in the numerical solution of high dimensional problems arising in a wide range of applications. Low-rank tensor formats based on subspaces are widely used for complexity reduction in the representation of high-order tensors. The construction of these formats are usually based on a hierarchy of tensor product subspaces spanned by orthonormal bases, because in most cases a hierarchical representation fits with the structure of the mathematical model and facilitates its computational implementation. Two of the most popular formats are the Tucker format and the Hierarchical Tucker format [18] (HT for short). It is possible to show that the Tensor Train format [29] (TT for short), introduced originally by Vidal [35], is a particular case of the HT format (see e.g. Chapter 12 in [19]). An important feature of these formats, in the framework of topological tensor spaces, is the existence of a best approximation in each fixed set of tensors with bounded rank [11]. In particular, it allows to construct, on a theoretical level, iterative minimisation methods for nonlinear convex problems over reflexive tensor Banach spaces [12].

Tucker tensors of fixed rank are also used for the discretisation of differential equations arising in quantum chemical problems or in the multireference Hartree and Hartree-Fock methods (MR-HF) in quantum dynamics [25]. In particular, for finite dimensional ambient tensor spaces, it can be shown that the set of Tucker tensors of fixed rank forms an immersed finite-dimensional quotient manifold [22]. A similar approach in a complex Hilbert space setting for Tucker tensors of fixed rank is given in [4]. Then the numerical treatment of this class of problems follows the general concepts of differential equations on manifolds [16]. Recently, similar results have been obtained for the TT format [20] and the HT format [33] (see also [3]). The term "matrix-product state" (MPS) was introduced in quantum physics (see, e.g., [34]). The related tensor representation can be found already in [35] without a special naming of the representation. The method has been reinvented by Oseledets and Tyrtyshnikov (see [28], [29], and [30]) and called "TT decomposition". For matrix product states (MPS), the differential geometry in a finite-dimensional complex Hilbert space setting is covered in [17].

As we will show below, the Tucker and the HT formats are completely characterised by a rooted tree together with a finite sequence of natural numbers associated to each node on the tree, denominated the tree-based rank. Each number in the tree-based rank is associated with a class of subspaces of fixed dimension. Moreover, it can be shown that for a given tree, every element in the tensor space possesses a unique tree-based rank. In consequence, given a tree, a tensor space is a union of sets indexed by the tree-based ranks. It allows to consider for a given tree two kinds of sets in a tensor space: the set of tensors of fixed tree-based rank and the set of tensors of bounded tree-based rank. Two commonly accepted facts are the following.

(a) Even if it can be shown in finite dimension that the set of Tucker (respectively, HT) tensors with bounded tree-based rank is closed, the existence of a manifold structure for this set is an open question. Thus the existence of minimisers over these sets can be shown, however, no first order optimality conditions are available from a geometric point of view.

(b) Even if either in finite dimension or in a Hilbert space setting it can be shown that the set of Tucker (respectively, in finite dimensions HT) tensors with fixed tree-based rank is a quotient manifold, the construction of an explicit parametrisation in order to provide a manifold structure is not known.

In our opinion, these two facts are due to the lack of a common mathematical frame for developing a mathematical analysis of these abstract objects. The main goal of this paper is to provide this common framework by means of the theory for algebraic and topological tensor spaces developed in [11] by some of the authors of this article.

Our starting point are the following natural questions that arise in the mathematical theory of tensor spaces. The first one is: is it possible to introduce a class of tensors containing Tucker, HT (and hence TT) tensors with fixed and bounded rank? A second question is: if such a class exists, is it possible to construct a parametrisation for the set of tensors of bounded (respectively, fixed) rank in order to show that it is a true manifold even in the infinite-dimensional case? Finally, if the answers to the first two questions are positive, we would like to ask: is the set of tensors of bounded (respectively, fixed) rank an immersed submanifold of the topological tensor space, as ambient manifold, under consideration?

The paper is organised as follows.

- In Sect. 2, we introduce the tree-based tensors as a generalisation, at algebraic level, of the hierarchical tensor format. This class contains the Tucker tensors (among others). Moreover, we characterise the minimal subspaces for tree-based tensors extending the previous results obtained in [11] and introducing the definition of tree-based rank. In particular, the main result of this section, Theorem 2.19, is a characterisation of the set of parameters needed to provide an explicit geometric representation for the set of tensors with fixed tree-based rank.

- In Sect. 3, by the help of Theorem 2.19, we show that in an algebraic tensor product of normed spaces the set of tensors with fixed tree-based rank is an analytic Banach manifold. Indeed, we give an explicit atlas and we prove that this atlas is induced by a fibre bundle structure. This result allows us to deduce that the set of tensors with bounded tree-based rank is also an analytic Banach manifold. An important fact is that the geometric structure of these manifolds is independent on the ambient tensor Banach space under consideration.

- In Sect. 4, we discuss the choice of a norm in the ambient tensor Banach space (a) to show the existence of a best approximation for the set of tensors with bounded tree-based rank and (b) to prove that the set of tensors with fixed tree-based rank is an immersed submanifold of that space (considered as Banach manifold). To this end we assume the existence of a norm at each node of the tree not weaker than the injective norm constructed from the Banach spaces associated with the sons of that node. This assumption generalises the condition used in [11] to prove the existence of a best approximation in the Tucker case. More precisely, under this assumption,

  – we provide a proof of the existence of best approximation in the manifold of tensors with bounded tree-based rank,

  – we construct a linear isomorphism, at each point in the manifold of tensors with fixed tree-based rank, from the tangent space at that point to a closed linear subspace of the ambient tensor Banach space, this subspace being given explicitly,

  – we show that the set of tensors with fixed tree-based rank is an immersed submanifold,

  – we also deduce that the set of tensors with bounded tree-based rank is an immersed submanifold.

- In Sect. 5, we give a formalisation in this framework of the multi–configuration time–dependent Hartree MCTDH method (see [25]) in tensor Banach spaces.

# 2 Algebraic tensors spaces in the tree-based Format

## 2.1 Preliminary definitions and notations

Concerning the definition of the algebraic tensor space ${}_a\bigotimes_{j=1}^d V_j$ generated from vector spaces $V_j$ $(1 \le j \le d)$, we refer to Greub [14]. As underlying field we choose $\mathbb{R}$, but the results hold also for $\mathbb{C}$. The suffix '$a$' in ${}_a\bigotimes_{j=1}^d V_j$ refers to the 'algebraic' nature. By definition, all elements of

$$\mathbf{V} := {}_a\bigotimes_{j=1}^{d} V_j$$

are *finite* linear combinations of elementary tensors $\mathbf{v} = \bigotimes_{j=1}^d v_j$ $(v_j \in V_j)$. Let $D := \{1, \ldots, d\}$ be the index set of the 'spatial directions'. In the sequel, the index sets $D\backslash\{j\}$ will appear. Here, we use the abbreviations

$$\mathbf{V}_{[j]} := {}_a\bigotimes_{k \neq j} V_k \,, \qquad \text{where } \bigotimes_{k\neq j} \text{ means } \bigotimes_{k \in D\backslash\{j\}} .$$

Similarly, elementary tensors $\bigotimes_{k\neq j} v_k$ are denoted by $\mathbf{v}_{[j]}$. The following notations and definitions will be useful.

For vector spaces $V_j$ and $W_j$ over $\mathbb{R}$, let linear mappings $A_j : V_j \to W_j$ $(1 \le j \le d)$ be given. Then the definition of the elementary tensor

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j : \ \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j \longrightarrow \mathbf{W} = {}_a\bigotimes_{j=1}^{d} W_j$$

is given by

$$\mathbf{A}\left(\bigotimes_{j=1}^{d} v_j\right) := \bigotimes_{j=1}^{d} (A_j v_j)\,. \tag{2.1}$$

Note that (2.1) uniquely defines the linear mapping $\mathbf{A} : \mathbf{V} \to \mathbf{W}$. We recall that $L(V, W)$ is the space of linear maps from $V$ into $W$, while $V' = L(V, \mathbb{R})$ is the algebraic dual of $V$. For metric spaces, $\mathcal{L}(V, W)$ denotes the continuous linear maps, while $V^* = \mathcal{L}(V, \mathbb{R})$ is the topological dual of $V$. Often, mappings $\mathbf{A} = \bigotimes_{j=1}^d A_j$ will appear, where most of the $A_j$ are the identity (and therefore $V_j = W_j$). If $A_k \in L(V_k, W_k)$ for one $k$ and $A_j = id$ for $j \neq k$, we use the following notation:

$$\mathbf{id}_{[k]} \otimes A_k := \underbrace{id \otimes \ldots \otimes id}_{k-1 \text{ factors}} \otimes A_k \otimes \underbrace{id \otimes \ldots \otimes id}_{d-k \text{ factors}} \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k),$$

provided that it is obvious which component $k$ is meant. By the multiplication rule $\left(\bigotimes_{j=1}^d A_j\right) \circ \left(\bigotimes_{j=1}^d B_j\right) = \bigotimes_{j=1}^d (A_j \circ B_j)$ and since $id \circ A_j = A_j \circ id$, the following identity[1] holds for $j \neq k$:

$$\begin{aligned} &id \otimes \ldots \otimes id \otimes A_j \otimes id \otimes \ldots \otimes id \otimes A_k \otimes id \otimes \ldots \otimes id \\ &= (\mathbf{id}_{[j]} \otimes A_j) \circ (\mathbf{id}_{[k]} \otimes A_k) \\ &= (\mathbf{id}_{[k]} \otimes A_k) \circ (\mathbf{id}_{[j]} \otimes A_j) \end{aligned}$$

(in the first line we assume $j < k$). Proceeding inductively with this argument over all indices, we obtain

$$\mathbf{A} = \bigotimes_{j=1}^{d} A_j = (\mathbf{id}_{[1]} \otimes A_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes A_d).$$

[1] Note that the meaning of $\mathbf{id}_{[j]}$ and $\mathbf{id}_{[k]}$ may differ: in the second line of (2.2), $(\mathbf{id}_{[k]} \otimes A_k) \in L(\mathbf{V}, \mathbf{V}_{[k]} \otimes_a W_k)$ and $(\mathbf{id}_{[j]} \otimes A_j) \in L\left(\mathbf{V}_{[k]} \otimes_a W_k, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$, whereas in the third one $(\mathbf{id}_{[j]} \otimes A_j) \in L(\mathbf{V}, \mathbf{V}_{[j]} \otimes_a W_j)$ and $(\mathbf{id}_{[k]} \otimes A_k) \in L\left(\mathbf{V}_{[j]} \otimes_a W_j, \mathbf{V}_{[j,k]} \otimes_a W_j \otimes_a W_k\right)$. Here $\mathbf{V}_{[j,k]} = {}_a\bigotimes_{l \in D\backslash\{j,k\}} V_l$.

If $W_j = \mathbb{R}$, i.e., if $A_j = \varphi_j \in V_j'$ is a linear form, then $\mathbf{id}_{[j]} \otimes \varphi_j \in L(\mathbf{V}, \mathbf{V}_{[j]})$ is used as symbol for $id \otimes \ldots \otimes id \otimes \varphi_j \otimes id \otimes \ldots \otimes id$ defined by

$$(\mathbf{id}_{[j]} \otimes \varphi_j)\left(\bigotimes_{k=1}^{d} v_k\right) = \varphi_j(v_j) \cdot \bigotimes_{k \neq j} v_k.$$

Thus, if $\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j \in \bigotimes_{j=1}^{d} V_j'$, we can also write

$$\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j = (\mathbf{id}_{[1]} \otimes \varphi_1) \circ \cdots \circ (\mathbf{id}_{[d]} \otimes \varphi_d). \tag{2.2a}$$

Consider again the splitting of $\mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$ into $\mathbf{V} = V_j \otimes_a \mathbf{V}_{[j]}$ with $\mathbf{V}_{[j]} := {}_a\bigotimes_{k \neq j} V_k$. For a linear form $\boldsymbol{\varphi}_{[j]} \in \mathbf{V}_{[j]}'$, the notation $id_j \otimes \boldsymbol{\varphi}_{[j]} \in L(\mathbf{V}, V_j)$ is used for the mapping

$$(id_j \otimes \boldsymbol{\varphi}_{[j]})\left(\bigotimes_{k=1}^{d} v_k\right) = \boldsymbol{\varphi}_{[j]}\left(\bigotimes_{k \neq j} v_k\right) \cdot v_j. \tag{2.2b}$$

If $\boldsymbol{\varphi}_{[j]} = \bigotimes_{k \neq j} \varphi_k \in {}_a\bigotimes_{k \neq j} V_k'$ is an elementary tensor, $\boldsymbol{\varphi}_{[j]}\left(\bigotimes_{k \neq j} v^{(k)}\right) = \prod_{k \neq j} \varphi_k\left(v^{(k)}\right)$ holds in (2.2b). Finally, we can write (2.2a) as

$$\boldsymbol{\varphi} = \otimes_{j=1}^{d} \varphi_j = \varphi_j \circ (id_j \otimes \boldsymbol{\varphi}_{[j]}) \qquad \text{for } 1 \leq j \leq d.$$

## 2.2 Algebraic tensor spaces in the tree-based format

We introduce the abbreviation TBF for 'tree-based format'. For instance, a TBF tensor is a tensor represented in the tree-based format, etc. The tree-based rank will be abbreviated by TB rank. To introduce the underlying tree we use the following example.

**Example 2.1** *Let us consider* $D = \{1, 2, 3, 4, 5, 6\}$, *then*

$$\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j = \left({}_a\bigotimes_{j=1}^{3} V_j\right) \otimes_a \left({}_a\bigotimes_{j=4}^{5} V_j\right) \otimes_a V_6 = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6.$$

*Observe that* $\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j$ *can be represented by the tree given in Figure 2.1 and* $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$ *by the tree given in Figure 2.2. We point out that there are other trees to describe the tensor representation* $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$, *because*

$$\mathbf{V}_D = \left({}_a\bigotimes_{j=1}^{3} V_j\right) \otimes_a \left({}_a\bigotimes_{j=4}^{5} V_j\right) \otimes_a V_6 = \left(V_1 \otimes_a \left({}_a\bigotimes_{j=2}^{3} V_j\right)\right) \otimes_a \left({}_a\bigotimes_{j=4}^{5} V_j\right) \otimes_a V_6,$$

*that is,* $\mathbf{V}_{123} = {}_a\bigotimes_{j=1}^{3} V_j = V_1 \otimes_a \mathbf{V}_{23}$ *(see Figure 2.3).*

The above example motivates the following definition.

**Definition 2.2** *The tree* $T_D$ *is called* a dimension partition tree of $D$ *if*

*(a) all vertices* $\alpha \in T_D$ *are non–empty subsets of* $D$,

*(b)* $D$ *is the root of* $T_D$,

*(c) every vertex* $\alpha \in T_D$ *with* $\#\alpha \geq 2$ *has at least two sons. Moreover, if* $S(\alpha) \subset 2^D$ *denotes the set of sons of* $\alpha$ *then* $\alpha = \cup_{\beta \in S(\alpha)} \beta$ *where* $\beta \cap \beta' = \emptyset$ *for all* $\beta, \beta' \in S(\alpha)$, $\beta \neq \beta'$,

*(d) every vertex* $\alpha \in T_D$ *with* $\#\alpha = 1$ *has no son.*

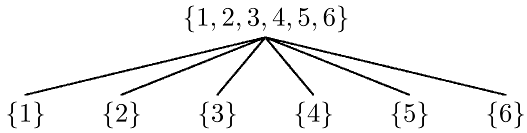

Figure 2.1: A dimension partition tree related to $\mathbf{V}_D = {}_a\bigotimes_{j=1}^{6} V_j$ .

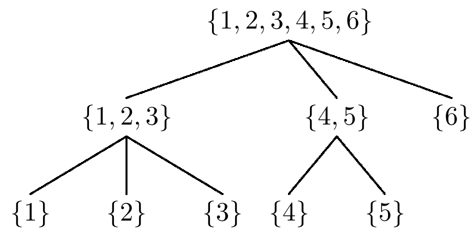

Figure 2.2: A dimension partition tree related to $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$.

If $S(\alpha) = \emptyset$, $\alpha$ is called a *leaf.* The set of leaves is denoted by $\mathcal{L}(T_D)$. An easy consequence of Definition 2.2 is that the set of leaves $\mathcal{L}(T_D)$ coincides with the singletons of $D$, i.e., $\mathcal{L}(T_D) = \{\{j\} : j \in D\}$.

**Example 2.3** *Consider* $D = \{1, 2, 3, 4, 5, 6\}$. *Take*

$$T_D = \{D, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\} \text{ and } S(D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$$

*(see Figure 2.1). Then* $S(D) = \mathcal{L}(T_D)$.

**Example 2.4** *In Figure 2.2 we have a tree which corresponds to* $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$. *Here* $D = \{1, 2, 3, 4, 5, 6\}$ *and*

$$T_D = \{D, \{1, 2, 3\}, \{4, 5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\},$$

$$S(D) = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}, S(\{4, 5\}) = \{\{4\}, \{5\}\}, S(\{1, 2, 3\}) = \{\{1\}, \{2\}, \{3\}\}.$$

*Moreover* $\mathcal{L}(T_D) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$.

Finally we give the definition of a TBF tensor.

**Definition 2.5** *Let $D$ be a finite index set and $T_D$ be a partition tree. Let $V_j$ be a vector space for $j \in D$, and consider for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$ a tensor space $\mathbf{V}_\alpha := {}_a\bigotimes_{\beta \in S(\alpha)} \mathbf{V}_\beta$ . Then the collection of vector spaces $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is called a representation of the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in* tree-based format.

Observe that we can write $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha = {}_a\bigotimes_{j \in D} V_j$ . A first property of TBF tensors is the independence of the representation of the algebraic tensor space $\mathbf{V}_D$ with respect to the tree $T_D$.

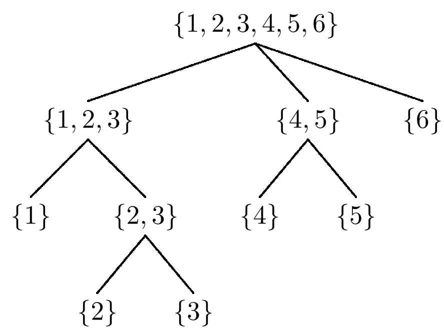

Figure 2.3: A dimension partition tree related to $\mathbf{V}_D = \mathbf{V}_{123} \otimes_a \mathbf{V}_{45} \otimes_a V_6$ where $\mathbf{V}_{123} = V_1 \otimes_a \mathbf{V}_{23}$.

**Lemma 2.6** *Let $D$ be a finite index set and $T_D$ be a partition tree. Let $V_j$ be a vector space for $j \in D$. Assume that $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a \bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in the tree-based format. Then for each $\alpha_1 \in T_D \setminus \{D\}$ there exist $\alpha_2, \ldots, \alpha_m \in T_D \setminus \{D, \alpha_1\}$ such that $D = \cup_{i=1}^m \alpha_i$, $\alpha_i \cap \alpha_j = \emptyset$ and $\mathbf{V}_D = {}_a \bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ .*

## 2.3 Minimal subspaces for TBF tensors

Let $V_j$ be a vector space for $j \in D$, where $D$ is a finite index set, and $\alpha_1, \ldots, \alpha_m \subset 2^D \setminus \{D, \emptyset\}$, be such that $\alpha_i \cap \alpha_j = \emptyset$ for all $i \neq j$ and $D = \bigcup_{j=1}^m \alpha_j$. For $\mathbf{v} \in {}_a \bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ we define the minimal subspace of $\mathbf{v}$ on each $\mathbf{V}_{\alpha_i} := {}_a \bigotimes_{j \in \alpha_i} V_j$ for $1 \leq i \leq m$, as follows.

**Definition 2.7** *For a tensor $\mathbf{v} \in {}_a \bigotimes_{j \in D} V_j = {}_a \bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ , the* minimal subspaces *denoted by $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_i}$, for $1 \leq i \leq m$, are defined by the properties that $\mathbf{v} \in {}_a \bigotimes_{i=1}^m U_{\alpha_i}^{\min}(\mathbf{v})$ and $\mathbf{v} \in {}_a \bigotimes_{i=1}^m \mathbf{U}_{\alpha_i}$ implies $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{U}_{\alpha_i}$.*

The minimal subspaces are useful to introduce the following sets of tensor representations based on subspaces. Fix $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{N}^d$. Then we define *the set of Tucker tensors with bounded rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a \bigotimes_{j=1}^d V_j$ by

$$\mathcal{T}_{\mathbf{r}}(\mathbf{V}) := \left\{ \mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) \leq r_j,\ 1 \leq j \leq d \right\},$$

and *the set of Tucker tensors with fixed rank* $\mathbf{r}$ *in* $\mathbf{V} = {}_a \bigotimes_{j=1}^d V_j$ by

$$\mathcal{M}_{\mathbf{r}}(\mathbf{V}) := \left\{ \mathbf{v} \in \mathbf{V} : \dim U_j^{\min}(\mathbf{v}) = r_j,\ 1 \leq j \leq d \right\}.$$

Then $\mathcal{M}_{\mathbf{r}}(\mathbf{V}) \subset \mathcal{T}_{\mathbf{r}}(\mathbf{V}) \subset \mathbf{V}$ holds.

The next characterisation of $U_{\alpha_j}^{\min}(\mathbf{v})$ for $1 \leq j \leq m$ is due to [19] (it is included in the proof of Lemma 6.12). Since we assume that $\mathbf{V}_{\alpha_j}$ are vector spaces for $1 \leq j \leq m$, then we may consider the subspaces

$$U_{\alpha_j}^{I}(\mathbf{v}) := \left\{ (id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a \bigotimes_{k \neq j} \mathbf{V}'_{\alpha_k} \right\}$$

and

$$U_{\alpha_j}^{II}(\mathbf{v}) := \left\{ (id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a \bigotimes_{k \neq j} U_{\alpha_k}^{\min}(\mathbf{v})' \right\},$$

for $1 \leq j \leq m$. Moreover, if $\mathbf{V}_{\alpha_j}$ are normed spaces for $1 \leq j \leq m$ we can also consider

$$U_{\alpha_j}^{III}(\mathbf{v}) := \left\{ (id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a \bigotimes_{k \neq j} \mathbf{V}^*_{\alpha_k} \right\},$$

and

$$U_{\alpha_j}^{IV}(\mathbf{v}) := \left\{ (id_{\alpha_j} \otimes \boldsymbol{\varphi}_{[\alpha_j]})(\mathbf{v}) :\ \boldsymbol{\varphi}_{[\alpha_j]} \in {}_a \bigotimes_{k \neq j} U_{\alpha_k}^{\min}(\mathbf{v})^* \right\},$$

**Theorem 2.8** *Assume that $\mathbf{V}_{\alpha_j}$ are vector spaces for $1 \leq j \leq m$. Then the following statements hold.*

*(a) For any $\mathbf{v} \in \mathbf{V} = {}_a \bigotimes_{j=1}^m \mathbf{V}_{\alpha_j}$ , it holds*

$$U_{\alpha_j}^{\min}(\mathbf{v}) = U_{\alpha_j}^{I}(\mathbf{v}) = U_{\alpha_j}^{II}(\mathbf{v}),$$

*for $1 \leq j \leq m$.*

*(b) Assume that $\mathbf{V}_{\alpha_j}$ are normed spaces for $1 \leq j \leq m$. Then for any $\mathbf{v} \in \mathbf{V} = {}_a \bigotimes_{j=1}^m \mathbf{V}_{\alpha_j}$ , it holds*

$$U_{\alpha_j}^{\min}(\mathbf{v}) = U_{\alpha_j}^{III}(\mathbf{v}) = U_{\alpha_j}^{IV}(\mathbf{v}),$$

*for $1 \leq j \leq m$.*

Let $D = \cup_{i=1}^m \alpha_i$ be a given partition. Assume that $\alpha_1 = \cup_{j=1}^n \beta_j$ is also a given partition, then we have minimal subspaces $U_{\beta_j}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\beta_j} = {}_a\bigotimes_{k\in\beta_j} V_k$ for $1 \le j \le n$ and $U_{\alpha_i}^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_i} = {}_a\bigotimes_{k\in\alpha_i} V_k$ for $1 \le i \le m$. Observe that $\mathbf{V}_{\alpha_1} = {}_a\bigotimes_{j=1}^n \mathbf{V}_{\beta_j}$ , where

$$\mathbf{v} \in {}_a\bigotimes_{i=1}^m U_{\alpha_i}^{\min}(\mathbf{v}) \text{ and } \mathbf{v} \in \left( {}_a\bigotimes_{j=1}^n U_{\beta_j}^{\min}(\mathbf{v}) \right) \otimes_a \left( {}_a\bigotimes_{i=2}^m U_{\alpha_i}^{\min}(\mathbf{v}) \right).$$

**Example 2.9** *Let us consider $D = \{1,2,3,4,5,6\}$ and the partition tree $T_D$ given in Figure 2.2. Take $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j = \mathbf{V}_{\alpha_1} \otimes_a \mathbf{V}_{\alpha_2} \otimes_a \mathbf{V}_{\alpha_3}$, where $\alpha_1 = \{1,2,3\}$, $\alpha_2 = \{4,5\}$, and $\alpha_3 = \{6\}$. Then we can conclude that there are minimal subspaces $U_{\alpha_\nu}^{\min}(\mathbf{v})$ for $\nu = 1,2,3$, such that $\mathbf{v} \in {}_a\bigotimes_{\nu=1}^3 U_{\alpha_\nu}^{\min}(\mathbf{v})$ and also minimal subspaces $U_j^{\min}(\mathbf{v})$ for $j \in D$, such that $\mathbf{v} \in {}_a\bigotimes_{j\in D} U_j^{\min}(\mathbf{v})$*

The relation between $U_j^{\min}(\mathbf{v})$ and $U_{\alpha_\nu}^{\min}(\mathbf{v})$ is as follows (see Corollary 2.9 of [11]).

**Proposition 2.10** *Let $V_j$ be a vector space for $j \in D$, where $D$ is a finite index set, and $D = \cup_{i=1}^m \alpha_i$ be a given partition. Let $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j$ . For a partition $\alpha_1 = \cup_{j=1}^m \beta_j$ it holds*

$$U_{\alpha_1}^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{j=1}^m U_{\beta_j}^{\min}(\mathbf{v}) .$$

The following result gives us the relationship between a basis of $U_{\alpha_1}^{\min}(\mathbf{v})$ and a basis of $U_{\beta_j}^{\min}(\mathbf{v})$ for $1 \le j \le m$.

**Proposition 2.11** *Let $V_j$ be a vector space for $j \in D$, where $D$ is a finite index set. Let $\alpha \subset D$ such that $\alpha = \bigcup_{i=1}^m \alpha_i$, where $\emptyset \ne \alpha_i$ are pairwise disjoint for $1 \le i \le m$. Let $\mathbf{v} \in {}_a\bigotimes_{j\in D} V_j$ . The following statements hold.*

*(a) For each $1 \le i \le m$, it holds*

$$\begin{aligned} U_{\alpha_i}^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} U_{\alpha_k}^{\min}(\mathbf{v})' \right\} \\ &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} \mathbf{V}'_{\alpha_k} \right\}. \end{aligned}$$

*(b) Assume that $\mathbf{V}_\alpha := {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}$ and $\mathbf{V}_{\alpha_i}$, for $1 \le i \le m$, are normed spaces. For each $1 \le i \le m$ it holds*

$$\begin{aligned} U_{\alpha_i}^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right\} \\ &= \operatorname{span}\left\{ \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\ne i} \mathbf{V}^*_{\alpha_k} \right\} \end{aligned}$$

*Proof.* Statements (a) and (b) are proved in a similar way. Let $\gamma = D \setminus \alpha$ and write $\gamma = \bigcup_{i=1}^n \gamma_i$, where $\emptyset \ne \gamma_i \subset D$ are pairwise disjoint for $i = 1,2,\ldots,n$. In particular, to prove (b), we observe that

$$\mathbf{V}_D = \mathbf{V}_\alpha \otimes_a \mathbf{V}_\gamma = \left( {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i} \right) \otimes_a \left( {}_a\bigotimes_{j=1}^n \mathbf{V}_{\gamma_j} \right).$$

Then, by Theorem 2.8(b), using $U_{\alpha_i}^{IV}(\mathbf{v})$, we have

$$U_{\alpha}^{\min}(\mathbf{v}) = \left\{ (id_\alpha \otimes \boldsymbol{\varphi}^{(\gamma)})(\mathbf{v}) : \boldsymbol{\varphi}^{(\gamma)} \in {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right\} \text{ and}$$
$$U_{\alpha_i}^{\min}(\mathbf{v}) = \left\{ (id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)})(\mathbf{v}) : \boldsymbol{\varphi}^{(D\setminus\alpha_i)} \in \left( {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right) \otimes_a \left( {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right) \right\}$$

for $1 \le i \le m$. Take $\mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v})$. Then there exists $\boldsymbol{\varphi}^{(\gamma)} \in {}_a\bigotimes_{j=1}^m U_{\gamma_j}^{\min}(\mathbf{v})^*$ such that $\mathbf{v}_\alpha = \left(id_\alpha \otimes \boldsymbol{\varphi}^{(\gamma)}\right)(\mathbf{v})$. Now, for $\boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^*$ , we have

$$\left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) = \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha)}\right)(\mathbf{v}),$$

and hence $\left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(\alpha\setminus\alpha_i)}\right)(\mathbf{v}_\alpha) \in U_{\alpha_i}^{\min}(\mathbf{v})$. Now, take $\mathbf{v}_{\alpha_i} \in U_{\alpha_i}^{\min}(\mathbf{v})$, then there exists

$$\boldsymbol{\varphi}^{(D\setminus\alpha_i)} \in \left( {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^* \right) \otimes_a \left( {}_a\bigotimes_{j=1}^{m} U_{\gamma_j}^{\min}(\mathbf{v})^* \right)$$

such that $\mathbf{v}_{\alpha_i} = \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)}\right)(\mathbf{v})$. Then $\boldsymbol{\varphi}^{(D\setminus\alpha_i)} = \sum_{l=1}^r \boldsymbol{\psi}_l^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\phi}_l^{(\gamma)}$, where $\boldsymbol{\phi}_l^{(\gamma)} \in {}_a\bigotimes_{j=1}^m U_{\gamma_j}^{\min}(\mathbf{v})^*$ and $\boldsymbol{\psi}_l^{(\alpha\setminus\alpha_i)} \in {}_a\bigotimes_{k\neq i} U_{\alpha_k}^{\min}(\mathbf{v})^*$ for $1 \le l \le r$. Thus,

$$\begin{aligned}
\mathbf{v}_{\alpha_i} &= \left(id_{\alpha_i} \otimes \boldsymbol{\varphi}^{(D\setminus\alpha_i)}\right)(\mathbf{v}) \\
&= \sum_{i=1}^r \left(id_{\alpha_i} \otimes \boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)} \otimes \boldsymbol{\phi}_i^{(\gamma)}\right)(\mathbf{v}) \\
&= \sum_{i=1}^r \left(id_{\alpha_i} \otimes \boldsymbol{\psi}_i^{(\alpha\setminus\alpha_i)}\right)\left((id_\alpha \otimes \boldsymbol{\phi}_i^{(\gamma)})(\mathbf{v})\right).
\end{aligned}$$

Observe that $(id_\alpha \otimes \boldsymbol{\phi}_l^{(\gamma)})(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v})$. Hence the other inclusion holds and the first equality of statement (b) is proved. To show the second inequality of statement (b), we proceed in a similar way by using Theorem 2.8(b) and the definition of $U_{\alpha_j}^{III}(\mathbf{v})$. ■

From now on, given $\emptyset \neq \alpha \subset D$, we will denote $\mathbf{V}_\alpha := {}_a\bigotimes_{j\in\alpha} V_j$ , $r_\alpha := \dim U_\alpha^{\min}(\mathbf{v})$ and $U_D^{\min}(\mathbf{v}) := \operatorname{span}\{\mathbf{v}\}$. Observe that for each $\mathbf{v} \in \mathbf{V}_D$ we have that $(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha\in 2^D\setminus\{\emptyset\}}$ is in $\mathbb{N}^{2^{\#D}-1}$.

**Definition 2.12** *Let $D$ be a finite index set and $T_D$ be a partition tree. Let $V_j$ be a vector space for $j \in D$, Assume that $\{\mathbf{V}_\alpha\}_{\alpha\in T_D\setminus\{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha$ in the tree-based format. Then for each $\mathbf{v} \in \mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ we define its* tree-based rank (TB rank) *by the tuple $(\dim U_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$.*

In order to characterise the tensors $\mathbf{v} \in \mathbf{V}_D$ satisfying $(\dim \mathbf{U}_\alpha^{\min}(\mathbf{v}))_{\alpha\in T_D} = \mathfrak{r}$, for a fixed $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$, we introduce the following definition.

**Definition 2.13** *We will say that $\mathfrak{r} := (r_\alpha)_{\alpha\in T_D} \in \mathbb{N}^{\#T_D}$ is an* admissible tuple for $T_D$, *if there exists $\mathbf{v} \in \mathbf{V}_D \setminus \{\mathbf{0}\}$ such that $\dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha$ for all $\alpha \in T_D \setminus \{D\}$.*

Necessary conditions for $\mathfrak{r} \in \mathbb{N}^{\#T_D}$ to be admissible are

$$\begin{array}{ll}
r_D = 1, & \\
r_{\{j\}} \le \dim V_j & \text{for } \{j\} \in \mathcal{L}(T_D), \\
r_\alpha \le \prod_{\beta\in S(\alpha)} r_\beta & \text{for } \alpha \in T_D \setminus \mathcal{L}(T_D), \\
r_\delta \le r_\alpha \prod_{\beta\in S(\alpha)\setminus\{\delta\}} r_\beta & \text{for } \alpha \in T_D \setminus \mathcal{L}(T_D) \text{ and } \delta \in S(\alpha).
\end{array}$$

## 2.4 The representations of tensors of fixed TB rank

Before introducing the representation of a tensor of fixed TB rank we need to define the set of coefficients of that tensors. To this end, we recall the definition of the 'matricisation' (or 'unfolding') of a tensor in a finite-dimensional setting.

**Definition 2.14** *For $\alpha \subset 2^D$, and $\beta \subset \alpha$ the map $\mathcal{M}_\beta$ is defined as the isomorphism*

$$\begin{array}{rccc} \mathcal{M}_\beta : & \mathbb{R}^{\times_{\mu\in\alpha} r_\mu} & \to & \mathbb{R}^{(\prod_{\mu\in\beta} r_\mu)\times(\prod_{\delta\in\alpha\setminus\beta} r_\delta)}, \\ & C_{(i_\mu)_{\mu\in\alpha}} & \mapsto & C_{(i_\mu)_{\mu\in\beta},(i_\delta)_{\delta\in\alpha\setminus\beta}} \end{array}$$

It allows to introduce the following definition.

**Definition 2.15** *For $\alpha \subset 2^D$, let $C^{(\alpha)} \in \mathbb{R}^{\times_{\mu\in\alpha} r_\mu}$. We say that $C^{(\alpha)} \in \mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}$ if and only if*

$$\prod_{\mu\in\alpha}\left(\det\left(\mathcal{M}_\mu(C^{(\alpha)})\mathcal{M}_\mu(C^{(\alpha)})^T\right) + \det\left(\mathcal{M}_\mu(C^{(\alpha)})^T\mathcal{M}_\mu(C^{(\alpha)})\right)\right) > 0,$$

*where $\mathcal{M}_\mu(C^{(\alpha)}) \in \mathbb{R}^{r_\mu\times(\prod_{\delta\in\alpha\setminus\{\mu\}} r_\delta)}$ for each $\mu \in \alpha$. We point out that this condition is equivalent to the condition that all $\mathcal{M}_\mu(C^{(\alpha)})$ have maximal rank.*

Since the determinant is a continuous function, $\mathbb{R}_*^{\times_{\mu\in\alpha} r_\mu}$ is an open set in $\mathbb{R}^{\times_{\mu\in\alpha} r_\mu}$.

**Definition 2.16** *Let $T_D$ be a given dimension partition tree and fix some tuple $\mathfrak{r} \in \mathbb{N}^{T_D}$. Then* the set of TBF tensors of fixed TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) := \left\{\mathbf{v} \in \mathbf{V}_D : \dim U_\alpha^{\min}(\mathbf{v}) = r_\alpha \text{ for all } \alpha \in T_D\right\} \tag{2.3}$$

*and the* set of TBF tensors of bounded TB rank $\mathfrak{r}$ *is defined by*

$$\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D) := \left\{\mathbf{v} \in \mathbf{V}_D : \ \dim U_\alpha^{\min}(\mathbf{v}) \leq r_\alpha \text{ for all } \alpha \in T_D \ \right\}. \tag{2.4}$$

Note that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) = \emptyset$ for an inadmissible tuple $\mathfrak{r}$. For $\mathfrak{r}, \mathfrak{s} \in \mathbb{N}^{T_D}$ we write $\mathfrak{s} \leq \mathfrak{r}$ if and only if $s_\alpha \leq r_\alpha$ for all $\alpha \in T_D$. Then we can also use the following notation

$$\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D) := \{\mathbf{0}\} \cup \bigcup_{\mathfrak{s}\leq\mathfrak{r}} \mathcal{FT}_{\mathfrak{s}}(\mathbf{V}_D). \tag{2.5}$$

Next we give some useful examples.

**Example 2.17 (Tucker format)** *Consider the dimension partition tree of $D := \{1, \ldots, d\}$, where $S(D) = \mathcal{L}(T_D) = \{\{j\} : 1 \leq j \leq d\}$. Let $(r_D, r_1, \ldots, r_d)$ be admissible, then $r_D = 1$ and $r_j \leq \dim V_j$ for $1 \leq j \leq d$. Thus we can write*

$$\mathcal{FT}_{\leq(1,r_1,\ldots,r_d)}(\mathbf{V}_D) = \mathcal{T}_{(r_1,\ldots,r_d)}(\mathbf{V}_D)$$

*and*

$$\mathcal{FT}_{(1,r_1,\ldots,r_d)}(\mathbf{V}_D) = \mathcal{M}_{(r_1,\ldots,r_d)}(\mathbf{V}_D).$$

**Example 2.18 (Tensor Train format)** *Consider a binary partition tree of $D := \{1, \ldots, d\}$ given by*

$$T_D = \{D, \{\{j\} : 1 \leq j \leq d\}, \{\{j+1, \ldots, d\} : 1 \leq j \leq d-2\}\}.$$

*In particular, $S(\{j, \ldots, d\}) = \{\{j\}, \{j+1, \ldots, d\}\}$ for $1 \leq j \leq d-1$. This tree-based format is related to the following chain of inclusions:*

$$\mathbf{U}_D^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{2\cdots d}^{\min}(\mathbf{v}) \subset \mathbf{U}_1^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_2^{\min}(\mathbf{v}) \otimes_a \mathbf{U}_{3\cdots d}^{\min}(\mathbf{v}) \subset \cdots \subset {}_a\bigotimes_{j\in D} \mathbf{U}_j^{\min}(\mathbf{v}).$$

The next result gives us a characterisation of the tensors in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

**Theorem 2.19** *Let $V_j$ be vector spaces for $j \in D$ and $T_D$ be a dimension partition tree of $D$. Then the following two statements are equivalent.*

*(a)* $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

*(b) There exists $\{u_{i_k}^{(k)} : 1 \leq i_k \leq r_k\}$ a basis of $U_k^{\min}(\mathbf{v})$ for $k \in \mathcal{L}(T_D)$ where for each $\mu \in T_D \setminus \mathcal{L}(T_D)$ there exists a unique $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \times_{\beta\in S(\mu)} r_\beta}$ such that the set $\{\mathbf{u}_{i_\mu}^{(\mu)} : 1 \leq i_\mu \leq r_\alpha\}$, with*

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1\leq i_\beta \leq r_\beta \\ \beta \in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} \mathbf{u}_{i_\beta}^{(\beta)} \tag{2.6}$$

*for $1 \leq i_\mu \leq r_\mu$, is a basis of $U_\mu^{\min}(\mathbf{v})$ and*

$$\mathbf{v} = \sum_{\substack{1\leq i_\alpha \leq r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)}. \tag{2.7}$$

*Furthermore, if $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ then (2.7) can be written for each $\alpha \in S(D)$ as*

$$\mathbf{v} = \sum_{1\leq i_\alpha \leq r_\alpha} \mathbf{u}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}. \tag{2.8}$$

*where $U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \leq i_\alpha \leq r_\alpha\}$, and for each $\mu \in T_D \setminus \mathcal{L}(T_D)$ we have*

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{1\leq i_\beta \leq r_\beta} \mathbf{u}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\mu,i_\beta}^{(\beta)},$$

*where*

$$\mathbf{U}_{i_\mu,i_\beta}^{(\beta)} := \sum_{\substack{1\leq i_\delta \leq r_\delta \\ \delta \in S(\mu) \\ \delta\neq\beta}} C^{(\mu)}_{i_\mu,(i_\delta)_{\delta\in S(\mu)}} \bigotimes_{\substack{\delta\in S(\mu) \\ \delta\neq\beta}} \mathbf{u}_{i_\delta}^{(\delta)}, \tag{2.9}$$

*and $U^{\min}_{S(\mu)\setminus\{\beta\}}(\mathbf{v}) = U^{\min}_{S(\mu)\setminus\{\beta\}}(\mathbf{u}_{i_\mu}^{(\mu)}) = \operatorname{span}\left\{\mathbf{U}_{i_\mu,i_\beta}^{(\beta)} : 1 \leq i_\beta \leq r_\beta\right\}$ for $1 \leq i_\mu \leq r_\mu$.*

*Proof.* Assuming first that (b) is true, (a) follows by the definition of $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Now, assume that (a) holds. Since $\mathbf{v} \in {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v})$ , there exists a unique $C^{(D)} \in \mathbb{R}^{\times_{\alpha\in S(D)} r_\alpha}$ such that

$$\mathbf{v} = \sum_{\substack{1\leq i_\alpha \leq r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)},$$

where $\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \leq i_\alpha \leq r_\alpha\}$ is a basis[2] of $U_\alpha^{\min}(\mathbf{v})$. For each $\alpha \in S(D)$ we set

$$\mathbf{U}_{i_\alpha}^{(\alpha)} := \sum_{\substack{1\leq i_\beta \leq r_\beta \\ \beta \in S(D) \\ \beta\neq\alpha}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\substack{\beta\in S(D) \\ \beta\neq\alpha}} u_{i_\beta}^{(\beta)}, \tag{2.10}$$

then (2.7) can be written as (2.8). From the definition of minimal subspaces we can write

$$U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) = \{(\mathbf{id}_{[\alpha]} \otimes \varphi_\alpha)(\mathbf{v}) : \varphi_\alpha \in U_\alpha^{\min}(\mathbf{v})^*\}.$$

[2]There are a small issue with the bold notation when $\alpha \in S(\mu)$ and $\alpha$ is a leaf, then $u_{i_\alpha}^{(\alpha)}$ should not be bold.

We claim that $\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ is a basis of $U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v})$. To prove the claim assume that $\mathbf{U}_1^{(\alpha)}$ is a linear combination of $\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 2 \le i_\alpha \le r_\alpha\}$, then $\mathbf{U}_1^{(\alpha)} = \sum_{2\le i_\alpha \le r_\alpha} \lambda_{i_\alpha} \mathbf{U}_{i_\alpha}^{(\alpha)}$ where $\lambda_{i_\alpha} \neq 0$ for some $2 \le i_\alpha \le r_\alpha$. Thus,

$$\mathbf{v} = \sum_{2\le i_\alpha \le r_\alpha} (\mathbf{u}_{i_\alpha}^{(\alpha)} + \lambda_{i_\alpha}\mathbf{u}_1^{(\alpha)}) \otimes \mathbf{U}_{i_\alpha}^{(\alpha)},$$

since $\{\mathbf{u}_{i_\alpha}^{(\alpha)} + \lambda_{i_\alpha}\mathbf{u}_1^{(\alpha)} : 2 \le i_\alpha \le r_\alpha\}$ are linearly independent we have $\dim U_\alpha^{\min}(\mathbf{v}) < r_\alpha$, a contradiction. Since $\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ are linearly independent for each $\alpha \in S(D)$, from (2.8) we have that

$$U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\},$$

and from (2.10), we deduce that $\mathcal{M}_\alpha(C^{(D)})$ maps a basis into another one for each $\alpha \in S(D)$ and hence $C^{(D)} \in \mathbb{R}_*^{\times_{\beta\in S(D)} r_\beta}$. In consequence, when $S(D) = \mathcal{L}(T_D)$ statement (a) holds and then (2.7) gives us the classical Tucker representation.

Next, assume $S(D) \neq \mathcal{L}(T_D)$. Then, for each $\mu \in T_D\setminus\{D\}$ such that $S(\mu) \neq \emptyset$, thanks to Proposition 2.10, we have

$$U_\mu^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\mu)} U_\beta^{\min}(\mathbf{v})\ .$$

Consider $\{\mathbf{u}_{i_\mu}^{(\mu)} : 1 \le i_\mu \le r_\mu\}$ a basis of $U_\mu^{\min}(\mathbf{v})$ and $\{\mathbf{u}_{i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta\}$ a basis of $U_\beta^{\min}(\mathbf{v})$ for $\beta \in S(\mu)$ and $1 \le i_\mu \le r_\mu$. Then, there exists a unique $C^{(\mu)} \in \mathbb{R}^{r_\mu \times (\times_{\beta\in S(\alpha)} r_\beta)}$ such that

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\mu)}} C_{i_\mu,(i_\beta)_{\beta\in S(\mu)}}^{(\mu)} \bigotimes_{\beta\in S(\mu)} \mathbf{u}_{i_\beta}^{(\beta)},$$

for $1 \le i_\mu \le r_\mu$. Since $\{\mathbf{u}_{i_\mu}^{(\mu)} : 1 \le i_\mu \le r_\mu\}$ is a basis, we can identify $C^{(\mu)}$ with the matrix $\mathcal{M}_\mu(C^{(\mu)})$, in the non-compact Stiefel manifold $\mathbb{R}_*^{r_\mu \times (\prod_{\beta\in S(\mu)} r_\beta)}$, which is the set of matrices in $\mathbb{R}^{r_\mu \times (\prod_{\beta\in S(\alpha)} r_\beta)}$ whose rows are linearly independent (see 3.1.5 in [1]). From (2.7) and (2.6) we obtain the Tucker representation of $\mathbf{v}$, when $S(D) \neq \mathcal{L}(T_D)$, as

$$\mathbf{v} = \sum_{\substack{1\le i_k\le r_k\\ k\in\mathcal{L}(T_D)}} \left( \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in T_D\setminus\{D\}\\ \alpha\notin\mathcal{L}(T_D)}} C_{(i_\alpha)_{\alpha\in S(D)}}^{(D)} \prod_{\substack{\mu\in T_D\setminus\{D\}\\ S(\mu)\neq\emptyset}} C_{i_\mu,(i_\beta)_{\beta\in S(\mu)}}^{(\mu)} \right) \bigotimes_{k\in\mathcal{L}(T_D)} u_{i_k}^{(k)}, \tag{2.11}$$

here $\{u_{i_k}^{(k)} : 1 \le i_k \le r_k\}$ is a basis of $U_k^{\min}(\mathbf{v})$ for each $k \in \mathcal{L}(T_D)$. To conclude, we claim that $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times (\times_{\beta\in S(\alpha)} r_\beta)}$ for all $\mu \in T_D \setminus \mathcal{L}(T_D)$. To prove the claim we proceed in a similar way as in the root case, for each fixed $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$, we introduce (2.9). Hence, we can write (2.6) as

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{1\le i_\beta\le r_\beta} \mathbf{u}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\mu,i_\beta}^{(\beta)},$$

where $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. From Proposition 2.11(a), we have

$$\begin{aligned} U_\beta^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ (id_\beta \otimes \varphi^{(\mu\setminus\beta)})(\mathbf{u}_{i_\mu}^{(\mu)}) : 1 \le i_\mu \le r_\mu \text{ and } \varphi^{(\mu\setminus\beta)} \in {}_a\bigotimes_{\delta\in S(\mu)\setminus\{\beta\}} U_\delta^{\min}(\mathbf{v})' \right\} \\ &= \operatorname{span}\left\{ (id_\beta \otimes \varphi^{(\mu\setminus\beta)})(\mathbf{u}_{i_\mu}^{(\mu)}) : 1 \le i_\mu \le r_\mu \text{ and } \varphi^{(\mu\setminus\beta)} \in {}_a\bigotimes_{\delta\in S(\mu)\setminus\{\beta\}} \mathbf{V}_\delta' \right\}, \end{aligned}$$

and hence $U_\beta^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) \subset U_\beta^{\min}(\mathbf{v})$ for $1 \le i_\mu \le r_\mu$. Let us consider $\{\varphi_{i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta\} \subset U_\beta^{\min}(\mathbf{v})'$ a dual basis of the finite-dimensional space $\{\mathbf{u}_{i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta\}$, that is, $\varphi_{i_\beta}^{(\beta)}(\mathbf{u}_{j_\beta}^{(\beta)}) = \delta_{i_\beta,j_\beta}$ for all $1 \le i_\beta, j_\beta \le r_\beta$, and $\beta \in S(\mu)$. Thus, we have

$$\left( id_\beta \otimes \bigotimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \varphi_{j_\delta}^{(\delta)} \right) (\mathbf{u}_{i_\mu}^{(\mu)}) = \sum_{1 \le j_\beta \le r_\beta} C^{(\mu)}_{i_\mu,(j_\delta)_{\delta\in S(\mu)}} \mathbf{u}_{j_\beta}^{(\beta)} \in U_\beta^{\min}(\mathbf{v})$$

for each multi-index $(j_\delta)_{\delta \in S(\mu)\setminus\beta} \in \times_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \{1,\ldots,r_\delta\}$. Then, for $\beta \in S(\mu)$,

$$U_\beta^{\min}(\mathbf{v}) = \operatorname{span}\left\{ \left( id_\beta \otimes \bigotimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \varphi_{j_\delta}^{(\delta)} \right) (\mathbf{u}_{i_\mu}^{(\mu)}) : (j_\delta)_{\delta\in S(\mu)\setminus\beta} \in \bigtimes_{\substack{\delta \in S(\mu) \\ \delta \neq \beta}} \{1,\ldots,r_\delta\},\, 1 \le i_\mu \le r_\mu \right\}$$

with $\dim U_\beta^{\min}(\mathbf{v}) = r_\beta$ if and only if $\operatorname{rank} \mathcal{M}_\beta(C^{(\mu)}) = r_\beta$ for $\beta \in S(\mu)$. Finally, we have $C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\times_{\delta\in S(\mu)} r_\delta\right)}$ for all $\mu \in T_D \setminus \mathcal{L}(T_D)$ and the claim follows. Thus, statement (b) holds.

To end the proof of the theorem, observe that in a similar way as above and by using $id_{S(\mu)\setminus\beta} \otimes \varphi_{j_\beta}^{(\beta)}$ for $1 \le j_\beta \le r_\beta$, over $\mathbf{u}_{i_\mu}^{(\mu)}$ it can be proved that

$$U_{S(\mu)\setminus\{\beta\}}^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) = \operatorname{span}\left\{ \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta \right\}$$

for $1 \le i_\mu \le r_\mu$ and also

$$U_{S(\mu)\setminus\{\beta\}}^{\min}(\mathbf{v}) = \operatorname{span}\left\{ \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta,\, 1 \le i_\mu \le r_\mu. \right\}.$$

Now, we claim that $\left\{ \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} : 1 \le i_\beta \le r_\beta \right\}$ are linearly independent in ${}_a\bigotimes_{\delta\neq\beta} \mathbf{V}_\delta$ for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. Otherwise, there exist $\lambda_{i_\beta}$ for $1 \le i_\beta \le r_\beta$ not all identically zero such that $\sum_{1\le i_\beta \le r_\beta} \lambda_{i_\beta} \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} = \mathbf{0}$. Take $\mathbf{w}_\beta \in \mathbf{V}_\beta \setminus \{\mathbf{0}\}$ and then

$$\mathbf{w}_\beta \otimes \left( \sum_{1\le i_\beta\le r_\beta} \lambda_{i_\beta} \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} \right) = \sum_{1\le i_\beta\le r_\beta} \lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} = \mathbf{0}.$$

Observe that

$$\sum_{1\le i_\beta\le r_\beta} \left( \lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} \right) = \sum_{\substack{1\le i_\delta \le r_\delta \\ \delta\in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\delta)_{\delta\in S(\mu)}} \lambda_{i_\beta} \mathbf{w}_\beta \otimes \left( \bigotimes_{\substack{\delta\neq\beta \\ \delta\in S(\mu)}} \mathbf{u}_{i_\delta}^{(\delta)} \right) = \mathbf{0},$$

for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$, take a dual basis of $\{\varphi_{i_\delta}^{(\delta)} : 1 \le i_\delta \le r_\delta\} \subset \mathbf{V}_\delta^*$ of $\{\mathbf{u}_{i_\delta}^{(\delta)} : 1 \le i_\delta \le r_\delta\} \subset \mathbf{V}_\delta$ where $\varphi_{i_\delta}^{(\delta)}(\mathbf{u}_{j_\delta}^{(\delta)}) = \delta_{i_\delta,j_\delta}$ for all $1 \le i_\delta, j_\delta \le r_\delta$. Then we obtain

$$id_\beta \otimes \left( \bigotimes_{\delta\in S(\mu)\setminus\{\beta\}} \varphi_{i_\delta}^{(\delta)} \right) \left( \sum_{1\le i_\beta\le r_\beta} \left( \lambda_{i_\beta} \mathbf{w}_\beta \otimes \mathbf{U}_{i_\mu,i_\beta}^{(\beta)} \right) \right) = \sum_{1\le i_\beta\le r_\beta} C^{(\mu)}_{i_\mu,(i_\delta)_{\delta\in S(\mu)}} \lambda_{i_\beta} \mathbf{w}_\beta = \mathbf{0},$$

that is, $\mathcal{M}_\beta(C^{(\mu)})^T \mathbf{z}_\beta = \mathbf{0}$, where $\mathbf{z}_\beta := (\lambda_{i_\beta}\mathbf{w}_\beta)_{i_\beta=1}^{r_\beta}$. Since $\operatorname{rank} \mathcal{M}_\beta(C^{(\mu)}) = r_\beta$, then $\dim \operatorname{Ker} \mathcal{M}_\beta(C^{(\mu)})^T = 0$, and hence $\mathbf{z}_\beta = (\lambda_{i_\beta}\mathbf{w}_\beta)_{i_\beta=1}^{r_\beta} = (\mathbf{0})_{i_\beta=1}^{r_\beta}$ for $\beta \in S(\gamma)$, a contradiction. In consequence,

$$\dim U_{S(\mu)\setminus\{\beta\}}^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) = \dim U_\beta^{\min}(\mathbf{u}_{i_\mu}^{(\mu)}) = r_\beta$$

for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. Hence $U_\beta^{\min}(\mathbf{v}) = U_\beta^{\min}(\mathbf{u}_{i_\mu}^{(\mu)})$ holds for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. ∎

# 3 Geometric structures for TBF tensors

Before characterising the "local coordinates" of a tensor $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ we need to introduce the Banach-Grassmann manifold and its relatives.

## 3.1 The Grassmann-Banach manifold and its relatives

In the following, $X$ is a Banach space with norm $\|\cdot\|$. The dual norm $\|\cdot\|_{X^*}$ of $X^*$ is

$$\|\varphi\|_{X^*} = \sup\{|\varphi(x)| : x \in X \text{ with } \|x\|_X \leq 1\} = \sup\{|\varphi(x)| / \|x\|_X : 0 \neq x \in X\}. \tag{3.1}$$

By $\mathcal{L}(X,Y)$ we denote the space of continuous linear mappings from $X$ into $Y$. The corresponding operator norm is written as $\|\cdot\|_{Y \leftarrow X}$.

**Definition 3.1** *Let $X$ be a Banach space. We say that $P \in \mathcal{L}(X,X)$ is a projection if $P \circ P = P$. In this situation we also say that $P$ is a projection from $X$ onto $P(X) := \operatorname{Im} P$ parallel to* $\operatorname{Ker} P$.

From now on, we will denote $P \circ P = P^2$. Observe that if $P$ is a projection then $I_X - P$ is also a projection. Moreover, $I_X - P$ is parallel to $\operatorname{Im} P$.

Observe that each projection gives rise to a pair of closed subspaces, namely $U = \operatorname{Im} P$ and $V = \operatorname{Ker} P$ such that $X = U \oplus V$. It allows us to introduce the following two definitions.

**Definition 3.2** *We will say that a subspace $U$ of a Banach space $X$ is a complemented subspace if $U$ is closed and there exists $V$ in $X$ such that $X = U \oplus V$ and $V$ is also a closed subspace of $X$. This subspace $V$ is called a (topological) complement of $U$ and $(U,V)$ is a pair of complementary subspaces.*

Corresponding to each pair $(U,V)$ of complementary subspaces, there is a projection $P$ mapping $X$ onto $U$ along $V$, defined as follows. Since for each $x$ there exists a unique decomposition $x = u + v$, where $u \in U$ and $v \in V$, we can define a linear map $P(u+v) := u$, where $\operatorname{Im} P = U$ and $\operatorname{Ker} P = V$. Moreover, $P^2 = P$.

**Definition 3.3** *The* Grassmann manifold *of a Banach space $X$, denoted by $\mathbb{G}(X)$, is the set of all complemented subspaces of $X$.*

$U \in \mathbb{G}(X)$ holds if and only if $U$ is a closed subspace and there exists a closed subspace $V$ in $X$ such that $X = U \oplus V$. Observe that $X$ and $\{0\}$ are in $\mathbb{G}(X)$. Moreover, by the proof of Proposition 4.2 of [10], the following result can be shown.

**Proposition 3.4** *Let $X$ be a Banach space. The following conditions are equivalent:*

*(a) $U \in \mathbb{G}(X)$.*

*(b) There exists $P \in \mathcal{L}(X,X)$ such that $P^2 = P$ and* $\operatorname{Im} P = U$.

*(c) There exists $Q \in \mathcal{L}(X,X)$ such that $Q^2 = Q$ and* $\operatorname{Ker} Q = U$.

Moreover, from Theorem 4.5 in [10], the following result can be shown.

**Proposition 3.5** *Let $X$ be a Banach space. Then every finite-dimensional subspace $U$ belongs to $\mathbb{G}(X)$.*

Let $V$ and $U$ be closed subspaces of a Banach space $X$ such that $X = U \oplus V$. From now on, we will denote by $P_{U \oplus V}$ the projection onto $U$ along $V$. Then we have $P_{V \oplus U} = I_X - P_{U \oplus V}$. Let $U, U' \in \mathbb{G}(X)$. We say that $U$ and $U'$ have a common complementary subspace in $X$, if $X = U \oplus W = U' \oplus W$ for some $W \in \mathbb{G}(X)$. The following result will be useful (see Lemma 2.1 in [8]).

**Lemma 3.6** *Let $X$ be a Banach space and assume that $W$, $U$, and $U'$ are in $\mathbb{G}(X)$. Then the following statements are equivalent:*

*(a) $X = U \oplus W = U' \oplus W$, i.e., $U$ and $U'$ have a common complement in $X$.*

(b) $P_{U\oplus W}|_{U'} : U' \to U$ *has an inverse.*

*Furthermore, if* $Q = \left(P_{U\oplus W}|_{U'}\right)^{-1}$, *then* $Q$ *is bounded and* $Q = P_{U'\oplus W}|_U$.

Next, we recall the definition of a Banach manifold.

**Definition 3.7** *Let* $\mathbb{M}$ *be a set. An atlas of class* $C^p$ $(p \geq 0)$ *on* $\mathbb{M}$ *is a family of charts with some indexing set* $A$, *namely* $\{(M_\alpha, u_\alpha) : \alpha \in A\}$, *having the following properties:*

*AT1* $\{M_\alpha\}_{\alpha\in A}$ *is a covering*[3] *of* $\mathbb{M}$, *that is,* $M_\alpha \subset \mathbb{M}$ *for all* $\alpha \in A$ *and* $\cup_{\alpha\in A} M_\alpha = \mathbb{M}$.

*AT2 For each* $\alpha \in A, (M_\alpha, u_\alpha)$ *stands for a bijection* $u_\alpha : M_\alpha \to U_\alpha$ *of* $M_\alpha$ *onto an open set* $U_\alpha$ *of a Banach space* $X_\alpha$, *and for any* $\alpha$ *and* $\beta$ *the set* $u_\alpha(M_\alpha \cap M_\beta)$ *is open in* $X_\alpha$.

*AT3 Finally, if we let* $M_\alpha \cap M_\beta = M_{\alpha\beta}$ *and* $u_\alpha(M_{\alpha\beta}) = U_{\alpha\beta}$, *the transition mapping* $u_\beta \circ u_\alpha^{-1} : U_{\alpha\beta} \to U_{\beta\alpha}$ *is a* $C^p$*-diffeomorphism.*

Since different atlases can give the same manifold, we say that two atlases are *compatible* if each chart of one atlas is compatible with the charts of the other atlas in the sense of AT3. One verifies that the relation of compatibility between atlases is an equivalence relation.

**Definition 3.8** *An equivalence class of atlases of class* $C^p$ *on* $\mathbb{M}$ *is said to define a structure of a* $C^p$*-Banach manifold on* $\mathbb{M}$, *and hence we say that* $\mathbb{M}$ *is a Banach manifold. In a similar way, if an equivalence class of atlases is given by analytic maps, then we say that* $\mathbb{M}$ *is an analytic Banach manifold. If* $X_\alpha$ *is a Hilbert space for all* $\alpha \in A$, *then we say that* $\mathbb{M}$ *is a Hilbert manifold.*

In condition AT2 we do not require that the Banach spaces are the same for all indices $\alpha$, or even that they are isomorphic. If $X_\alpha$ is linearly isomorphic to some Banach space $X$ for all $\alpha$, we have the following definition.

**Definition 3.9** *Let* $\mathbb{M}$ *be a set and* $X$ *be a Banach space. We say that* $\mathbb{M}$ *is a* $C^p$ *Banach manifold modelled on* $X$ *if there exists an atlas of class* $C^p$ *over* $\mathbb{M}$ *with* $X_\alpha$ *linearly isomorphic to* $X$ *for all* $\alpha \in A$.

**Example 3.10** *Every Banach space is a Banach manifold modelled on itself (for a Banach space* $Y$, *simply take* $(Y, I_Y)$ *as atlas, where* $I_Y$ *is the identity map on* $Y$*). In particular, the set of all bounded linear maps* $\mathcal{L}(X,X)$ *of a Banach space* $X$ *is also a Banach manifold modelled on itself.*

If $X$ is a Banach space, then the set of all bounded linear automorphisms of $X$ will be denoted by

$$\mathrm{GL}(X) := \{A \in \mathcal{L}(X,X) : A \text{ invertible }\}.$$

**Example 3.11** *If* $X$ *is a Banach space, then* $\mathrm{GL}(X)$ *is a Banach manifold modelled on* $\mathcal{L}(X,X)$, *because it is an open set in* $\mathcal{L}(X,X)$. *Moreover, the map* $A \mapsto A^{-1}$ *is analytic (see 2.7 in [32]).*

The next example is a Banach manifold not modelled on a particular Banach space.

**Example 3.12 (Grassmann–Banach manifold)** *Let* $X$ *be a Banach space. Then, following [9] (see also [32] and [26]), it is possible to construct an atlas for* $\mathbb{G}(X)$. *To do this, denote one of the complements of* $U \in \mathbb{G}(X)$ *by* $W$, *i.e.,* $X = U \oplus W$. *Then we define the* Banach Grassmannian of $U$ relative to $W$ *by*

$$\mathbb{G}(W,X) := \{V \in \mathbb{G}(X) : X = V \oplus W\}.$$

*By using Lemma 3.6 it is possible to introduce a bijection*

$$\Psi_{U\oplus W} : \mathbb{G}(W,X) \longrightarrow \mathcal{L}(U,W)$$

*defined by*

$$\Psi_{U\oplus W}(U') = P_{W\oplus U}|_{U'} \circ P_{U'\oplus W}|_U = P_{W\oplus U}|_{U'} \circ (P_{U\oplus W}|_{U'})^{-1}.$$

[3]The condition of an *open* covering is not needed, see [23].

*It can be shown that the inverse*

$$\Psi_{U\oplus W}^{-1} : \mathcal{L}(U,W) \longrightarrow \mathbb{G}(W,X),$$

*is given by*

$$\Psi_{U\oplus W}^{-1}(L) = G(L) := \{u + L(u) : u \in U\}.$$

*Observe that* $G(0) = U$ *and* $G(L) \oplus W = X$ *for all* $L \in \mathcal{L}(U,W)$. *Now, to prove that this manifold is analytic we need to describe the overlap maps. To explain the behaviour of one overlap map, assume that* $X = U \oplus W = U' \oplus W'$ *and the existence of* $U'' \in \mathbb{G}(W,X) \cap \mathbb{G}(W',X)$. *Let* $L \in \mathcal{L}(U,W)$ *be such that*

$$U'' = G(L) = \Psi_{U\oplus W}^{-1}(L).$$

*and then*

$$X = U \oplus W = U' \oplus W' = G(L) \oplus W = G(L) \oplus W'.$$

*Since* $(id + L)$ *is a linear isomorphism from* $U$ *to* $U'' = G(L)$ *then* $T := P_{U'\oplus W'} \circ (id + L)$ *is a linear isomorphism from* $U$ *to* $U'$. *It follows that the map* $(\Psi_{U'\oplus W'} \circ \Psi_{U\oplus W}^{-1}) : \mathcal{L}(U,W) \to \mathcal{L}(U',W')$ *given by*

$$\begin{aligned}(\Psi_{U'\oplus W'} \circ \Psi_{U\oplus W}^{-1})(L) &= \Psi_{U'\oplus W'}(G(L)) = P_{W'\oplus U'}|_{G(L)} \circ (P_{U'\oplus W'}|_{G(L)})^{-1} \\ &= \Psi_{U'\oplus W'}(G(L)) = P_{W'\oplus U'}|_{G(L)} \circ P_{G(L)\oplus W'}|_{U'} \circ T \circ T^{-1} \\ &= P_{W'\oplus U'}|_{G(L)} \circ P_{G(L)\oplus W'}|_{U'} \circ P_{U'\oplus W'} \circ (id + L) \circ T^{-1} \\ &= P_{W'\oplus U'} \circ (id + L) \circ (P_{U'\oplus W'} \circ (id + L))^{-1}.\end{aligned}$$

*is analytic. Then we say that the collection* $\{\Psi_{U\oplus W}, \mathbb{G}(W,X)\}_{U\in\mathbb{G}(X)}$ *is an analytic atlas, and therefore,* $\mathbb{G}(X)$ *is an analytic Banach manifold. In particular, for each* $U \in \mathbb{G}(X)$ *the set* $\mathbb{G}(W,X) \overset{\Psi_{U\oplus W}}{\cong} \mathcal{L}(U,W)$ *is a Banach manifold modelled on* $\mathcal{L}(U,W)$. *Observe that if* $U$ *and* $U'$ *are finite-dimensional subspaces of* $X$ *such that* $\dim U \neq \dim U'$ *and* $X = U \oplus W = U' \oplus W'$, *then* $\mathcal{L}(U,W)$ *is not linearly isomorphic to* $\mathcal{L}(U',W')$.

**Example 3.13** *Let* $X$ *be a Banach space. From Proposition 3.5, every finite-dimensional subspace belongs to* $\mathbb{G}(X)$. *It allows to introduce* $\mathbb{G}_n(X)$, *the space of all* $n$*-dimensional subspaces of* $X$ $(n \geq 0)$. *It can be shown (see [26]) that* $\mathbb{G}_n(X)$ *is a connected component of* $\mathbb{G}(X)$, *and hence it is also a Banach manifold modelled on* $\mathcal{L}(U,W)$, *here* $U \in \mathbb{G}_n(X)$ *and* $X = U \oplus W$. *Moreover,*

$$\mathbb{G}_{\leq r}(X) := \bigcup_{n\leq r} \mathbb{G}_n(X)$$

*is also a Banach manifold for each fixed* $r < \infty$.

The next example introduce the Banach-Grassmannian manifold for a normed (non-Banach) space. To the authors knowledge there are not references in the literature about this (nontrivial) Banach manifold structure. We need the following lemma.

**Lemma 3.14** *Assume that* $(X, \|\cdot\|)$ *is a normed space and let* $\overline{X}$ *be the Banach space obtained as the completion of* $X$. *Let* $U \in \mathbb{G}_n(\overline{X})$ *be such that* $U \subset X$ *and* $\overline{X} = U \oplus W$ *for some* $W \in \mathbb{G}(\overline{X})$. *Then every subspace* $U' \in \mathbb{G}(W, \overline{X})$ *is a subspace of* $X$, *that is,* $U' \subset X$.

*Proof.* First at all observe that $X = U \oplus (W \cap X)$ where $W \cap X$ is a linear subspace dense in $W = W \cap \overline{X}$. Assume that the lemma is not true. Then there exists $U' \in \mathbb{G}(W, \overline{X})$ such that $U' \oplus W = \overline{X}$ and $U' \cap X \neq U'$. Clearly $U' \cap X \neq \{0\}$, otherwise $W \cap X = X$ a contradiction. We have $X = (U' \cap X) \oplus (W \cap X)$, which implies $\overline{X} = (U' \cap X) \oplus W$, a contradiction and the lemma follows. ■

**Example 3.15** *Assume that* $(X, \|\cdot\|)$ *is a normed space and let* $\overline{X}$ *be the Banach space obtained as the completion of* $X$. *We define the set* $\mathbb{G}_n(X)$ *as follows. We say that* $U \in \mathbb{G}_n(X)$ *if and only if* $U \in \mathbb{G}_n(\overline{X})$ *and* $U \subset X$. *Then* $\mathbb{G}_n(X)$ *is also a Banach manifold. To see this observe that, by Lemma 3.14, for each* $U \in \mathbb{G}_n(X)$ *such that* $\overline{X} = U \oplus W$ *for some* $W \in \mathbb{G}(\overline{X})$, *we have* $\mathbb{G}(W, \overline{X}) \subset \mathbb{G}_n(X)$. *Then the collection* $\{\Psi_{U\oplus W}, \mathbb{G}(W, \overline{X})\}_{U\in\mathbb{G}_n(X)}$ *is an analytic atlas on* $\mathbb{G}_n(X)$, *and therefore,* $\mathbb{G}_n(X)$ *is an analytic Banach manifold modelled on* $\mathcal{L}(U,W)$, *here* $U \in \mathbb{G}_n(X)$ *and* $\overline{X} = U \oplus W$. *Moreover, as in Example 3.13, we can define a Banach manifold* $\mathbb{G}_{\leq r}(X)$ *for each fixed* $r < \infty$.

Let $\mathbb{M}$ be a Banach manifold of class $\mathcal{C}^p$, $p \geq 1$. Let $m$ be a point of $\mathbb{M}$. We consider triples $(U, \varphi, v)$ where $(U, \varphi)$ is a chart at $m$ and $v$ is an element of the vector space in which $\varphi(U)$ lies. We say that two of such triples $(U, \varphi, v)$ and $(V, \psi, w)$ are *equivalent* if the derivative of $\psi\varphi^{-1}$ at $\varphi(m)$ maps $v$ on $w$. Thanks to the chain rule it is an equivalence relation. An equivalence class of such triples is called a *tangent vector of* $\mathbb{M}$ *at* $m$.

**Definition 3.16** *The set of such tangent vectors is called the* tangent space of $\mathbb{M}$ at $m$ *and it is denoted by* $\mathbb{T}_m(\mathbb{M})$.

Each chart $(U, \varphi)$ determines a bijection of $\mathbb{T}_m(\mathbb{M})$ on a Banach space, namely the equivalence class of $(U, \varphi, v)$ corresponds to the vector $v$. By means of such a bijection it is possible to equip $\mathbb{T}_m(\mathbb{M})$ with the structure of a topological vector space given by the chart, and it is immediate that this structure is independent of the selected chart.

**Example 3.17** *If $X$ is a Banach space, then $\mathbb{T}_x(X) = X$ for all $x \in X$.*

**Example 3.18** *Let $X$ be a Banach space and take $A \in \mathrm{GL}(X)$. Then $\mathbb{T}_A(\mathrm{GL}(X)) = \mathcal{L}(X, X)$.*

**Example 3.19** *For $U \in \mathbb{G}(X)$ such that $X = U \oplus W$ for some $W \in \mathbb{G}(X)$, we have $\mathbb{T}_U(\mathbb{G}(X)) = \mathcal{L}(U, W)$.*

**Example 3.20** *We point out that for a Hilbert space $X$ with associated inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, its unit sphere denoted by*

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\},$$

*is a Hilbert manifold of codimension one. Moreover, for each $x \in \mathbb{S}_X$, its tangent space is*

$$\mathbb{T}_x(\mathbb{S}_X) = \mathrm{span}\,\{x\}^{\perp} = \{x' \in X : \langle x, x' \rangle = 0\}.$$

## 3.2 The manifold of TBF tensors of fixed TB rank

Assume that $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in the tree-based format where for each $k \in \mathcal{L}(T_D)$ the vector space $V_k$ is a normed space with a norm $\| \cdot \|_k$. As usual $V_{k_{\| \cdot \|_k}}$ denotes the corresponding Banach space obtained from $V_k$ for $k \in \mathcal{L}(T_D)$. From now on, to simplify the notation, we introduce for an admissible $\mathfrak{r} \in \mathbb{N}^{T_D}$ the product vector space

$$\mathbb{R}^{\mathfrak{r}} := \bigtimes_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \mathbb{R}^{r_\alpha \times \left(\times_{\beta \in S(\alpha)} r_\beta\right)},$$

with $r_D = 1$. It allows us to introduce its open subset $\mathbb{R}^{\mathfrak{r}}_*$, and hence a manifold, defined as

$$\mathbb{R}^{\mathfrak{r}}_* := \left\{ \mathfrak{C} \in \mathbb{R}^{\mathfrak{r}} : \begin{array}{l} C^{(D)} \in \mathbb{R}_*^{\times_{\alpha \in S(D)} r_\alpha} \text{and } C^{(\mu)} \in \mathbb{R}_*^{r_\mu \times \left(\times_{\beta \in S(\mu)} r_\beta\right)} \\ \text{for each } \mu \in T_D \setminus \{D\} \text{ such that } S(\mu) \neq \emptyset. \end{array} \right\}.$$

From Theorem 2.19 we know that each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is totally characterised by $\mathfrak{C} = \mathfrak{C}(\mathbf{v}) \in \mathbb{R}^{\mathfrak{r}}_*$ and a basis $\{u^{(k)}_{i_k} : 1 \leq i_k \leq r_k\}$ of $U^{\min}_k(\mathbf{v})$ for $k \in \mathcal{L}(T_D)$. Recall that in Example 3.15, the finite-dimensional subspace $U^{\min}_k(\mathbf{v}) \subset V_k \subset V_{k_{\| \cdot \|_k}}$ belongs to the Banach manifold $\mathbb{G}_{r_k}(V_k)$ for $k \in \mathcal{L}(T_D)$ (see also Example 3.13) and for each $k \in \mathcal{L}(T_D)$, there exists a bijection (local chart)

$$\Psi_{U^{\min}_k(\mathbf{v}) \oplus W^{\min}_k(\mathbf{v})} : \mathbb{G}(W^{\min}_k(\mathbf{v}), \mathbf{V}_{k_{\| \cdot \|_k}}) \to \mathcal{L}(U^{\min}_k(\mathbf{v}), W^{\min}_k(\mathbf{v}))$$

given by

$$\Psi_{U^{\min}_k(\mathbf{v}) \oplus W^{\min}_k(\mathbf{v})}(U_k) = L_k := P_{W^{\min}_k(\mathbf{v}) \oplus U^{\min}_k(\mathbf{v})}|_{U_k} \circ (P_{U^{\min}_k(\mathbf{v}) \oplus W^{\min}_k(\mathbf{v})}|_{U_k})^{-1}.$$

Moreover, $U_k = G(L_k) = \mathrm{span}\{u_k + L_k(u_k) : u_k \in U^{\min}_k(\mathbf{v})\}$. Clearly, the map

$$\mathbf{\Psi}_{\mathbf{v}} : \bigtimes_{k \in \mathcal{L}(T_D)} \mathbb{G}(W^{\min}_k(\mathbf{v}), \mathbf{V}_{k_{\| \cdot \|_k}}) \to \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U^{\min}_k(\mathbf{v}), W^{\min}_k(\mathbf{v})),$$

defined as $\mathbf{\Psi_v} := \bigtimes_{k\in\mathcal{L}(T_D)} \Psi_{U_k^{\min}(\mathbf{v})\oplus W_k^{\min}(\mathbf{v})}$ is also bijective. Furthermore, it is a local chart for an element $\mathfrak{U}(\mathbf{v}) = \{U_k^{\min}(\mathbf{v})\}_{k\in\mathcal{L}(T_D)}$ in the product manifold such that $\mathbf{\Psi_v}(\mathfrak{U}(\mathbf{v})) = \mathfrak{o} := (0)_{k\in\mathcal{L}(T_D)}$. It allows us to introduce the surjective map

$$\varrho_{\mathfrak{r}} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \to \bigtimes_{j\in\mathcal{L}(T_D)} \mathbb{G}_{r_j}(V_j)$$

defined by $\varrho_{\mathfrak{r}}(\mathbf{v}) = \mathfrak{U}(\mathbf{v}) := (U_k^{\min}(\mathbf{v}))_{k\in\mathcal{L}(T_D)}$. Now, for each $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ introduce the set

$$\mathcal{U}(\mathbf{v}) := \varrho_{\mathfrak{r}}^{-1}\left(\bigtimes_{j\in\mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j)\right) = \left\{\mathbf{w}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) : U_k^{\min}(\mathbf{w}) \in \mathbb{G}(W_k^{\min}(\mathbf{v}), V_k),\ 1\le k\le d\right\}.$$

Our next step is to construct the following natural bijection. Let

$$\chi_{\mathfrak{r}}(\mathbf{v}) : \mathcal{U}(\mathbf{v}) \to \left(\bigtimes_{j\in\mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j)\right)\times\mathbb{R}_*^{\mathfrak{r}}, \quad \mathbf{w}\mapsto(\chi_1(\mathbf{v})(\mathbf{w}), \chi_2(\mathbf{v})(\mathbf{w}))$$

defined as follows. Let $\mathbf{w}\in\mathcal{U}(\mathbf{v})$. From Theorem 2.19 we have the following.

(a) There exists a basis of $U_k^{\min}(\mathbf{w}) \in \mathbb{G}(W_k^{\min}(\mathbf{v}), V_{k_{\|\cdot\|_k}})$, for each $k\in\mathcal{L}(T_D)$ and hence a unique

$$\mathfrak{L} = (L_k)_{k\in\mathcal{L}(T_D)} \in \bigtimes_{k\in\mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v}))$$

such that $\mathbf{\Psi_v}(\varrho_{\mathfrak{r}}(\mathbf{w})) = \mathfrak{L}$, that is, $U_k^{\min}(\mathbf{w}) = G(L_k)$ for all $k\in\mathcal{L}(T_D)$. Then $\chi_1(\mathbf{v})(\mathbf{w}) := \mathbf{\Psi_v}^{-1}(\mathfrak{L})$ and $U_k^{\min}(\mathbf{w}) = G(L_k) = \operatorname{span}\{(id_k+L_k)(u_{i_k}^{(k)}) : 1\le i_k\le r_k\}$ where $U_k^{\min}(\mathbf{v}) = \operatorname{span}\{u_{i_k}^{(k)} : 1\le i_k\le r_k\}$ is a fixed basis for $k\in\mathcal{L}(T_D)$ and hence $\mathbf{\Psi_v}(\varrho_{\mathfrak{r}}(\mathbf{v})) = (0)_{k\in\mathcal{L}(T_D)}$.

(b) There exists a unique $\chi_2(\mathbf{v})(\mathbf{w}) := \mathfrak{C} = (C^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)} \in \mathbb{R}_*^{\mathfrak{r}}$ such that

$$\mathbf{w} = \sum_{\substack{1\le i_\alpha\le r_\alpha\\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{w}_{i_\alpha}^{(\alpha)},. \tag{3.2}$$

and where for each $\beta\in T_D\setminus(\{D\}\cup\mathcal{L}(T_D))$ we have

$$U_\beta^{\min}(\mathbf{w}) = \operatorname{span}\{\mathbf{w}_{i_\beta} : 1\le i_\beta\le r_\beta\}$$

with

$$\mathbf{w}_{i_\beta}^{(\beta)} = \begin{cases} (id_\beta + L_\beta)(u_{i_\beta}^{(\beta)}) & \text{if } \beta\in\mathcal{L}(T_D) \\ \sum_{\substack{1\le i_\delta\le r_\delta\\ \delta\in S(\beta)}} C^{(\beta)}_{i_\beta,(i_\delta)_{\delta\in S(\beta)}} \bigotimes_{\delta\in S(\beta)} \mathbf{w}_{i_\delta}^{(\delta)} & \text{otherwise.}\end{cases}$$

Finally, let

$$p_{\mathbf{v}} : \left(\bigtimes_{j\in\mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j)\right)\times\mathbb{R}_*^{\mathfrak{r}} \to \bigtimes_{j\in\mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j)$$

be the projection $p_{\mathbf{v}}(\mathfrak{U},\mathfrak{C}) = \mathfrak{U}$ then $p_{\mathbf{v}}\circ\chi_{\mathfrak{r}}(\mathbf{v}) = \varrho_{\mathfrak{r}}$.

A very useful remark is the following. Recall that $(id_k+L_k)$ is a linear isomorphism from $U_k^{\min}(\mathbf{v})$ to $U_k^{\min}(\mathbf{w}) = G(L_k)$ for all $k\in\mathcal{L}(T_D)$. From Proposition 3.49 of [19] we have

$$_a\bigotimes_{k\in\mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), U_k^{\min}(\mathbf{w})) = \mathcal{L}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)} U_k^{\min}(\mathbf{v}),\ {}_a\bigotimes_{k\in\mathcal{L}(T_D)} U_k^{\min}(\mathbf{w})\right)$$

and denote by $\mathrm{GL}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\,,\,{}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{w})\right)$ the set of linear isomorphisms of

$$\mathcal{L}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\,,\,{}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{w})\right).$$

Let us define

$$\mathrm{GL}_{\mathbf{1}}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\,,\,{}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{w})\right):=$$

$$\mathrm{GL}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\,,\,{}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{w})\right)\cap\mathcal{M}_{\leq\mathbf{1}}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}\mathcal{L}(U_k^{\min}(\mathbf{v}),U_k^{\min}(\mathbf{w}))\right).$$

Then

$$\bigotimes_{k\in\mathcal{L}(T_D)}(id_k+L_k)\in\mathrm{GL}_{\mathbf{1}}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\,,\,{}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{w})\right).$$

Observe that for each given $\mathbf{v}\in\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ the map

$$\Theta_{\mathbf{v}}:\left(\bigtimes_{j\in\mathcal{L}(T_D)}\mathbb{G}(W_j^{\min}(\mathbf{v}),V_j)\right)\times\mathbb{R}_*^{\mathfrak{r}}\to\left(\bigtimes_{j\in\mathcal{L}(T_D)}\mathcal{L}(U_j^{\min}(\mathbf{v}),W_j^{\min}(\mathbf{v}))\right)\times\mathbb{R}_*^{\mathfrak{r}}$$

where $\Theta_{\mathbf{v}}:=\mathbf{\Psi}_{\mathbf{v}}\times id$ is a bijection. Then

$$\Theta_{\mathbf{v}}\circ\chi_{\mathfrak{r}}(\mathbf{v}):\mathcal{U}(\mathbf{v})\to\left(\bigtimes_{j\in\mathcal{L}(T_D)}\mathcal{L}(U_j^{\min}(\mathbf{v}),W_j^{\min}(\mathbf{v}))\right)\times\mathbb{R}_*^{\mathfrak{r}}$$

is also a bijection where

$$(\Theta_{\mathbf{v}}\circ\chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{L},\mathfrak{C})=\mathbf{w}=\left(\bigotimes_{k\in\mathcal{L}(T_D)}(id_k+L_k)\right)(\mathbf{u})=\left(\bigotimes_{k\in\mathcal{L}(T_D)}(id_k+L_k)\right)(\Theta_{\mathbf{v}}\circ\chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{o},\mathfrak{C}).$$

We can interpret this last equality as follows. $\mathbf{w}\in\mathcal{U}(\mathbf{v})$ holds if and only if

$$\mathbf{w}\in\mathcal{FT}_{\mathfrak{r}}\left(\left(\bigotimes_{k\in\mathcal{L}(T_D)}(id_k+L_k)\right)\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\right)\right)$$

for some $\mathfrak{L}\in\bigtimes_{k\in\mathcal{L}(T_D)}\mathcal{L}(U_k^{\min}(\mathbf{v}),W_k^{\min}(\mathbf{v}))$. In consequence, each neighbourhood of $\mathbf{v}$ in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ can be written as

$$\mathcal{U}(\mathbf{v})=\bigcup_{\mathfrak{L}\in\bigtimes_{k\in\mathcal{L}(T_D)}\mathcal{L}(U_k^{\min}(\mathbf{v}),W_k^{\min}(\mathbf{v}))}\mathcal{FT}_{\mathfrak{r}}\left(\left(\bigotimes_{k\in\mathcal{L}(T_D)}(id_k+L_k)\right)\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}U_k^{\min}(\mathbf{v})\right)\right),$$

that is, a union of copies of $\mathcal{FT}_{\mathfrak{r}}\left({}_a\bigotimes_{k\in\mathcal{L}(T_D)}\mathbb{R}^{r_k}\right)$ indexed by a Banach manifold. Before stating the next result, we introduce the following definition.

**Definition 3.21** *Let $X$ and $Y$ be two Banach manifolds. Let $F:X\to Y$ be a map. We shall say that $F$ is a $\mathcal{C}^r$ (respectively, analytic)* morphism *if given $x\in X$ there exists a chart $(U,\varphi)$ at $x$ and a chart $(W,\psi)$ at $F(x)$ such that $F(U)\subset W$, and the map*

$$\psi\circ F\circ\varphi^{-1}:\varphi(U)\to\psi(W)$$

*is a $\mathcal{C}^r$-Fréchet differentiable (respectively, analytic) map.*

**Lemma 3.22** *Let* $\mathbf{v}, \mathbf{v}' \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ *be such that* $\mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{v}') \neq \emptyset$. *Then the bijective map*

$$\chi_{\mathfrak{r}}(\mathbf{v}') \circ \chi_{\mathfrak{r}}(\mathbf{v})^{-1} : \left( \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j) \right) \times \mathbb{R}_*^{\mathfrak{r}} \to \left( \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}'), V_j) \right) \times \mathbb{R}_*^{\mathfrak{r}}$$

*is an analytic morphism. Furthermore, it is an analytic diffeomorphism.*

*Proof.* Let $\mathbf{v}, \mathbf{v}' \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ be given. To prove the lemma we need to check that the map

$$\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}') \circ \chi_{\mathfrak{r}}(\mathbf{v})^{-1} \circ \Theta_{\mathbf{v}}^{-1} : \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})) \times \mathbb{R}_*^{\mathfrak{r}} \to \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}'), W_k^{\min}(\mathbf{v}')) \times \mathbb{R}_*^{\mathfrak{r}}$$

is analytic whenever $\mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{v}') \neq \emptyset$. Let $\mathbf{w} \in \mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{v}')$ be such that $(\chi_{\mathfrak{r}}(\mathbf{v}) \circ \Theta_{\mathbf{v}})(\mathbf{w}) = (\mathfrak{L}, \mathfrak{C})$ and $(\chi_{\mathfrak{r}}(\mathbf{v}') \circ \Theta_{\mathbf{v}'})(\mathbf{w}) = (\mathfrak{L}', \mathfrak{D})$, that is,

$$(\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}') \circ \chi_{\mathfrak{r}}(\mathbf{v})^{-1} \circ \Theta_{\mathbf{v}}^{-1})(\mathfrak{L}, \mathfrak{C}) = (\mathfrak{L}', \mathfrak{D}).$$

Since $\mathbf{w} \in \mathcal{U}(\mathbf{v}) \cap \mathcal{U}(\mathbf{v}')$ then

$$\varrho_{\mathfrak{r}}(\mathbf{w}) = (U_k^{\min}(\mathbf{w}))_{k \in \mathcal{L}(T_D)} \in \left( \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}), V_j) \right) \cap \left( \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}(W_j^{\min}(\mathbf{v}'), V_j) \right)$$

and

$$(\Psi_{\mathbf{v}'} \circ \Psi_{\mathbf{v}}^{-1})(\Psi_{\mathbf{v}}((U_k^{\min}(\mathbf{w}))_{k \in \mathcal{L}(T_D)})) = \Psi_{\mathbf{v}'}(U_k^{\min}(\mathbf{w}))_{k \in \mathcal{L}(T_D)}),$$

that is,

$$(\Psi_{\mathbf{v}'} \circ \Psi_{\mathbf{v}}^{-1})(\mathfrak{L}) = \mathfrak{L}'.$$

Hence

$$(\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}'))\,(\mathbf{w}) = ((\Psi_{\mathbf{v}'} \circ \Psi_{\mathbf{v}}^{-1})(\mathfrak{L}), \mathfrak{D}),$$

where $\Psi_{\mathbf{v}'} \circ \Psi_{\mathbf{v}}^{-1}$ is an analytic map. Let $\mathbf{u} = (\chi_{\mathfrak{r}}(\mathbf{v})^{-1} \circ \Theta_{\mathbf{v}}^{-1})(\mathfrak{o}, \mathfrak{C})$ and $\mathbf{u}' = (\chi_{\mathfrak{r}}(\mathbf{v}'^{-1} \circ \Theta_{\mathbf{v}'}^{-1})(\mathfrak{o}, \mathfrak{D})$. Then $\mathbf{u} \in {}_a\bigotimes_{k \in \mathcal{L}(T_D)} U_k^{\min}(\mathbf{v})$ , $\mathbf{u}' \in {}_a\bigotimes_{k \in \mathcal{L}(T_D)} U_k^{\min}(\mathbf{v}')$ and

$$\mathbf{w} = (\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{L}, \mathfrak{C}) = \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k) \right)(\mathbf{u}) = \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k) \right) \circ (\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{o}, \mathfrak{C})$$

$$= (\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}'))^{-1}(\mathfrak{L}', \mathfrak{D}) = \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k') \right)(\mathbf{u}') = \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k') \right) \circ (\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}'))^{-1}(\mathfrak{o}, \mathfrak{D}).$$

Hence,

$$(\mathfrak{o}, \mathfrak{D}) = (\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}')) \circ \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k')^{-1} \circ (id_k + L_k) \right) \circ (\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{o}, \mathfrak{C}).$$

In consequence, we can write

$$(\mathfrak{o}, \mathfrak{D}) = f(\mathfrak{L}, \mathfrak{C}) := (\Theta_{\mathbf{v}'} \circ \chi_{\mathfrak{r}}(\mathbf{v}')) \circ \left( \bigotimes_{k \in \mathcal{L}(T_D)} (id_k + L_k')^{-1} \circ (id_k + L_k) \right) \circ (\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))^{-1}(\mathfrak{o}, \mathfrak{C}).$$

and the map

$$f : \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})) \times \mathbb{R}_*^{\mathfrak{r}} \to \{\mathfrak{o}\} \times \mathbb{R}_*^{\mathfrak{r}}$$

is an analytic morphism. Thus the lemma is proved. ■

The next result will help us to show that the collection $\{\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}), \mathcal{U}(\mathbf{v})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Indeed, it is the unique manifold structure for which $\varrho_{\mathfrak{r}} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \to \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}_{r_j}(V_j)$ defines a locally trivial fibre bundle with typical fibre $\mathbb{R}_*^{\mathfrak{r}}$. To this end we will use Lemma 3.22 and the following classical result (see Proposition 3.4.28 in [26]).

**Theorem 3.23** *Let $E$ be a set, $B$ and $F$ be $\mathcal{C}^k$ manifolds, and let $\pi : E \to B$ be a surjective map. Assume that*

*(a) there is a $\mathcal{C}^k$ atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ of $B$ and a family of bijective maps $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ satisfying $p_\alpha \circ \chi_\alpha = \pi$, where $p_\alpha : U_\alpha \times F \to U_\alpha$ is the projection, and that*

*(b) the maps $\chi_{\alpha'} \circ \chi_\alpha^{-1} : U_\alpha \times F \to U_{\alpha'} \times F$ are $\mathcal{C}^k$ diffeomorphisms whenever $U_\alpha \cap U_{\alpha'} \neq \emptyset$.*

*Then there is a $\mathcal{C}^k$ atlas $\{(V_\beta, \psi_\alpha) : \beta \in J\}$ of $F$ and a unique $\mathcal{C}^k$ manifold structure on $E$ given by*

$$\{(\chi_\alpha^{-1}(U_\alpha \times V_\beta), (\varphi_\alpha \times \psi_\beta)) \circ \chi_\alpha : \alpha \in I, \, \beta \in J\}$$

*for which $\pi : E \to B$ is a $\mathcal{C}^k$ locally trivial fibre bundle with typical fibre $F$.*

Let us mention the following two mathematical objects related to the above theorem. Let $B$ and $F$ be $\mathcal{C}^k$ manifolds, and let $\pi : E \to B$ be a surjective map satisfying the conditions (a)-(b) of Theorem 3.23. Then $(E, B, \pi)$ is called a *fibre bundle* with typical fibre $F$, and if $F$ is also a Banach space, then it is called a *vector bundle* (see Chapters 6 and 7 in [6]). In consequence, we can state the following result.

**Theorem 3.24** *Assume that $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a\bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in the tree-based format where for each $k \in \mathcal{L}(T_D)$ the vector space $V_k$ is a normed space with a norm $\|\cdot\|_k$. Then the collection $\{\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}), \mathcal{U}(\mathbf{v})\}_{\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)}$ is an analytic atlas for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Furthermore, the set $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ of TBF tensors with fixed TB rank is an analytic Banach manifold and*

$$\left( \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D), \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}_{r_j}(V_j), \varrho_{\mathfrak{r}} \right)$$

*is a fibre bundle with typical fibre $\mathbb{R}^{\mathfrak{r}}_*$.*

*Proof.* Take the set $E = \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ and the analytic Banach manifolds $B = \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}_{r_j}(V_j)$ and $F = \mathbb{R}^{\mathfrak{r}}_*$. Let us consider the surjective map $\varrho_{\mathfrak{r}} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \to \bigtimes_{j \in \mathcal{L}(T_D)} \mathbb{G}_{r_j}(V_j)$. The theorem follows from Theorem 3.23 because Theorem 3.23(a) is true by the definition of $\chi_{\mathfrak{r}}(\mathbf{v})$ and Theorem 3.23(b) is a consequence of Lemma 3.22. ∎

**Remark 3.25** *We observe that the geometric structure of manifold is independent of the choice of the norm $\|\cdot\|_D$ over the tensor space $\mathbf{V}_D$.*

**Corollary 3.26** *Assume that $V_{k_{\|\cdot\|_k}}$ is a Hilbert space with norm $\|\cdot\|_k$ for $k \in \mathcal{L}(T_D)$. Then $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an analytic Hilbert manifold.*

*Proof.* We can identify each $L_k \in \mathcal{L}\left(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})\right)$ with a $(\mathbf{w}_{s_k}^{(k)})_{s_k=1}^{s_k=r_k} \in W_k^{\min}(\mathbf{v})^{r_k}$, where $\mathbf{w}_{s_k}^{(k)} = L_k(\mathbf{u}_{(s_k)}^k)$ and $U_k^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{u}_{(1)}^k, \ldots, \mathbf{u}_{(r_k)}^k\}$ for $k \in \mathcal{L}(T_D)$. Thus we can identify each $(\mathfrak{L}, \mathfrak{C}) \in \mathcal{U}(\mathbf{v})$ with a pair

$$(\mathfrak{W}, \mathfrak{C}) \in \bigtimes_{k \in \mathcal{L}(T_D)} W_k^{\min}(\mathbf{v})^{r_k} \times \mathbb{R}^{\mathfrak{r}}_*,$$

where $\mathfrak{W} := ((\mathbf{w}_{s_k}^{(k)})_{s_k=1}^{s_k=r_k})_{k \in \mathcal{L}(T_D)}$. Take $\bigtimes_{k \in \mathcal{L}(T_D)} W_k^{\min}(\mathbf{v})^{r_k} \times \mathbb{R}^{\mathfrak{r}}_*$ an open subset of the Hilbert space $\bigtimes_{k \in \mathcal{L}(T_D)} W_\alpha^{\min}(\mathbf{v})^{r_k} \times \mathbb{R}^{\mathfrak{r}}$ endowed with the product norm

$$\|\,(\mathfrak{W}, \mathfrak{C})\,\|_\times := \sum_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \|C^\alpha\|_F + \sum_{k \in \mathcal{L}(T_D)} \sum_{s_k=1}^{r_k} \|\mathbf{w}_{s_k}^{(k)}\|_k,$$

with $\|\cdot\|_F$ the Frobenius norm. It allows us to define local charts, also denoted by $\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v})$, by

$$\chi_{\mathfrak{r}}^{-1}(\mathbf{v}) \circ \Theta_{\mathbf{v}}^{-1} : \bigtimes_{k \in \mathcal{L}(T_D)} W_\alpha^{\min}(\mathbf{v})^{r_k} \times \mathbb{R}^{\mathfrak{r}}_* \longrightarrow \mathcal{U}(\mathbf{v}),$$

where $(\chi_{\mathfrak{r}}^{-1}(\mathbf{v}) \circ \Theta_{\mathbf{v}}^{-1})(\mathfrak{W}, \mathfrak{C}) = \mathbf{w}$ putting $L_k(\mathbf{u}_{i_k}^{(k)}) = \mathbf{w}_{i_k}^{(k)}$, $1 \le i_k \le r_k$ and $k \in \mathcal{L}(T_D)$. Since each local chart is defined over an open subset of the Hilbert space $\times_{k \in \mathcal{L}(T_D)} W_k^{\min}(\mathbf{v})^{r_k} \times \mathbb{R}^{\mathfrak{r}}$, the corollary follows. ■

Using the geometric structure of local charts for the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, we can identify its tangent space at $\mathbf{v}$ with $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)) := \times_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}}$. We will consider $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ endowed with the product norm

$$|||(\mathfrak{L}, \mathfrak{C})||| := \sum_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \|C^{(\alpha)}\|_F + \sum_{k \in \mathcal{L}(T_D)} \|L_k\|_{W_k^{\min}(\mathbf{v}) \leftarrow U_k^{\min}(\mathbf{v})}.$$

Moreover, the map $\varrho_{\mathfrak{r}}$ is an analytic morphism and

$$\mathrm{T}_{\mathbf{v}} \varrho_{\mathfrak{r}} : \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}} \to \bigtimes_{k \in \mathcal{L}(T_D)} \mathcal{L}(U_k^{\min}(\mathbf{v}), W_k^{\min}(\mathbf{v})), \quad (\mathfrak{L}, \mathfrak{C}) \mapsto \mathfrak{L}.$$

Finally, the same argument used to provide a Banach manifold structure to the set $\mathbb{G}_{\le n}(X)$ used with $\mathcal{FT}_{\le \mathfrak{r}}(\mathbf{V}_D)$ and (2.5), allows us to state the following.

**Theorem 3.27** *Assume that $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a \bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in the tree-based format where for each $k \in \mathcal{L}(T_D)$ the vector space $V_k$ is a normed space with a norm $\| \cdot \|_k$. Then the set $\mathcal{FT}_{\le \mathfrak{r}}(\mathbf{V}_D)$ of TBF tensors with bounded TB rank is an analytic Banach (Hilbert) manifold.*

# 4 The TBF tensors and its natural ambient tensor Banach space

Assume that $\{\mathbf{V}_\alpha\}_{\alpha \in T_D \setminus \{D\}}$ is a representation of the tensor space $\mathbf{V}_D = {}_a \bigotimes_{\alpha \in S(D)} \mathbf{V}_\alpha$ in the tree-based format and that for each $k \in \mathcal{L}(T_D)$ the vector space $V_k$ is a normed space with a norm $\| \cdot \|_k$. We start with a brief discussion about the choice of the ambient manifold for $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. To this end assume the existence of two norms $\| \cdot \|_{D,1}$ and $\| \cdot \|_{D,2}$ on $\mathbf{V}_D$. Then we have $\mathbf{V}_D \subset \overline{\mathbf{V}_D}^{\| \cdot \|_{D,1}}$ and $\mathbf{V}_D \subset \overline{\mathbf{V}_D}^{\| \cdot \|_{D,2}}$. The next example illustrates this situation.

**Example 4.1** *Let $V_{1_{\| \cdot \|_1}} := H^{1,p}(I_1)$ and $V_{2_{\| \cdot \|_2}} = H^{1,p}(I_2)$. Take $\mathbf{V}_D := H^{1,p}(I_1) \otimes_a H^{1,p}(I_2)$. From Theorem 3.24 we obtain that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is a Banach manifold. However, we can consider as ambient manifold either $\overline{\mathbf{V}_D}^{\| \cdot \|_{D,1}} := H^{1,p}(I_1 \times I_2)$ or $\overline{\mathbf{V}_D}^{\| \cdot \|_{D,2}} = H^{1,p}(I_1) \otimes_{\| \cdot \|_{(0,1),p}} H^{1,p}(I_2)$, where $\| \cdot \|_{(0,1),p}$ is the norm given by*

$$\|f\|_{(0,1),p} := \left( \|f\|_p^p + \left\| \frac{\partial f}{\partial x_2} \right\|_p^p \right)^{1/p}$$

*for $1 \le p < \infty$.*

In this context two questions about the choice of a norm $\| \cdot \|_\alpha$ for each algebraic tensor space $\mathbf{V}_\alpha = {}_a \bigotimes_{\beta \in S(\alpha)} \mathbf{V}_\beta$ , where $\alpha \in T_D \setminus \mathcal{L}(T_D)$ appears:

1. What is the good choice for these norms to show that $\mathcal{FT}_{\le \mathfrak{r}}(\mathbf{V}_D)$ is proximinal?
2. What is the good choice for these norms to show that $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold?

To see this we need to introduce the topological tensor spaces in the tree-based format.

## 4.1 Topological tensor spaces in the tree-based Format

First, we recall the definition of some topological tensor spaces and we will give some examples.

**Definition 4.2** *We say that* $\mathbf{V}_{\|\cdot\|}$ *is a* Banach tensor space *if there exists an algebraic tensor space* $\mathbf{V}$ *and a norm* $\|\cdot\|$ *on* $\mathbf{V}$ *such that* $\mathbf{V}_{\|\cdot\|}$ *is the completion of* $\mathbf{V}$ *with respect to the norm* $\|\cdot\|$*, i.e.*

$$\mathbf{V}_{\|\cdot\|} := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j = \overline{{}_a\bigotimes_{j=1}^{d} V_j}^{\|\cdot\|}.$$

*If* $\mathbf{V}_{\|\cdot\|}$ *is a Hilbert space, we say that* $\mathbf{V}_{\|\cdot\|}$ *is a* Hilbert tensor space.

Next, we give some examples of Banach and Hilbert tensor spaces.

**Example 4.3** *For* $I_j \subset \mathbb{R}$ $(1 \le j \le d)$ *and* $1 \le p < \infty$*, the Sobolev space* $H^{N,p}(I_j)$ *consists of all univariate functions* $f$ *from* $L^p(I_j)$ *with bounded norm*[4]

$$\|f\|_{N,p;I_j} := \Big( \sum_{n=0}^{N} \int_{I_j} |\partial^n f|^p \, \mathrm{d}x \Big)^{1/p},$$

*whereas the space* $H^{N,p}(\mathbf{I})$ *of* $d$*-variate functions on* $\mathbf{I} = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d$ *is endowed with the norm*

$$\|f\|_{N,p} := \Big( \sum_{0 \le |\mathbf{n}| \le N} \int_{\mathbf{I}} |\partial^{\mathbf{n}} f|^p \, \mathrm{d}\mathbf{x} \Big)^{1/p}$$

*with* $\mathbf{n} \in \mathbb{N}_0^d$ *being a multi-index of length* $|\mathbf{n}| := \sum_{j=1}^{d} n_j$*. For* $p > 1$ *it is well known that* $H^{N,p}(I_j)$ *and* $H^{N,p}(\mathbf{I})$ *are reflexive and separable Banach spaces. Moreover, for* $p = 2$*, the Sobolev spaces* $H^N(I_j) := H^{N,2}(I_j)$ *and* $H^N(\mathbf{I}) := H^{N,2}(\mathbf{I})$ *are Hilbert spaces. As a first example,*

$$H^{N,p}(\mathbf{I}) = {}_{\|\cdot\|_{N,p}}\bigotimes_{j=1}^{d} H^{N,p}(I_j)$$

*is a Banach tensor space. Examples of Hilbert tensor spaces are*

$$L^2(\mathbf{I}) = {}_{\|\cdot\|_{0,2}}\bigotimes_{j=1}^{d} L^2(I_j) \quad \text{and} \quad H^N(\mathbf{I}) = {}_{\|\cdot\|_{N,2}}\bigotimes_{j=1}^{d} H^N(I_j) \ \text{ for } N \in \mathbb{N}.$$

In the definition of a tensor Banach space ${}_{\|\cdot\|}\bigotimes_{j\in D} V_j$ we have not fixed, whether $V_j$, for $j \in D$, are complete or not. This leads us to introduce the following definition.

**Definition 4.4** *Let* $D$ *be a finite index set and* $T_D$ *be a dimension partition tree. Let* $(V_j, \|\cdot\|_j)$ *be a normed space such that* $V_{j_{\|\cdot\|_j}}$ *is a Banach space obtained as the completion of* $V_j$*, for* $j \in D$*, and consider a representation* $\{\mathbf{V}_\alpha\}_{\alpha\in T_D\setminus\{D\}}$ *of the tensor space* $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ *where for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *we have a tensor space* $\mathbf{V}_\alpha = {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta$ *. If for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *there exists a norm* $\|\cdot\|_\alpha$ *defined on* $\mathbf{V}_\alpha$ *such that* $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta$ *is a tensor Banach space, we say that* $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ *is a representation of the tensor Banach space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ *in the* topological tree-based format.

Since $\mathbf{V}_\alpha = {}_a\bigotimes_{j\in\alpha} V_j$ ,

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j$$

holds for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$.

**Example 4.5** *Figure 4.1 gives an example of a representation in the topological tree-based format for an anisotropic Sobolev space.*

[4] It suffices to have in (4.1) the terms $n = 0$ and $n = N$. The derivatives are to be understood as weak derivatives.

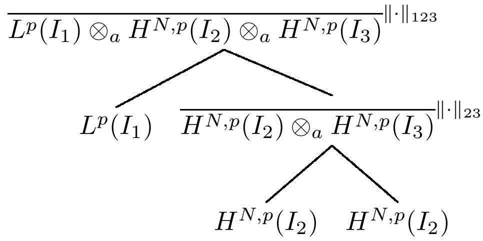


Figure 4.1: A representation in the topological tree-based format for the tensor Banach space $\overline{L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{123}}$. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

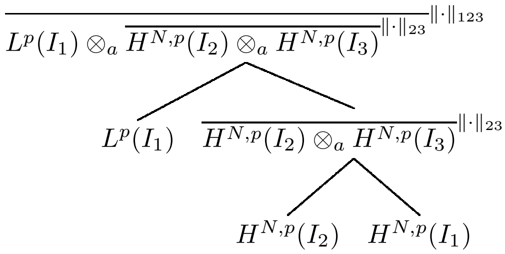


Figure 4.2: A representation for the tensor Banach space $\overline{L^p(I_1) \otimes_a \overline{H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)}^{\|\cdot\|_{23}}}^{\|\cdot\|_{123}}$, using a tree. Here $\|\cdot\|_{23}$ and $\|\cdot\|_{123}$ are given norms.

**Remark 4.6** *Observe that a tree as given in Figure 4.2 is not included in the definition of the topological tree-based format. Moreover, for a tensor* $\mathbf{v} \in L^p(I_1) \otimes_a (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3))$*, we have* $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3)$*. However, in the topological tree-based representation of Figure 4.1, for a given* $\mathbf{v} \in L^p(I_1) \otimes_a H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$ *we have* $U_{23}^{\min}(\mathbf{v}) \subset H^{N,p}(I_2) \otimes_a H^{N,p}(I_3)$*, and hence* $U_{23}^{\min}(\mathbf{v}) \subset U_2^{\min}(\mathbf{v}) \otimes_a U_3^{\min}(\mathbf{v})$.

The difference between the tensor spaces involved in Figure 4.1 and Figure 4.2 is the following. For all $\beta \in T_D \setminus \mathcal{L}(T_D)$, if $\|\cdot\|_\beta$ is also a norm on the tensor space ${}_a\bigotimes_{\eta\in S(\beta)} \mathbf{V}_{\eta_{\|\cdot\|_\eta}}$, we have

$$_{\|\cdot\|_\beta}\bigotimes_{\eta\in S(\beta)} \mathbf{V}_{\eta_{\|\cdot\|_\eta}} \supset \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\beta}\bigotimes_{\eta\in S(\beta)} \mathbf{V}_\eta = {}_{\|\cdot\|_\beta}\bigotimes_{j\in\beta} V_j .$$

A desirable property for the tensor product is that if $\|\cdot\|_\alpha$ is also a norm on the tensor space ${}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, then

$$_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j \tag{4.2}$$

must be true for all $\alpha \in T_D \setminus \mathcal{L}(T_D)$. To precise these ideas, we introduce the following definitions and results.

Let $\|\cdot\|_j$, $1 \le j \le d$, be the norms of the vector spaces $V_j$ appearing in $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$. By $\|\cdot\|$ we denote the norm on the tensor space $\mathbf{V}$. Note that $\|\cdot\|$ is not determined by $\|\cdot\|_j$, for $j \in D$, but there are relations which are 'reasonable'. Any norm $\|\cdot\|$ on ${}_a\bigotimes_{j=1}^d V_j$ satisfying

$$\left\|\bigotimes_{j=1}^d v_j\right\| = \prod_{j=1}^d \|v_j\|_j \qquad \text{for all } v_j \in V_j \ (1 \le j \le d) \tag{4.3}$$

is called a *crossnorm.* As usual, the dual norm of $\|\cdot\|$ is denoted by $\|\cdot\|^*$. If $\|\cdot\|$ is a crossnorm and also $\|\cdot\|^*$ is a crossnorm on ${}_a\bigotimes_{j=1}^d V_j^*$, i.e.,

$$\left\|\bigotimes_{j=1}^d \varphi^{(j)}\right\|^* = \prod_{j=1}^d \|\varphi^{(j)}\|_j^* \qquad \text{for all } \varphi^{(j)} \in V_j^* \ (1 \le j \le d), \tag{4.4}$$

then $\|\cdot\|$ is called a *reasonable crossnorm.*

**Remark 4.7** *Eq.* (4.3) *implies the inequality* $\| \bigotimes_{j=1}^d v_j \| \lesssim \prod_{j=1}^d \|v_j\|_j$ *which is equivalent to the continuity of the multilinear tensor product mapping*[5] *between normed spaces:*

$$\bigotimes : \bigtimes_{j=1}^{d} \left(V_j, \|\cdot\|_j\right) \longrightarrow \left( {}_a\bigotimes_{j=1}^{d} V_j \ , \|\cdot\| \right), \tag{4.5}$$

*defined by* $\bigotimes((v_1, \ldots, v_d)) = \bigotimes_{j=1}^d v_j$*, the product space being equipped with the product topology induced by the maximum norm* $\|(v_1, \ldots, v_d)\| = \max_{1 \le j \le d} \|v_j\|_j$*.*

The following result is a consequence of Lemma 4.34 of [19].

**Lemma 4.8** *Let* $(V_j, \|\cdot\|_j)$ *be normed spaces for* $1 \le j \le d$*. Assume that* $\|\cdot\|$ *is a norm on the tensor space* ${}_a\bigotimes_{j=1}^d V_{j_{\|\cdot\|_j}}$ *such that the tensor product map*

$$\bigotimes : \bigtimes_{j=1}^{d} \left(V_{j_{\|\cdot\|_j}}, \|\cdot\|_j\right) \longrightarrow \left( {}_a\bigotimes_{j=1}^{d} V_{j_{\|\cdot\|_j}} \ , \|\cdot\| \right) \tag{4.6}$$

*is continuous. Then* (4.5) *is also continuous and*

$$_{\|\cdot\|}\bigotimes_{j=1}^{d} V_{j_{\|\cdot\|_j}} = {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j$$

*holds.*

**Definition 4.9** *Assume that for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *there exists a norm* $\|\cdot\|_\alpha$ *defined on* ${}_a\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$ *. We will say that the tensor product map* $\bigotimes$ *is* $T_D$*-continuous if the map*

$$\bigotimes : \bigtimes_{\beta\in S(\alpha)} (V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \to \left( {}_a \bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}} \ , \|\cdot\|_\alpha \right)$$

*is continuous for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$*.*

The next result gives the conditions to have (4.2).

**Theorem 4.10** *Assume that we have a representation* $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ *in the topological tree-based format of the tensor Banach space* $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{\alpha\in S(D)} \mathbf{V}_\alpha$ *, such that for each* $\alpha \in T_D \setminus \mathcal{L}(T_D)$*, the norm* $\|\cdot\|_\alpha$ *is also defined on* ${}_a\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$ *and the tensor product map* $\bigotimes$ *is* $T_D$*-continuous. Then*

$$_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_{\|\cdot\|_\alpha}\bigotimes_{j\in\alpha} V_j \ ,$$

*holds for all* $\alpha \in T_D \setminus \mathcal{L}(T_D)$*.*

*Proof.* From Lemma 4.8, if the tensor product map

$$\bigotimes : \bigtimes_{\beta\in S(\alpha)} (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \longrightarrow ({}_a \bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} \ , \|\cdot\|_\alpha)$$

is continuous, then

$$_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_\beta \ ,$$

holds. Since $\mathbf{V}_\alpha = {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta = {}_a\bigotimes_{j\in\alpha} V_j$ , the theorem follows. ■

[5] Recall that a multilinear map $T$ from $\bigtimes_{j=1}^d (V_j, \|\cdot\|_j)$ equipped with the product topology to a normed space $(W, \|\cdot\|)$ is continuous if and only if $\|T\| < \infty$, with

$$\|T\| := \sup_{\substack{(v_1,\ldots,v_d) \\ \|(v_1,\ldots,v_d)\|\le 1}} \|T(v_1, \ldots, v_d)\| = \sup_{\substack{(v_1,\ldots,v_d) \\ \|v_1\|_1\le 1,\ldots,\|v_d\|_d\le 1}} \|T(v_1, \ldots, v_d)\| = \sup_{(v_1,\ldots,v_d)} \frac{\|T(v_1, \ldots, v_d)\|}{\|v_1\|_1 \cdots \|v_d\|_d}.$$

**Example 4.11** *Assume that the tensor product maps*

$$\bigotimes : (L^p(I_1), \|\cdot\|_{0,p;I_1}) \times (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3), \|\cdot\|_{23}) \to (L^p(I_1) \otimes_a (H^{N,p}(I_2) \otimes_{\|\cdot\|_{23}} H^{N,p}(I_3)), \|\cdot\|_{123})$$

*and*

$$\bigotimes : (H^{N,p}(I_2), \|\cdot\|_{N,p;I_2}) \times (H^{N,p}(I_3), \|\cdot\|_{N,p;I_3}) \to (H^{N,p}(I_2) \otimes_a H^{N,p}(I_3), \|\cdot\|_{23})$$

*are continuous. Then the trees of Figure 4.1 and Figure 4.2 are the same.*

The next result is a consequence of the well-known fact that every continuous multilinear map between normed spaces is also Fréchet differentiable (see (2.1.22) in [5]).

**Proposition 4.12** *Let $(V_j, \|\cdot\|_j)$ be normed spaces for $1 \le j \le d$. Assume that $\|\cdot\|$ is a norm onto the tensor space ${}_a\bigotimes_{j=1}^d V_{j_{\|\cdot\|_j}}$ such that the tensor product map (4.6) is continuous. Then it is also Fréchet differentiable and its differential is given by*

$$D\left(\bigotimes(v_1, \ldots, v_d)\right)(w_1, \ldots, w_d) = \sum_{j=1}^d v_1 \otimes \ldots \otimes v_{j-1} \otimes w_j \otimes v_{j+1} \otimes \cdots v_d.$$

### 4.1.1 On the best approximation in $\mathcal{FT}_{\le \mathfrak{r}}(\mathbf{V}_D)$

Now we discuss about the best approximation problem in $\mathcal{FT}_{\le \mathfrak{r}}(\mathbf{V}_D)$. For this, we need a stronger condition than the $T_D$-continuity of the tensor product. Grothendieck [15] named the following norm $\|\cdot\|_\vee$ the *injective norm.*

**Definition 4.13** *Let $V_i$ be a Banach space with norm $\|\cdot\|_i$ for $1 \le i \le d$. Then for $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ define $\|\cdot\|_{\vee(V_1,\ldots,V_d)}$ by*

$$\|\mathbf{v}\|_{\vee(V_1,\ldots,V_d)} := \sup\left\{ \frac{|(\varphi_1 \otimes \varphi_2 \otimes \ldots \otimes \varphi_d)(\mathbf{v})|}{\prod_{j=1}^d \|\varphi_j\|_j^*} : 0 \ne \varphi_j \in V_j^*, 1 \le j \le d \right\}. \tag{4.7}$$

It is well known that the injective norm is a reasonable crossnorm (see Lemma 1.6 in [24] and (4.3)-(4.4)). Further properties are given by the next proposition (see Lemma 4.96 and 4.2.4 in [19]).

**Proposition 4.14** *Let $V_i$ be a Banach space with norm $\|\cdot\|_i$ for $1 \le i \le d$, and $\|\cdot\|$ be a norm on $\mathbf{V} := {}_a\bigotimes_{j=1}^d V_j$. The following statements hold.*

*(a) For each $1 \le j \le d$ introduce the tensor Banach space $\mathbf{X}_j := {}_{\|\cdot\|_{\vee(V_1,\ldots,V_{j-1},V_{j+1},\ldots,V_d)}}\bigotimes_{k\ne j} V_k$. Then*

$$\|\cdot\|_{\vee(V_1,\ldots,V_d)} = \|\cdot\|_{\vee(V_j,\mathbf{X}_j)} \tag{4.8}$$

*holds for $1 \le j \le d$.*

*(b) The injective norm is the weakest reasonable crossnorm on $\mathbf{V}$, i.e., if $\|\cdot\|$ is a reasonable crossnorm on $\mathbf{V}$, then*

$$\|\cdot\| \gtrsim \|\cdot\|_{\vee(V_1,\ldots,V_d)}. \tag{4.9}$$

*(c) For any norm $\|\cdot\|$ on $\mathbf{V}$ satisfying $\|\cdot\|_{\vee(V_1,\ldots,V_d)} \lesssim \|\cdot\|$, the map (4.5) is continuous, and hence Fréchet differentiable.*

In Corollary 4.4 in [11] the following result, which is re-stated here using the notations of the present paper, is proved as a consequence of a similar result showed for tensors in Tucker format with bounded rank.

**Theorem 4.15** *Let $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ and let $\{\mathbf{V}_{\alpha_j\,\|\cdot\|_{\alpha_j}} : 2 \le j \le d\} \cup \{V_{j_{\|\cdot\|_j}} : 1 \le j \le d\}$ for $d \ge 3$, be a representation of a reflexive Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in topological tree-based format such that* *(a)* $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(V_{1_{\|\cdot\|_j}},\ldots,V_{d_{\|\cdot\|_d}})}$,

*(b)* $\mathbf{V}_{\alpha_d} = V_{d-1} \otimes_a V_d$, *and* $\mathbf{V}_{\alpha_j} = V_{j-1} \otimes_a \mathbf{V}_{\alpha_{j+1}}$, *for* $2 \leq j \leq d-1$, *and*

*(c)* $\|\cdot\|_{\alpha_j} := \|\cdot\|_{\vee(V_{j-1}{}_{\|\cdot\|_{j-1}},\ldots,V_{d_{\|\cdot\|_d}})}$ *for* $2 \leq j \leq d$.

*Then for each* $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ *there exists* $\mathbf{u}_{best} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D)$ *such that*

$$\|\mathbf{v} - \mathbf{u}_{best}\|_D = \min_{\mathbf{u} \in \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D)} \|\mathbf{v} - \mathbf{u}\|_D.$$

It seems clear that tensor Banach spaces as we show in Example 4.2 are not included in this framework. So a natural question is if for a representation in the topological tree-based format of a reflexive Banach space the statement of Theorem 4.15 is also true. To prove this, we will reformulate some of the results given in [11]. In the aforementioned paper, the milestone to prove the existence of a best approximation is the extension of the definition of minimal subspace for tensors $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$. To do this the authors use a similar result to the following lemma (see Lemma 3.8 in [11]).

**Lemma 4.16** *Let* $V_{j_{\|\cdot\|_j}}$ *be a Banach space for* $j \in D$, *where* $D$ *is a finite index set, and* $\alpha_1, \ldots, \alpha_m \subset 2^D \setminus \{D, \emptyset\}$, *be such that* $\alpha_i \cap \alpha_j = \emptyset$ *for all* $i \neq j$ *and* $D = \bigcup_{i=1}^m \alpha_i$. *Assume that if* $\#\alpha_i \geq 2$ *for some* $1 \leq i \leq m$, *then* $\mathbf{V}_{\alpha_i}{}_{\|\cdot\|_{\alpha_i}}$ *is a tensor Banach space. Consider the tensor space*

$$\mathbf{V}_D := {}_a\bigotimes_{i=1}^m \mathbf{V}_{\alpha_i}{}_{\|\cdot\|_{\alpha_i}}$$

*endowed with the injective norm* $\|\cdot\|_{\vee(\mathbf{V}_{\alpha_1}{}_{\|\cdot\|_{\alpha_1}},\ldots,\mathbf{V}_{\alpha_m}{}_{\|\cdot\|_{\alpha_m}})}$. *Fix* $1 \leq k \leq m$, *then given* $\boldsymbol{\varphi}_{[\alpha_k]} \in {}_a\bigotimes_{i\neq k} \mathbf{V}^*_{\alpha_i}{}_{\|\cdot\|_{\alpha_i}}$ *the map* $id_{\alpha_k} \otimes \boldsymbol{\varphi}_{[\alpha_k]}$ *belongs to* $\mathcal{L}\left(\mathbf{V}_D, \mathbf{V}_{\alpha_k}{}_{\|\cdot\|_{\alpha_k}}\right)$. *Moreover,* $\overline{id_{\alpha_k} \otimes \boldsymbol{\varphi}_{[\alpha_k]}} \in \mathcal{L}(\overline{\mathbf{V}_D}^{\|\cdot\|}, \mathbf{V}_{\alpha_k}{}_{\|\cdot\|_{\alpha_k}})$ *for any norm satisfying*

$$\|\cdot\| \gtrsim \|\cdot\|_{\vee(\mathbf{V}_{\alpha_1}{}_{\|\cdot\|_{\alpha_1}},\ldots,\mathbf{V}_{\alpha_m}{}_{\|\cdot\|_{\alpha_m}})}.$$

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ , in the topological tree-based format and assume that the tensor product map $\bigotimes$ is $T_D$-continuous. From Theorem 4.10, we may assume that we have a tensor Banach space

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha}\bigotimes_{\beta\in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}}$$

for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$, and a Banach space $V_{j_{\|\cdot\|_j}}$ for $j \in \mathcal{L}(T_D)$. Let $\alpha \in T_D \setminus \mathcal{L}(T_D)$. To simplify the notation we write for $A, B \subset S(\alpha)$

$$\|\cdot\|_{\vee(A)} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_\delta}}:\delta\in A\})},$$

and

$$\|\cdot\|_{\vee(A,\vee(B))} := \|\cdot\|_{\vee(\{\mathbf{V}_{\delta_{\|\cdot\|_\delta}}:\delta\in A\},\mathbf{X}_B)}$$

where

$$\mathbf{X}_B := {}_{\|\cdot\|_{\vee(B)}}\bigotimes_{\beta\in B} \mathbf{V}_{\beta_{\|\cdot\|_\beta}}.$$

From Proposition 4.14(a), we can write

$$\|\cdot\|_{\vee(S(\alpha))} = \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$$

for each $\beta \in S(\alpha)$. From now on, we assume that

$$\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(S(\alpha))} \text{ for each } \alpha \in T_D \setminus \mathcal{L}(T_D), \tag{4.10}$$

holds. Recall that Proposition 4.14(c) implies that the tensor product map $\bigotimes$ is $T_D$-continuous. Since $\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(\beta,\vee(S(\alpha)\setminus\beta))}$ holds for each $\beta \in S(\alpha)$, the tensor product map

$$\bigotimes : (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \times \left( {}_{\|\cdot\|_{\vee(S(\alpha)\setminus\beta)}} \bigotimes_{\delta \in S(\alpha)\setminus\{\beta\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}, \|\cdot\|_{\vee(S(\alpha)\setminus\beta)} \right) \to (\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \|\cdot\|_\alpha)$$

is also continuous for each $\beta \in S(\alpha)$. Moreover, by Theorem 4.10,

$$\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}} = {}_{\|\cdot\|_\alpha} \bigotimes_{\beta \in S(\alpha)} V_{\beta_{\|\cdot\|_\beta}} = {}_{\|\cdot\|_\alpha} \bigotimes_{\beta \in S(\alpha)} V_\beta = {}_{\|\cdot\|_\alpha} \bigotimes_{j \in \alpha} V_j \,,$$

holds for each $\alpha \in T_D \setminus \mathcal{L}(T_D)$. Observe, that $\mathbf{V}^*_{\alpha_{\|\cdot\|_\alpha}} \subset \mathbf{V}^*_\alpha$ for all $\alpha \in S(D)$. Take $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ . Since $\|\cdot\|_D \gtrsim \|\cdot\|_{\vee(S(D))}$, from Lemma 4.16 and Proposition 2.11(b), we can extend for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$, the definition of minimal subspace for each $\alpha \in S(D)$ as

$$U_\alpha^{\min}(\mathbf{v}) := \left\{ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a \bigotimes_{\beta \in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\beta \right\}.$$

Observe that $\overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$. Recall that if $\mathbf{v} \in \mathbf{V}_D$ and $\alpha \notin \mathcal{L}(T_D)$, from Proposition 2.10, we have $U_\alpha^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \subset {}_a\bigotimes_{\beta\in S(\alpha)} \mathbf{V}_\beta$ . Moreover, by Proposition 2.11(b), for $\beta \in S(\alpha)$ we have

$$\begin{aligned} U_\beta^{\min}(\mathbf{v}) &= \operatorname{span}\left\{ (id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})(\mathbf{v}_\alpha) : \mathbf{v}_\alpha \in U_\alpha^{\min}(\mathbf{v}) \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a \bigotimes_{\delta \in S(\alpha)\setminus\{\beta\}} \mathbf{V}^*_\delta \right\} \\ &= \operatorname{span}\left\{ (id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}) \circ (id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a \bigotimes_{\mu \in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\mu \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a \bigotimes_{\delta \in S(\alpha)\setminus\{\beta\}} \mathbf{V}^*_\delta \right\}. \end{aligned}$$

Thus, $(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v}) \subset \mathbf{V}_\alpha \subset \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$, and hence

$$(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}) \circ (id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})(\mathbf{v}) \in U_\beta^{\min}(\mathbf{v}) \subset \mathbf{V}_\beta \subset \mathbf{V}_{\beta_{\|\cdot\|_\beta}},$$

when $\#\beta \geq 2$. However, if $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$ then $\overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) \in U_\alpha^{\min}(\mathbf{v}) \subset \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$. Since $\|\cdot\|_\alpha \gtrsim \|\cdot\|_{\vee(S(\alpha))}$ also by Lemma 4.16 we have $\overline{id_\beta \otimes \boldsymbol{\varphi}_{[\beta]}} \in \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\beta_{\|\cdot\|_\beta}})$. In consequence, a natural extension of the definition of minimal subspace $U_\beta^{\min}(\mathbf{v})$, for $\mathbf{v} \in \mathbf{V}_{D_{\|\cdot\|_D}} \setminus \mathbf{V}_D$, is given by

$$U_\beta^{\min}(\mathbf{v}) := \operatorname{span}\left\{ \overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})} \circ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[\alpha]} \in {}_a \bigotimes_{\mu \in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\mu \text{ and } \boldsymbol{\varphi}_{[\beta]} \in {}_a \bigotimes_{\delta \in S(\alpha)\setminus\{\beta\}} \mathbf{V}^*_\delta \right\}.$$

To simplify the notation, we can write

$$\overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta,\alpha]})}(\mathbf{v}) := \overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta]})} \circ \overline{(id_\alpha \otimes \boldsymbol{\varphi}_{[\alpha]})}(\mathbf{v})$$

where $\boldsymbol{\varphi}_{[\beta,\alpha]} := \boldsymbol{\varphi}_{[\alpha]} \otimes \boldsymbol{\varphi}_{[\beta]} \in \left( {}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\mu \right) \otimes_a \left( {}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}^*_\delta \right)$ and $\overline{(id_\beta \otimes \boldsymbol{\varphi}_{[\beta,\alpha]})} \in \mathcal{L}(\mathbf{V}_{D_{\|\cdot\|_D}}, \mathbf{V}_{\beta_{\|\cdot\|_\beta}})$. Proceeding inductively, from the root to the leaves, we define the minimal subspace $U_j^{\min}(\mathbf{v})$ for each $j \in \mathcal{L}(T_D)$ such that there exists $\eta \in T_D \setminus \{D\}$ with $j \in S(\eta)$ as

$$U_j^{\min}(\mathbf{v}) := \operatorname{span}\left\{ \overline{(id_j \otimes \boldsymbol{\varphi}_{[j,\eta,\ldots,\beta,\alpha]})}(\mathbf{v}) : \boldsymbol{\varphi}_{[j,\eta,\ldots,\beta,\alpha]} \in \mathbf{W}_j \right\},$$

where

$$\mathbf{W}_j := \left( {}_a \bigotimes_{\mu \in S(D)\setminus\{\alpha\}} \mathbf{V}^*_\mu \right) \otimes_a \left( {}_a \bigotimes_{\delta \in S(\alpha)\setminus\{\beta\}} \mathbf{V}^*_\delta \right) \otimes_a \cdots \otimes_a \left( {}_a \bigotimes_{k \in S(\eta)\setminus\{j\}} V^*_k \right).$$

With this extension the following result can be shown (see Lemma 3.13 in [11]).

**Lemma 4.17** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree-based format and assume that (4.10) holds. Let $\{\mathbf{v}_n\}_{n\geq 0}\subset \mathbf{V}_{D_{\|\cdot\|_D}}$ with $\mathbf{v}_n \rightharpoonup \mathbf{v}$, and $\mu\in T_D\setminus(\{D\}\cup\mathcal{L}(T_D))$. Then for each $\gamma\in S(\mu)$ we have*

$$\overline{(id_\gamma\otimes\boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}_n)\rightharpoonup\overline{(id_\gamma\otimes\boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]})}(\mathbf{v}) \text{ in } \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}},$$

*for all* $\boldsymbol{\varphi}_{[\gamma,\mu,\cdots,\beta,\alpha]}\in\left({}_a\bigotimes_{\mu\in S(D)\setminus\{\alpha\}}\mathbf{V}_\mu^*\right)\otimes_a\left({}_a\bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}}\mathbf{V}_\delta^*\right)\otimes_a\cdots\otimes_a\left({}_a\bigotimes_{\eta\in S(\mu)\setminus\{\gamma\}}V_\eta^*\right)$.

Then in a similar way as Theorem 3.15 in [11] the following theorem can be shown.

**Theorem 4.18** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree-based format and assume that (4.10) holds. Let $\{\mathbf{v}_n\}_{n\geq 0}\subset \mathbf{V}_{D_{\|\cdot\|_D}}$ with $\mathbf{v}_n \rightharpoonup \mathbf{v}$, then*

$$\dim\overline{U_\alpha^{\min}(\mathbf{v})}^{\|\cdot\|_\alpha} = \dim U_\alpha^{\min}(\mathbf{v}) \leq \liminf_{n\to\infty}\dim U_\alpha^{\min}(\mathbf{v}_n),$$

*for all $\alpha\in T_D\setminus\{D\}$.*

Now, following the proof of Theorem 4.1 in [11] we obtain the final theorem.

**Theorem 4.19** *Let $\mathbf{V}_D = {}_a\bigotimes_{j\in D}V_j$ and let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of a reflexive Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree-based format and assume that (4.10) holds. Then the set $\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D)$ is weakly closed in $\mathbf{V}_{D_{\|\cdot\|_D}}$ and hence for each $\mathbf{v}\in\mathbf{V}_{D_{\|\cdot\|_D}}$ there exists $\mathbf{u}_{best}\in\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D)$ such that*

$$\|\mathbf{v}-\mathbf{u}_{best}\|_D = \min_{\mathbf{u}\in\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D)}\|\mathbf{v}-\mathbf{u}\|_D.$$

## 4.2 Is $\mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)$ an immersed submanifold?

Assume that the tensor product map $\bigotimes$ is $T_D$-continuous and that we have a natural ambient space for $\mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)$ given by a Banach tensor space $\overline{\mathbf{V}_D}^{\|\cdot\|_D} = \mathbf{V}_{D_{\|\cdot\|_D}}$. Since the natural inclusion

$$\mathfrak{i}:\mathcal{FT}_\mathfrak{r}(\mathbf{V}_D)\longrightarrow\mathbf{V}_{D_{\|\cdot\|_D}},$$

given by $\mathfrak{i}(\mathbf{v})=\mathbf{v}$, is an injective map we will study $\mathfrak{i}$ as a function between Banach manifolds. To this end we recall the definition of an immersion between manifolds.

**Definition 4.20** *Let $F:X\to Y$ be a morphism between Banach manifolds and let $x\in X$. We shall say that $F$ is an* immersion at $x$, *if there exists an open neighbourhood $X_x$ of $x$ in $X$ such that the restriction of $F$ to $X_x$ induces an isomorphism from $X_x$ onto a submanifold of $Y$. We say that $F$ is an* immersion *if it is an immersion at each point of $X$.*

Our next step is to recall the definition of the differential as a morphism which gives a linear map between the tangent spaces of the manifolds involved with the morphism.

**Definition 4.21** *Let $X$ and $Y$ be two Banach manifolds. Let $F:X\to Y$ be a $\mathcal{C}^r$ morphism, i.e.,*

$$\psi\circ F\circ\varphi^{-1}:\varphi(U)\to\psi(W)$$

*is a $\mathcal{C}^r$-Fréchet differentiable map, where $(U,\varphi)$ is a chart in $X$ at $x$ and $(W,\psi)$ is a chart in $Y$ at $F(x)$. For $x\in X$, we define*

$$\mathrm{T}_xF:\mathbb{T}_x(X)\longrightarrow\mathbb{T}_{F(x)}(Y),\quad v\mapsto[(\psi\circ F\circ\varphi^{-1})'(\varphi(x))]v.$$

For Banach manifolds we have the following criterion for immersions (see Theorem 3.5.7 in [26]).

**Proposition 4.22** *Let $X,Y$ be Banach manifolds of class $\mathcal{C}^p$ $(p\geq 1)$. Let $F:X\to Y$ be a $\mathcal{C}^p$ morphism and $x\in X$. Then $F$ is an immersion at $x$ if and only if $\mathrm{T}_xF$ is injective and $\mathrm{T}_xF(\mathbb{T}_x(X))\in\mathbb{G}(\mathbb{T}_{F(x)}(Y))$.*

A concept related to an immersion between Banach manifolds is the following definition.

**Definition 4.23** *Assume that $X$ and $Y$ are Banach manifolds and let $f : X \longrightarrow Y$ be a $\mathcal{C}^r$ morphism. If $f$ is an injective immersion, then $f(X)$ is called an* immersed submanifold of $Y$.

Recall that there exists injective immersions which are not isomorphisms onto manifolds. It allows us to introduce the following definition.

**Definition 4.24** *An injective immersion $f : X \longrightarrow Y$ which is a homeomorphism onto $f(X)$ with the relative topology induced from $Y$ is called an* embedding. *Moreover, if $f : X \longrightarrow Y$ is an embedding, then $f(X)$ is called an* embedded submanifold of $Y$.

A classical example of an immersed submanifold which is not an embedded submanifold is given by the map $f : (3\pi/4, 7\pi/4) \longrightarrow \mathbb{R}^2$, written in polar coordinates by $r = \cos 2\theta$. It can be see that $f$ is an injective immersion however $f(3\pi/4, 7\pi/4)$ is not an open set in $\mathbb{R}^2$, because any neighbourhood of 0 in $\mathbb{R}^2$ intersects $f(3\pi/4, 7\pi/4)$ in a set with "corners" which is not homeomorphic to an open interval (see Figure 4.3). Before to give an example with tensors we need the following lemma.

**Lemma 4.25** *For each $\alpha \in T_D \setminus \{D\}$, the set $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ is a complemented subspace of $\mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$. Hence for each $\mathbf{v} \in \mathbf{V}_D$ and $\beta \notin \mathcal{L}(T_D)$ the set $\bigtimes_{\alpha \in S(\beta)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})))$ is a complemented subspace of the Banach space $\bigtimes_{\alpha \in S(\beta)} \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$.*

*Proof.* Observe that the map

$$\Pi_\alpha : \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right) \to \mathcal{L}\left(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\right)$$

defined by

$$\Pi_\alpha(L_\alpha) = P_{W_\alpha^{\min}(\mathbf{v}) \oplus U_\alpha^{\min}(\mathbf{v})} L_\alpha P_{U_\alpha^{\min}(\mathbf{v}) \oplus W_\alpha^{\min}(\mathbf{v})}$$

is a projection onto $\mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$. ■

**Example 4.26** *Consider the morphism*

$$f : \mathcal{U}(\mathbf{v}) \subset \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(V_{\alpha_{\|\cdot\|_\alpha}}, V_{\alpha_{\|\cdot\|_\alpha}}) \times \mathbb{R}^{\mathfrak{r}}$$

*defined locally for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ by $f(\mathbf{w}) = (\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))(\mathbf{w}) = (\mathfrak{L}, \mathfrak{C})$. Then in local coordinates we have that $f$ is the identity map. Clearly, $f$ is injective and*

$$\mathrm{T}_{\mathbf{v}} f(\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}}) = \bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}}.$$

*From Lemma 4.25 we have that*

$$\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \in \mathbb{G}(\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(V_{\alpha_{\|\cdot\|_\alpha}}, V_{\alpha_{\|\cdot\|_\alpha}}))$$

*and hence*

$$\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}} \in \mathbb{G}(\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(V_{\alpha_{\|\cdot\|_\alpha}}, V_{\alpha_{\|\cdot\|_\alpha}}) \times \mathbb{R}^{\mathfrak{r}}).$$

*Then by Proposition 4.22 $f$ is an immersion. Moreover, $f(\mathcal{U}(\mathbf{v}))$ with the topology induced by*

$$\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(V_{\alpha_{\|\cdot\|_\alpha}}, V_{\alpha_{\|\cdot\|_\alpha}}) \times \mathbb{R}^{\mathfrak{r}}$$

*is homeomorphic to $\mathcal{U}(\mathbf{v})$ when we consider in $\mathcal{U}(\mathbf{v})$ the initial topology induced by $f$. We point out that we can consider $\{\mathcal{U}(\mathbf{v}) : \mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)\}$ as a basis for a topology in $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Then, $f$ is an embedding and $f(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$ is an embedded submanifold of $\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \times \mathbb{R}^{\mathfrak{r}}$.*

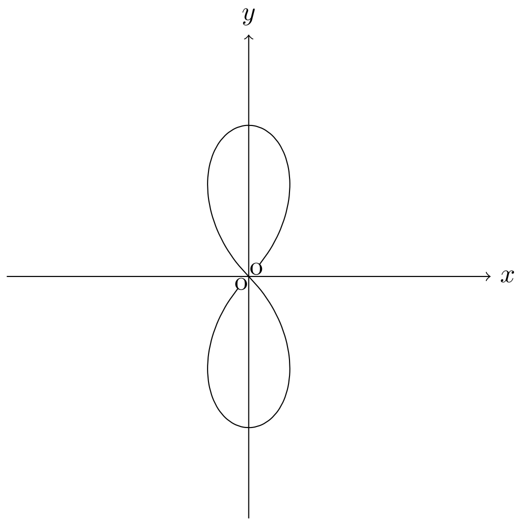


Figure 4.3: The set $f(3\pi/4, 7\pi/4)$ in $\mathbb{R}^2$. The "o" means that the lines approach without touch.

From the above example we have that even the manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is a subset of $\mathbf{V}_{D_{\|\cdot\|_D}}$ its geometric structure is fully compatible with the topology of the Banach space $\times_{\alpha\in S(\beta)} \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) \times \mathbb{R}^{\mathfrak{r}}$. Moreover, it is not difficult to see that the same argument runs for the manifold $\mathcal{FT}_{\leq\mathfrak{r}}(\mathbf{V}_D)$.

In consequence, to prove that the standard inclusion map $\mathfrak{i}$ is an immersion we shall prove, under the appropriate conditions, that if $\mathfrak{i}$ is a differentiable morphism then for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ the linear map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective and $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)))$ belongs to $\mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$.

### 4.2.1 The derivative as a morphism of the standard inclusion map

To describe $\mathfrak{i}$ as a morphism, we proceed as follows. Given $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, we consider $\mathcal{U}(\mathbf{v})$, a local neighbourhood of $\mathbf{v}$, and then

$$(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v})) : \bigtimes_{\alpha\in\mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}_*^{\mathfrak{r}} \to \mathbf{V}_{\|\cdot\|_D}$$

is given by

$$(\mathfrak{L}, \mathfrak{C}) \mapsto \sum_{\substack{1\leq i_k\leq r_k\\ k\in\mathcal{L}(T_D)}} \left( \sum_{\substack{1\leq i_\alpha\leq r_\alpha\\ \alpha\in T_D\setminus\{D\}\\ \alpha\notin\mathcal{L}(T_D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \prod_{\substack{\mu\in T_D\setminus\{D\}\\ S(\mu)\neq\emptyset}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \right) \bigotimes_{k\in\mathcal{L}(T_D)} (u^{(k)}_{i_k} + L_k(u^{(k)}_{i_k})),$$

that is,

$$(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))\,(\mathfrak{L}, \mathfrak{C}) = \mathbf{w} = \sum_{\substack{1\leq i_\alpha\leq r_\alpha\\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{w}^{(\alpha)}_{i_\alpha}$$

where for each $\mu \in T_D \setminus \{D\}$ we write

$$\mathbf{w}^{(\mu)}_{i_\mu} = \begin{cases} (id + L_\mu)(\mathbf{u}^{(\mu)}_{i_\mu}) & \text{if } \mu \in \mathcal{L}(T_D) \\ \sum_{\substack{1\leq i_\beta\leq r_\beta\\ \beta\in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} \mathbf{w}^{(\beta)}_{i_\beta} & \text{otherwise,} \end{cases}$$

for $1 \le i_\mu \le r_\mu$.

Assume that $(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))$ is Fréchet differentiable, then

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} : \bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}_*^{\mathfrak{r}} \longrightarrow \mathbf{V}_{\|\cdot\|_D}$$

is given by

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{L}}, \dot{\mathfrak{C}}) = [(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))'((\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))(\mathbf{v}))](\dot{\mathfrak{L}}, \dot{\mathfrak{C}}) = [(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))'(\mathfrak{o}, \mathfrak{C})](\dot{\mathfrak{L}}, \dot{\mathfrak{C}}),$$

where $(\Theta_{\mathbf{v}} \circ \chi_{\mathfrak{r}}(\mathbf{v}))(\mathbf{v}) = (\mathfrak{o}, \mathfrak{C})$, because $\Psi_{\mathbf{v}}((U_k^{\min}(\mathbf{v}))_{k \in \mathcal{L}(T_D)}) = (0)_{k \in \mathcal{L}(T_D)} = \mathfrak{o}$.

The next lemma describes the tangent map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$.

**Proposition 4.27** *Assume that the tensor product map $\bigotimes$ is $T_D$-continuous. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ be such that $\Theta_{\mathbf{v}}(\mathbf{v}) = (\mathfrak{o}, \mathfrak{C}(\mathbf{v}))$, where $\mathfrak{C}(\mathbf{v}) = (C^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$, $\mathfrak{o} = (0)_{\alpha \in \mathcal{L}(T_D)} \in \bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ and*

$$U_\alpha^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{u}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$$

*for $\alpha \in T_D \setminus \{D\}$. Then the following statements hold.*

*(a) The map $(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}(\mathbf{v}))$ from $\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \times \mathbb{R}_*^{\mathfrak{r}}$ to $\mathbf{V}_{D_{\|\cdot\|_D}}$ is Fréchet differentiable, and hence*

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} \in \mathcal{L}\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)), \mathbf{V}_{D_{\|\cdot\|_D}}\right).$$

*(b) Assume $(\dot{\mathfrak{L}}, \dot{\mathfrak{C}}) \in \mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$, where $\dot{\mathfrak{C}} = (\dot{C}^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$ and $\dot{\mathfrak{L}} = (\dot{L}_\alpha)_{\alpha \in \mathcal{L}(T_D)}$ is in $\bigtimes_{\alpha \in \mathcal{L}(T_D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$. Then $\dot{\mathbf{w}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$ if and only*

$$\dot{\mathbf{w}} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha \in S(D)} \sum_{1 \le i_\alpha \le r_\alpha} \left(\dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}\right), \tag{4.11}$$

*where*

$$\mathbf{U}_{i_\alpha}^{(\alpha)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(D) \\ \beta \ne \alpha}} C^{(D)}_{(i_\beta)_{\beta \in S(D)}} \bigotimes_{\beta \in S(D)} \mathbf{u}_{i_\beta}^{(\beta)}, \tag{4.12}$$

*and for each $\gamma \in T_D \setminus \{D\}$ we have*

$$\dot{\mathbf{u}}_{i_\gamma}^{(\gamma)} = \begin{cases} \dot{L}_\mu(\mathbf{u}_{i_\gamma}^{(\gamma)}) & \text{if } \gamma \in \mathcal{L}(T_D) \\ \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma, (i_\beta)_{\beta \in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta \in S(\gamma)} \sum_{1 \le i_\beta \le r_\beta} \left(\dot{\mathbf{u}}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\gamma, i_\beta}^{(\beta)}\right) & \text{otherwise,} \end{cases}$$

*where*

$$\mathbf{U}_{i_\gamma, i_\beta}^{(\beta)} = \sum_{\substack{1 \le i_\delta \le r_\delta \\ \delta \in S(\gamma) \\ \delta \ne \beta}} C^{(\gamma)}_{i_\mu, (i_\delta)_{\delta \in S(\gamma)}} \bigotimes_{\substack{\delta \ne \beta \\ \delta \in S(\gamma)}} \mathbf{u}_{i_\delta}^{(\delta)}, \tag{4.13}$$

*for $1 \le i_\gamma \le r_\gamma$ and $1 \le i_\beta \le r_\beta$.*

*Proof.* To prove statement (a), observe that for each $\mathbf{u}_\alpha \in U_\alpha^{\min}(\mathbf{v})$, $\alpha \in \mathcal{L}(T_D)$, the map

$$\Phi_{\mathbf{u}_\alpha} : \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})) \to W_\alpha^{\min}(\mathbf{v}), \quad L_\alpha \mapsto L_\alpha(\mathbf{u}_\alpha),$$

is linear and continuous, and hence Fréchet differentiable. Clearly, its differential is given by

$$[\Phi'_{\mathbf{u}_\alpha}(L_\alpha)](H_\alpha) = H_\alpha(\mathbf{u}_\alpha).$$

Also, if the tensor product map $\bigotimes$ is $T_D$-continuous, by Proposition 4.12, the tensor product map

$$\bigotimes : \bigtimes_{\beta\in S(\gamma)} (V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \to \left( {}_a\bigotimes_{\beta\in S(\gamma)} V_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\gamma \right),$$

for $\gamma \in T_D \setminus \mathcal{L}(T_D)$, is also Fréchet differentiable. Then, by the chain rule, the map $\Theta_{\mathbf{v}}^{-1}$ is Fréchet differentiable. Since $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}}) = [(\mathfrak{i} \circ \Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))'(\mathfrak{C}, \mathfrak{o})](\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$, (a) follows. Using the chain rule, we obtain (b). ■

Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ be such that

$$\mathbf{v} = \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha},$$

where for each $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ we have

$$\mathbf{u}^{(\mu)}_{i_\mu} = \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(\mu)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta\in S(\mu)} \mathbf{u}^{(\beta)}_{i_\beta}$$

for $1 \le i_\mu \le r_\mu$. Recall that for $\alpha \in S(D)$ we have

$$U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}^{(\alpha)}_{i_\alpha} : 1 \le i_\alpha \le r_\alpha\},$$

and for $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ we know that $U^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu}) = U^{\min}_\beta(\mathbf{v})$ and

$$U^{\min}_{S(\mu)\setminus\{\beta\}}(\mathbf{u}^{(\mu)}_{i_\mu}) = \operatorname{span}\{\mathbf{U}^{(\beta)}_{i_\mu,i_\beta} : 1 \le i_\beta \le r_\beta\}$$

for $1 \le i_\mu \le r_\mu$ and $\beta \in S(\mu)$. Hence

$$W^{\min}_\beta(\mathbf{v}) = W^{\min}_\beta(\mathbf{u}^{(\mu)}_{i_\mu}) \text{ for } 1 \le i_\mu \le r_\mu \text{ and } \beta \in S(\mu).$$

In the next proposition we prove that $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ injective when we consider $\mathbf{v}$ in the manifold $\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$. It allows us to characterise the tangent space for Tucker tensors inside the tensor space $\mathbf{V}_{D_{\|\cdot\|_D}}$.

**Proposition 4.28** *Assume that $S(D) = \mathcal{L}(T_D)$ and the tensor product map $\bigotimes$ is $T_D$-continuous. Let $\mathbf{v} \in \mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$, then the linear map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective and*

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D))) = {}_a\bigotimes_{\alpha\in S(D)} U^{\min}_\alpha(\mathbf{v}) \oplus \left( \bigoplus_{\alpha\in S(D)} W^{\min}_\alpha(\mathbf{v}) \otimes_a U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}) \right)$$

*is linearly isomorphic to $\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D))$.*

*Proof.* First, observe that if $\mathbf{v} \in \mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)$ and $\dot{\mathbf{w}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$, then by Proposition 4.27(b)

$$\dot{\mathbf{w}} = \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left( \dot{\mathbf{u}}^{(\alpha)}_{i_\alpha} \otimes \mathbf{U}^{(\alpha)}_{i_\alpha} \right),$$

where

$$\mathbf{U}^{(\alpha)}_{i_\alpha} = \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(D) \\ \beta\ne\alpha}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\beta\in S(D)} \mathbf{u}^{(\beta)}_{i_\beta} \in U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v}),$$

and $\dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} = \dot{L}(\mathbf{u}_{i_\alpha}^{(\alpha))}) \in W_\alpha^{\min}(\mathbf{v})$ for all $\alpha \in \mathcal{L}(T_D)$. Hence $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D))) \subset \mathbf{Z}^{(D)}(\mathbf{v})$ where

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} W_\alpha^{\min}(\mathbf{v}) \otimes_a U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v})\right).$$

Next, we claim that $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{M}_{\mathbf{r}}(\mathbf{V}_D)))$. To prove the claim take $\mathbf{w} \in \mathbf{Z}^{(D)}(\mathbf{v})$. Then we can write

$$\mathbf{w} = \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left(\mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}\right),$$

where $\mathbf{w}_{i_\alpha}^{(\alpha)} = W_\alpha^{\min}(\mathbf{v})$ for $1 \le i_\alpha \le r_\alpha$ and $\alpha \in S(D)$. Recall that

$$U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) = \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}.$$

Now, define $\dot{L}_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ by $\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) := \mathbf{w}_{i_\alpha}^{(\alpha)}$ for $1 \le i_\alpha \le r_\alpha$ and $\alpha \in S(D)$. Then the claim follows from $\mathbf{w} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}((\dot{L}_\alpha)_{\alpha\in S(D)}, \dot{C}^{(D)})$. To conclude the proof of the proposition we need to show that the map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is an injective linear operator. To prove this consider that

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}\left((\dot{L}_\beta)_{\beta\in\mathcal{L}(T_D)}, \dot{C}^{(D)}\right) = \mathbf{0},$$

that is,

$$\mathbf{0} = \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left(\dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}\right).$$

Thus,

$$\sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} = \mathbf{0},$$

$$\sum_{1\le i_\alpha\le r_\alpha} \left(\dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}\right) = \mathbf{0} \text{ for } \alpha \in S(D),$$

and hence $\dot{C}^{(D)} = \mathfrak{o}$, because $\left\{\bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)}\right\}$ is a basis of ${}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v})$ , and $\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} = \mathbf{0}$ for $1 \le i_\alpha \le r_\alpha$, because the $\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ are linearly independent for $\alpha \in S(D)$. Then $\dot{L}_\alpha = 0$ for all $\alpha \in S(D)$. We conclude that

$$\left((\dot{L}_\beta)_{\beta\in\mathcal{L}(T_D)}, \dot{C}^{(D)}\right) = ((0)_{\beta\in\mathcal{L}(T_D)}, \mathfrak{o})$$

and, in consequence, $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is injective. ■

Our next step is to show, by using the above proposition, that if the tensor product map $\bigotimes$ is $T_D$-continuous then the linear map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}$ is always injective for all $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

**Proposition 4.29** *Assume that the tensor product map* $\bigotimes$ *is* $T_D$*-continuous. Let* $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$*, then the linear map* $\mathrm{T}_{\mathbf{v}}\mathfrak{i} : \times_{\beta\in\mathcal{L}(T_D)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}} \to \mathbf{V}_{D_{\|\cdot\|_D}}$ *is injective.*

*Proof.* From Proposition 4.28 the statement holds when $S(D) = \mathcal{L}(T_D)$. Thus assume that $S(D) \neq \mathcal{L}(T_D)$. Then we can write the standard inclusion map $\mathfrak{i} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$ as $\mathfrak{i}_D \circ \mathfrak{i}_{\mathfrak{r},D}$ where

$$\mathfrak{i}_{\mathfrak{r},D} : \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \longrightarrow \mathcal{M}_{(r_\beta)_{\beta\in S(D)}}\left({}_a\bigotimes_{\beta\in S(D)} \mathbf{V}_\beta\right), \quad \mathbf{v} \mapsto \mathbf{v}$$

is a standard inclusion map and

$$\mathfrak{i}_D : \mathcal{M}_{(r_\beta)_{\beta\in S(D)}}\left( {}_a \bigotimes_{\beta\in S(D)} \mathbf{V}_\beta \right) \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$$

is given by

$$\mathbf{v} = \mathfrak{i}_D(\mathbf{v}) = \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(\alpha)}} C^{(D)}_{(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(D)} \mathbf{u}^{(\beta)}_{i_\beta}.$$

Using the chain rule, we have

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}_D \circ \mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D},$$

where

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}_D : \bigtimes_{\beta\in S(D)} \mathcal{L}(U^{\min}_\beta(\mathbf{v}), W^{\min}_\beta(\mathbf{v})) \times \mathbb{R}^{\times_{\beta\in S(D)} r_\beta} \to \mathbf{V}_{D_{\|\cdot\|_D}},$$

is given by

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}_D((\dot{L}_\alpha)_{\alpha\in S(D)}, \dot{C}^{(D)}) = \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}^{(\alpha)}_{i_\alpha} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left( \dot{L}_\alpha(\mathbf{u}^{(\alpha)}_{i_\alpha}) \otimes \mathbf{U}^{(\alpha)}_{i_\alpha} \right),$$

and

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D} : \bigtimes_{\beta\in \mathcal{L}(T_D)} \mathcal{L}(U^{\min}_\beta(\mathbf{v}), W^{\min}_\beta(\mathbf{v})) \times \mathbb{R}^{\mathfrak{r}} \to \bigtimes_{\beta\in S(D)} \mathcal{L}(U^{\min}_\beta(\mathbf{v}), W^{\min}_\beta(\mathbf{v})) \times \mathbb{R}^{\times_{\beta\in S(D)} r_\beta}$$

is given by

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D}((\dot{L}_\beta)_{\beta\in\mathcal{L}(T_D)}, (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}) = ((\dot{S}_\beta)_{\beta\in S(D)}, \dot{C}^{(D)}),$$

where $\dot{S}_\gamma = \dot{L}_\gamma$ if $\gamma \in \mathcal{L}(T_D)$, otherwise

$$\dot{S}_\gamma(\mathbf{u}^{(\gamma)}_{i_\gamma}) = \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}^{(\beta)}_{i_\beta} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta\le r_\beta} \dot{\mathbf{u}}^{(\gamma)}_{i_\gamma} \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}$$

and where for each $\gamma \in T_D \setminus \{D\}$ we have

$$\dot{\mathbf{u}}^{(\gamma)}_{i_\gamma} = \begin{cases} \dot{L}_\gamma(\mathbf{u}^{(\gamma)}_{i_\gamma}) & \text{if} \quad \gamma\in\mathcal{L}(T_D) \\ \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}^{(\beta)}_{i_\beta} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta\le r_\beta} \left( \dot{\mathbf{u}}^{(\beta)}_{i_\beta} \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} \right) & \text{otherwise.} \end{cases}$$

Let $\dot{\mathbf{w}} = T_{\mathbf{v}}\mathfrak{i}((\dot{L}_\beta)_{\beta\in\mathcal{L}(T_D)}, (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}) = \mathbf{0}$. Since $\mathrm{T}_{\mathbf{v}}\mathfrak{i} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}_D \circ \mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D}$ and, by Proposition 4.28, the linear map $\mathrm{T}_{\mathbf{v}}\mathfrak{i}_D$ is injective, then

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D}((\dot{L}_\beta)_{\beta\in\mathcal{L}(T_D)}, (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}) = ((0)_{\beta\in\mathcal{L}(T_D)}, 0).$$

In particular $\dot{C}^{(D)} = 0$ and by Proposition 4.27(b), we have

$$\dot{\mathbf{w}} = \mathbf{0} = \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \left( \dot{\mathbf{u}}^{(\alpha)}_{i_\alpha} \otimes \mathbf{U}^{(\alpha)}_{i_\alpha} \right), \tag{4.14}$$

where

$$\mathbf{U}^{(\alpha)}_{i_\alpha} = \sum_{\substack{1\le i_\beta\le r_\beta \\ \beta\in S(D) \\ \beta\neq\alpha}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\beta\in S(D)} \mathbf{u}^{(\beta)}_{i_\beta},$$

and for each $\gamma \in T_D \setminus \{D\}$ we have

$$\dot{\mathbf{u}}_{i_\gamma}^{(\gamma)} = \begin{cases} \dot{L}_\mu(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \dot{S}_\mu(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \mathbf{0} & \text{if} \quad \gamma \in \mathcal{L}(T_D) \\ \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta \le r_\beta} \left( \dot{\mathbf{u}}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\gamma,i_\beta}^{(\beta)} \right) & \text{otherwise,} \end{cases}$$

where

$$\mathbf{U}_{i_\gamma,i_\beta}^{(\beta)} = \sum_{\substack{1 \le i_\delta \le r_\delta \\ \delta \in S(\mu) \\ \delta \neq \beta}} C^{(\gamma)}_{i_\mu,(i_\delta)_{\delta \in S(\gamma)}} \bigotimes_{\substack{\delta \neq \beta \\ \delta \in S(\gamma)}} \mathbf{u}_{i_\delta}^{(\delta)},$$

for $1 \le i_\gamma \le r_\gamma$ and $1 \le i_\beta \le r_\beta$. We remark that if $S(\gamma) \subset \mathcal{L}(T_D)$ then

$$\dot{\mathbf{u}}_{i_\gamma}^{(\gamma)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)}.$$

From (4.14) and the fact that $\sum_{\alpha \in S(D)} \sum_{1 \le i_\alpha \le r_\alpha} \left( \dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \right) \in \bigoplus_{\alpha\in S(D)} W_\alpha^{\min}(\mathbf{v}) \otimes_a U^{\min}_{S(D)\setminus\{\alpha\}}(\mathbf{v})$ we obtain that

$$\sum_{1 \le i_\alpha \le r_\alpha} \left( \dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \right) = 0$$

for each $\alpha \in S(D)$. Finally, $\dot{\mathbf{u}}_{i_\alpha}^{(\alpha)} = \mathbf{0}$, because $\{\mathbf{U}_{i_\alpha}^{(\alpha)} : 1 \le i_\alpha \le r_\alpha\}$ are linearly independent vectors for each $\alpha \in S(D)$. In consequence, if $\alpha \in \mathcal{L}(T_D)$ then nothing has to be done, otherwise we have that for all $\gamma \notin \mathcal{L}(T_D)$ the equality

$$\mathbf{0} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta\in S(\gamma)} \sum_{1\le i_\beta \le r_\beta} \left( \dot{\mathbf{u}}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\gamma,i_\beta}^{(\beta)} \right)$$

holds for all $1 \le i_\gamma \le r_\gamma$. We remark that when $S(\gamma) \subset \mathcal{L}(T_D)$ we have

$$\mathbf{0} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)}$$

and hence we obtain that $\dot{C}^{(\gamma)} = 0$. Proceeding from the leaves to the root in the tree, we check that $\dot{C}^{(\gamma)} = 0$ holds for all $\gamma \in T_D \setminus \mathcal{L}(T_D)$ and the proposition follows. ■

Now, we want to construct for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ a linear subspace $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ to prove that $\mathbf{Z}^{(D)}(\mathbf{v}) = \mathrm{T}_{\mathbf{v}}\mathfrak{i}\left(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))\right)$. To this end assume that

$$\mathbf{v} = (\Theta_{\mathbf{v}}^{-1} \circ \chi_{\mathfrak{r}}^{-1}(\mathbf{v}))(\mathfrak{o}, \mathfrak{C}) = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} C^{(D)}_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)},$$

where for each $\mu \in T_D \setminus (\{D\} \cup \mathcal{L}(T_D))$ we have

$$\mathbf{u}_{i_\mu}^{(\mu)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\alpha)}} C^{(\mu)}_{i_\mu,(i_\beta)_{\beta\in S(\mu)}} \bigotimes_{\beta \in S(\mu)} \mathbf{u}_{i_\beta}^{(\beta)}.$$

Then to define $\mathbf{Z}^{(D)}(\mathbf{v})$ we proceed by the following steps.

**Step 1:** For $\gamma \in T_D \setminus \mathcal{L}(T_D)$ we observe that

$$\mathbf{u}_{i_\gamma}^{(\gamma)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\gamma)}} C^{(\alpha)}_{i_\gamma,(i_\beta)_{\beta\in S(\alpha)}} \bigotimes_{\beta \in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} \in \mathcal{M}_{(r_\beta)_{\beta\in S(\gamma)}}\left( {}_a \bigotimes_{\beta \in S(\gamma)} \mathbf{V}_\beta \right)$$

for $1 \le i_\gamma \le r_\gamma$ and $\beta \in S(\gamma)$. In particular, $\mathbf{u}_1^{(D)} = \mathbf{v}$. Let

$$\mathfrak{i}_\gamma : \mathcal{M}_{(r_\beta)_{\beta\in S(\gamma)}}\left({}_a\bigotimes_{\beta\in S(\gamma)} \mathbf{V}_\beta\right) \longrightarrow \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}}, \quad \mathbf{u}_\gamma \mapsto \mathbf{u}_\gamma,$$

be the standard inclusion map. Thanks to the proof of Proposition 4.28 we have a linear injective map

$$\mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}}\mathfrak{i}_\gamma : \bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})) \times \mathbb{R}^{\times_{\beta\in S(\gamma)} r_\beta} \to \mathbf{V}_{\gamma_{\|\cdot\|_\gamma}}$$

given by

$$\mathrm{T}_{\mathbf{u}_{i_\gamma}^{(\gamma)}}\mathfrak{i}_\gamma((\dot{L}_\beta)_{\beta\in S(\gamma)}, \dot{C}_{i_\gamma}^{(\gamma)}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta\in S(\gamma)}\sum_{1\le i_\beta\le r_\beta} \dot{L}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) \otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta},$$

where $\mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} = \sum_{\substack{1\le i_\delta\le r_\delta\\ \delta\in S(\gamma)\\ \delta\neq\beta}} C^{(\alpha)}_{i_\gamma,(i_\delta)_{\delta\in S(\gamma)}} \bigotimes_{\delta\in S(\gamma)} \mathbf{u}_{i_\delta}^{(\delta)}$ for $1 \le i_\beta \le r_\beta$ and $\beta \in S(\gamma)$ and also a linear subspace

$$\begin{aligned}\mathbf{Z}^{(\gamma)}(\mathbf{u}_{j_\gamma}^{(\gamma)}) &:= \mathrm{T}_{\mathbf{u}_{j_\gamma}^{(\gamma)}}\mathfrak{i}_\gamma\left(\bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})) \times \mathbb{R}^{\times_{\beta\in S(\gamma)} r_\beta}\right)\\ &\cong \bigtimes_{\beta\in S(\gamma)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})) \times \mathbb{R}^{\times_{\beta\in S(\gamma)} r_\beta}\end{aligned}$$

for $1 \le j_\gamma \le r_\gamma$ such that

$$\mathbf{Z}^{(\gamma)}(\mathbf{u}_{j_\gamma}^{(\gamma)}) = {}_a\bigotimes_{\beta\in S(\gamma)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\beta\in S(\gamma)} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\left\{\mathbf{U}^{(\beta)}_{j_\gamma,i_\beta} : 1 \le i_\beta \le r_\beta\right\}\right)$$

for $1 \le j_\gamma \le r_\gamma$. Since for each $\gamma \in T_D \setminus \mathcal{L}(T_D)$ we can write

$$\mathfrak{i}_\gamma(\mathbf{u}_{i_\gamma}^{(\gamma)}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\alpha)}} C^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{z}^{(\beta)}(\mathbf{u}_{i_\beta}^{(\beta)}) \tag{4.15}$$

for $1 \le i_\gamma \le r_\gamma$, where

$$\mathbf{z}^{(\beta)}(\mathbf{u}_{i_\beta}^{(\beta)}) := \begin{cases} \mathbf{u}_{i_\beta}^{(\beta)} & \text{if } \beta \in \mathcal{L}(T_D) \\ \mathfrak{i}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) = \sum_{\substack{1\le i_\mu\le r_\mu\\ \mu\in S(\beta)}} C^{(\beta)}_{i_\beta,(i_\mu)_{\mu\in S(\beta)}} \bigotimes_{\mu\in S(\beta)} \mathbf{u}_{i_\mu}^{(\mu)} & \text{otherwise,} \end{cases}$$

represents that either $\mathbf{u}_{i_\beta}^{(\beta)} \in \mathbf{V}_{\beta_{\|\cdot\|_\beta}}$ if $\beta \in \mathcal{L}(T_D)$ or $\mathbf{u}_{i_\beta}^{(\beta)} \in \mathcal{M}_{(r_\gamma)_{\gamma\in S(\beta)}}(\mathbf{V}_\beta)$, otherwise. We remark that in any case $\mathbf{z}^{(\beta)}(\mathbf{u}_{i_\beta}^{(\beta)}) = \mathbf{u}_{i_\beta}^{(\beta)}$. In particular, for each $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ we have

$$\mathfrak{i}_D(\mathbf{v}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(D)}} C^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\beta\in S(D)} \mathbf{z}^{(\beta)}(\mathbf{u}_{i_\beta}^{(\beta)}). \tag{4.16}$$

Assume that

$$\dot{\mathbf{w}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}((\dot{L}_k)_{k\in\mathcal{L}(T_D)}, (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)}) = \mathrm{T}_{\mathbf{v}}\mathfrak{i}_D((\dot{L}_\beta)_{\beta\in S(D)}, \dot{C}^{(D)}),$$

where $((\dot{L}_\beta)_{\beta\in S(D)}, \dot{C}^{(D)}) = \mathrm{T}_{\mathbf{v}}\mathfrak{i}_{\mathfrak{r},D}((\dot{L}_k)_{k\in\mathcal{L}(T_D)}, (\dot{C}^{(\alpha)})_{\alpha\in T_D\setminus\mathcal{L}(T_D)})$. Then, using the chain rule in (4.16) and taking into account (4.15), we have

$$\dot{\mathbf{w}} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}_D((\dot{L}_\beta)_{\beta\in S(D)}, \dot{C}^{(D)}) = \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(D)}} \dot{C}^{(D)}_{(i_\beta)_{\beta\in S(D)}} \bigotimes_{\beta\in S(D)} \mathbf{u}^{(\beta)}_{i_\beta} + \sum_{\beta\in S(D)}\sum_{1\le i_\beta\le r_\beta} \left(\dot{\mathbf{u}}^{(\beta)}_{i_\beta}\otimes \mathbf{U}^{(\beta)}_{i_\beta}\right)$$

where for all $\mu\in T_D\setminus\{D\}$ either $\dot{\mathbf{u}}^{(\mu)}_{i_\mu} = \dot{L}_\mu(\mathbf{u}^{(\mu)}_{i_\mu})$ if $\mu\in\mathcal{L}(T_D)$ or there exists a unique

$$(\dot{L}_\gamma)_{\substack{\gamma\in S(\mu)\\ \gamma\notin\mathcal{L}(T_D)}} \in \bigtimes_{\substack{\gamma\in S(\mu)\\ \gamma\notin\mathcal{L}(T_D)}} \mathcal{L}(U_\gamma^{\min}(\mathbf{v}), W_\gamma^{\min}(\mathbf{v}))$$

such that

$$\begin{aligned}
\dot{\mathbf{u}}^{(\mu)}_{i_\mu} &= \mathrm{T}_{\mathbf{u}^{(\mu)}_{i_\mu}}\mathfrak{i}_\mu((\dot{L}_\gamma)_{\gamma\in S(\mu)}, \dot{C}^{(\mu)}_{i_\mu})\\
&= \sum_{\substack{1\le i_\mu\le r_\mu\\ \mu\in S(D)}} \dot{C}^{(D)}_{i_\mu,(i_\gamma)_{\gamma\in S(\mu)}} \bigotimes_{\gamma\in S(\mu)} \mathbf{u}^{(\gamma)}_{i_\gamma} + \sum_{\gamma\in S(\mu)}\sum_{1\le i_\gamma\le r_\gamma}\left(\dot{L}_\gamma(\mathbf{u}^{(\gamma)}_{i_\gamma})\otimes \mathbf{U}^{(\gamma)}_{i_\mu,i_\gamma}\right)\\
&= \sum_{\substack{1\le i_\mu\le r_\mu\\ \mu\in S(D)}} \dot{C}^{(D)}_{i_\mu,(i_\gamma)_{\gamma\in S(\mu)}} \bigotimes_{\gamma\in S(\mu)} \mathbf{u}^{(\gamma)}_{i_\gamma} + \sum_{\gamma\in S(\mu)}\sum_{1\le i_\gamma\le r_\gamma}\left(\dot{\mathbf{u}}^{(\gamma)}_{i_\gamma}\otimes \mathbf{U}^{(\gamma)}_{i_\mu,i_\gamma}\right),
\end{aligned}$$

where the last equality is given by Lemma 4.27(b). In consequence, we obtain that

$$\dot{\mathbf{u}}^{(\gamma)}_{i_\gamma} \in W_\gamma^{\min}(\mathbf{v}) \text{ for all } \gamma\in T_D\setminus\{D\}.$$

**Step 2:** Now, for each $\gamma\in T_D\setminus\{D\}$ we define a linear subspace $\mathcal{H}_\gamma(\mathbf{v})\subset W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$ as follows. Let $\mathcal{H}_\gamma(\mathbf{v}) := W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$ if $\gamma\in\mathcal{L}(T_D)$. For $\gamma\notin\mathcal{L}(T_D)$ we construct $\mathcal{H}_\gamma(\mathbf{v})$ in the following way. Let

$$\Upsilon_{\gamma,\mathbf{v}} : \mathbb{R}^{r_\gamma\times \bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{H}_\beta(\mathbf{v}) \longrightarrow W_\gamma^{\min}(\mathbf{v})^{r_\gamma}$$

be a linear map defined by

$$\Upsilon_{\gamma,\mathbf{v}}(\dot{C}^{(\gamma)}, ((\mathbf{w}^{(\beta)}_{i_\beta})_{i_\beta=1}^{r_\beta})_{\beta\in S(\gamma)}) := (\mathbf{w}^{(\gamma)}_{i_\gamma})_{i_\gamma=1}^{r_\gamma},$$

where

$$\mathbf{w}^{(\gamma)}_{i_\gamma} := \sum_{\substack{1\le i_\beta\le r_\beta\\ \beta\in S(\gamma)}} \dot{C}^{(\gamma)}_{i_\gamma,(i_\beta)_{\beta\in S(\gamma)}} \bigotimes_{\beta\in S(\gamma)} \mathbf{u}^{(\beta)}_{i_\beta} + \sum_{\beta\in S(\gamma)}\sum_{1\le i_\beta\le r_\beta} \mathbf{w}^{(\beta)}_{i_\beta}\otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}$$

for $1\le i_\gamma\le r_\gamma$. Let $\pi_{i_\gamma} : \mathbb{R}^{r_\gamma\times \bigtimes_{\beta\in S(\gamma)} r_\beta} \to \mathbb{R}^{\bigtimes_{\beta\in S(\gamma)} r_\beta}$ be given by $\pi_{i_\gamma}(\dot{C}^{(\gamma)}) = \dot{C}^{(\gamma)}_{i_\gamma}$, for $1\le i_\gamma\le r_\gamma$. Observe that if we define $\dot{L}_\gamma(\mathbf{u}^{(\gamma)}_{i_\gamma}) := \mathbf{w}^{(\gamma)}_{i_\gamma}$ for $1\le i_\gamma\le r_\gamma$ and $\dot{L}_\beta(\mathbf{u}_{i_\beta}) := \mathbf{w}^{(\beta)}_{i_\beta}$ for $1\le i_\beta\le r_\beta$ and $\beta\in S(\gamma)$, then

$$\mathbf{w}^{(\gamma)}_{i_\gamma} = \mathrm{T}_{\mathbf{u}^{(\gamma)}_{i_\gamma}}\mathfrak{i}_\gamma(\pi_{i_\gamma}(\dot{C}^{(\gamma)}), (\dot{L}_\beta)_{\beta\in S(\gamma)}) \in \mathbf{Z}^{(\gamma)}(\mathbf{u}^{(\gamma)}_{i_\gamma})$$

for $1\le i_\gamma\le r_\gamma$, and hence by Proposition 4.27 the map $\Upsilon_{\gamma,\mathbf{v}}$ is injective. Finally, we define the linear subspace

$$\mathcal{H}_\gamma(\mathbf{v}) := \Upsilon_{\gamma,\mathbf{v}}\left(\mathbb{R}^{r_\gamma\times \bigtimes_{\beta\in S(\gamma)} r_\beta} \times \bigtimes_{\beta\in S(\gamma)} \mathcal{H}_\beta(\mathbf{v})\right).$$

For $\delta\in T_D\setminus\{D\}$ let $\Pi_{i_\delta} : W_\delta^{\min}(\mathbf{v})^{r_\delta} \to W_\delta^{\min}(\mathbf{v})$ be given by $\Pi_{i_\delta}((\mathbf{w}^{(\delta)}_{k_\delta})_{k_\delta=1}^{r_\delta}) := \mathbf{w}^{(\delta)}_{i_\delta}$ for $1\le i_\delta\le r_\delta$. Observe, that for each $\beta\in S(\gamma)$, we can identify $(\mathbf{w}^{(\beta)}_{i_\beta})_{i_\beta=1}^{r_\beta}\in\mathcal{H}_\beta(\mathbf{v})$ with

$$\sum_{1\le i_\beta\le r_\beta} \mathbf{w}^{(\beta)}_{i_\beta}\otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta} = \sum_{1\le i_\beta\le r_\beta} \Pi_{i_\beta}((\mathbf{w}^{(\beta)}_{k_\beta})_{k_\beta=1}^{r_\beta})\otimes \mathbf{U}^{(\beta)}_{i_\gamma,i_\beta}$$

for $1 \le i_\gamma \le r_\gamma$. It allows us to construct an injective linear map

$$f_{\beta,i_\gamma} : \mathcal{H}_\beta(\mathbf{v}) \longrightarrow V_{\gamma_{\|\cdot\|_\gamma}}, \quad (\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta} \mapsto \sum_{1\le i_\beta \le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\gamma,i_\beta}^{(\beta)},$$

for $1 \le i_\gamma \le r_\gamma$. Hence $f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v}))$ is a linear subspace of $V_{\gamma_{\|\cdot\|_\gamma}}$ linearly isomorphic to $\mathcal{H}_\beta(\mathbf{v})$ for $1 \le i_\gamma \le i_\gamma$. Thus,

$$\Pi_{i_\gamma}(\mathcal{H}_\gamma(\mathbf{v})) = \begin{cases} {}_a\bigotimes_{\beta\in S(\gamma)} U_\beta^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\beta\in S(\gamma)} f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v}))\right) & \text{if } \gamma \notin \mathcal{L}(T_D), \\ W_\gamma^{\min}(\mathbf{v}) & \text{if } \gamma \in \mathcal{L}(T_D), \end{cases}$$

where

$$f_{\beta,i_\gamma}(\mathcal{H}_\beta(\mathbf{v})) = \begin{cases} \bigoplus_{i_\beta=1}^{r_\beta} \Pi_{i_\beta}(\mathcal{H}_\beta(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\gamma,i_\beta}^{(\beta)}\} & \text{if } \beta \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\beta=1}^{r_\beta} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\gamma,i_\beta}^{(\beta)}\} & \text{if } \beta \in \mathcal{L}(T_D) \end{cases}$$

for $1 \le i_\gamma \le r_\gamma$.

**Step 3:** Finally, we construct a linear subspace $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathbf{V}_{D_{\|\cdot\|_D}}$ by using a linear injective map

$$\Upsilon_{D,\mathbf{v}} : \mathbb{R}^{\times_{\alpha\in S(D)} r_\alpha} \times \bigtimes_{\alpha\in S(D)} \mathcal{H}_\alpha(\mathbf{v}) \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$$

defined by

$$\Upsilon_{\gamma,\mathbf{v}}(\dot{C}^{(D)}, ((\mathbf{w}_{i_\alpha}^{(\alpha)})_{i_\alpha=1}^{r_\alpha})_{\alpha\in S(D)}) := \mathbf{w}$$

where

$$\mathbf{w} := \sum_{\substack{1\le i_\alpha\le r_\alpha \\ \alpha\in S(D)}} \dot{C}^{(D)}_{(i_\alpha)_{\alpha\in S(D)}} \bigotimes_{\alpha\in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha\in S(D)} \sum_{1\le i_\alpha\le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}.$$

Then $\mathbf{Z}^{(D)}(\mathbf{v}) := \Upsilon_{D,\mathbf{v}}\left(\mathbb{R}^{\times_{\alpha\in S(D)} r_\alpha} \times \bigtimes_{\alpha\in S(D)} \mathcal{H}_\alpha(\mathbf{v})\right)$ and from Step 1 we have that

$$\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \subset \mathbf{Z}^{(D)}(\mathbf{v})$$

holds. Moreover, we can introduce for each $\alpha \in S(D)$ a linear injective map

$$f_{D,\alpha} : \mathcal{H}_\alpha(\mathbf{v}) \to \mathbf{V}_{D_{\|\cdot\|_D}}, \quad (\mathbf{w}_{i_\alpha})_{i_\alpha=1}^{r_\alpha} \mapsto \sum_{1\le i_\alpha\le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)}.$$

Then $f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v}))$ is a linear subspace in $\mathbf{V}_{D_{\|\cdot\|_D}}$ linearly isomorphic to $\mathcal{H}_\alpha(\mathbf{v})$. It is not difficult to show that

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \begin{cases} \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\alpha=1}^{r_\alpha} W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \in \mathcal{L}(T_D) \end{cases}$$

for $\alpha \in S(D)$. By construction, we have

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a\bigotimes_{\alpha\in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left(\bigoplus_{\alpha\in S(D)} f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v}))\right).$$

**Proposition 4.30** *Assume that $S(D) \neq \mathcal{L}(T_D)$ and the tensor product map $\bigotimes$ is $T_D$-continuous. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, then $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) = \mathbf{Z}^{(D)}(\mathbf{v})$ and hence it is linearly isomorphic to $\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$.*

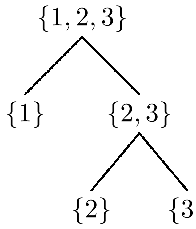


Figure 4.4: A binary tree $T_D$.

*Proof.* From Step 1 and the construction of $\mathbf{Z}^{(D)}(\mathbf{v})$, the inclusion $\mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \subset \mathbf{Z}^{(D)}(\mathbf{v})$ holds. Now, take $\mathbf{w} \in \mathbf{Z}^{(D)}(\mathbf{v})$. Then we can write

$$\mathbf{w} = \sum_{\substack{1 \le i_\alpha \le r_\alpha \\ \alpha \in S(D)}} (\dot{C}^{(D)})_{(i_\alpha)_{\alpha \in S(D)}} \bigotimes_{\alpha \in S(D)} \mathbf{u}_{i_\alpha}^{(\alpha)} + \sum_{\alpha \in S(D)} \sum_{1 \le i_\alpha \le r_\alpha} \left( \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \right),$$

where $\dot{C}^{(D)} \in \mathbb{R}^{\times_{\alpha \in S(D)} r_\alpha}$ and $\mathbf{w}_{i_\alpha}^{(\alpha)} \in W_\alpha^{\min}(\mathbf{v})$ for $1 \le i_\alpha \le r_\alpha$. Then we can define $\dot{L}_\alpha \in \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v}))$ by $\dot{L}_\alpha(\mathbf{u}_{i_\alpha}^{(\alpha)}) := \mathbf{w}_{i_\alpha}^{(\alpha)}$ for $1 \le i_\alpha \le r_\alpha$, and we have

$$(\dot{C}^{(D)}, (\dot{L}_\alpha)_{\alpha \in S(D)}) \in \mathbb{R}^{\times_{\alpha \in S(D)} r_\alpha} \times \bigtimes_{\alpha \in S(D)} \mathcal{L}(U_\alpha^{\min}(\mathbf{v}), W_\alpha^{\min}(\mathbf{v})).$$

Moreover, $\sum_{1 \le i_\alpha \le r_\alpha} \mathbf{w}_{i_\alpha}^{(\alpha)} \otimes \mathbf{U}_{i_\alpha}^{(\alpha)} \in f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v}))$ for $\alpha \in S(D)$. If $\alpha \notin \mathcal{L}(T_D)$, then $(\mathbf{w}_{i_\alpha}^{(\alpha)})_{i_\alpha=1}^{r_\alpha} \in \mathcal{H}_\alpha(\mathbf{v}) = \Upsilon_{\alpha,\mathbf{v}} \left( \mathbb{R}^{r_\alpha \times \times_{\beta \in S(\gamma)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{H}_\beta(\mathbf{v}) \right)$. Hence there exists

$$(\dot{C}^{(\alpha)}, ((\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta})_{\beta \in S(\alpha)}) \in \mathbb{R}^{r_\alpha \times \times_{\beta \in S(\alpha)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{H}_\beta(\mathbf{v})$$

such that

$$\mathbf{w}_{i_\alpha}^{(\alpha)} = \sum_{\substack{1 \le i_\beta \le r_\beta \\ \beta \in S(\alpha)}} \dot{C}^{(\alpha)}_{i_\alpha,(i_\beta)_{\beta \in S(\alpha)}} \bigotimes_{\beta \in S(\alpha)} \mathbf{u}_{i_\beta}^{(\beta)} + \sum_{\beta \in S(\alpha)} \sum_{1 \le i_\beta \le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}$$

for $1 \le i_\alpha \le r_\alpha$. Define $\dot{L}_\beta(\mathbf{u}_{i_\beta}^{(\beta)}) := \mathbf{w}_{i_\beta}^{(\beta)}$ for $1 \le i_\beta \le r_\beta$ and $\beta \in S(\alpha)$. Then

$$(\dot{C}^{(\alpha)}, (\dot{L}_\beta)_{\beta \in S(\alpha)}) \in \mathbb{R}^{r_\gamma \times \times_{\beta \in S(\gamma)} r_\beta} \times \bigtimes_{\beta \in S(\alpha)} \mathcal{L}(U_\beta^{\min}(\mathbf{v}), W_\beta^{\min}(\mathbf{v})).$$

Moreover, $\sum_{1 \le i_\beta \le r_\beta} \mathbf{w}_{i_\beta}^{(\beta)} \otimes \mathbf{U}_{i_\alpha,i_\beta}^{(\beta)} \in f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v}))$ for $1 \le i_\alpha \le r_\alpha$. If $\beta \notin \mathcal{L}(T_D)$, then $(\mathbf{w}_{i_\beta}^{(\beta)})_{i_\beta=1}^{r_\beta} \in \mathcal{H}_\beta(\mathbf{v}) = \Upsilon_{\beta,\mathbf{v}} \left( \mathbb{R}^{r_\beta \times \times_{\gamma \in S(\beta)} r_\gamma} \times \bigtimes_{\gamma \in S(\beta)} \mathcal{H}_\gamma(\mathbf{v}) \right)$. Proceeding in a similar way from the root to the leaves, we construct $(\dot{\mathfrak{L}}, \dot{\mathfrak{C}}) \in \mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))$, where $\dot{\mathfrak{C}} = (\dot{C}^{(\alpha)})_{\alpha \in T_D \setminus \mathcal{L}(T_D)} \in \mathbb{R}^{\mathfrak{r}}$ and $\dot{\mathfrak{L}} = (\dot{L}_\alpha)_{\alpha \in T_D \setminus \{D\}} \in \mathcal{L}_{T_D}(\mathbf{v})$ such that $\mathbf{w} = \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\dot{\mathfrak{C}}, \dot{\mathfrak{L}})$. Thus, we can conclude that $\mathbf{Z}^{(D)}(\mathbf{v}) \subset \mathrm{T}_{\mathbf{v}}\mathfrak{i}(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)))$ and the equality follows. ∎

**Example 4.31** *Consider the binary tree $T_D$ given in Figure 4.4 and consider TB ranks $\mathfrak{r} = (1, r_1, r_{23}, r_2, r_3)$. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(V_1 \otimes_a V_2 \otimes_a V_3)$ and assume that the tensor product map $\bigotimes$ is $T_D$-continuous. Then*

$$\mathbf{Z}^{(123)}(\mathbf{v}) = \left( U_1^{\min}(\mathbf{v}) \otimes_a U_{23}^{\min}(\mathbf{v}) \right) \oplus f_{123,1}(\mathcal{H}_1(\mathbf{v})) \oplus f_{123,23}(\mathcal{H}_{23}(\mathbf{v})),$$

*where*

$$f_{123,1}(\mathcal{H}_1(\mathbf{v})) = \bigoplus_{i_1=1}^{r_1} W_1^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_1}^{(1)}\} \subset V_{1_{\|\cdot\|_1}} \otimes_a \left( V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}} \right),$$

$$f_{123,23}(\mathcal{H}_{23}(\mathbf{v})) = \bigoplus_{i_{23}=1}^{r_{23}} \operatorname{span}\{\mathbf{U}_{i_{23}}^{(23)}\} \otimes_a \Pi_{i_{23}}(\mathcal{H}_{23}(\mathbf{v})) \subset V_{1_{\|\cdot\|_1}} \otimes_a \left( V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}} \right),$$

*and*

$$\Pi_{i_{23}}(\mathcal{H}_{23}(\mathbf{v})) = \left(U_2^{\min}(\mathbf{v}) \otimes_a U_3^{\min}(\mathbf{v})\right) \oplus \left(\bigoplus_{i_2=1}^{r_2} W_2^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_{23},i_2}^{(2)}\}\right) \oplus \left(\bigoplus_{i_3=1}^{r_3} \operatorname{span}\{\mathbf{U}_{i_{23},i_3}^{(3)}\} \otimes_a W_3^{\min}(\mathbf{v})\right),$$

*which is a linear subspace in* $V_{2_{\|\cdot\|_2}} \otimes_a V_{3_{\|\cdot\|_3}}$.

### 4.2.2 Is the standard inclusion map an immersion?

Finally, to show that $\mathfrak{i}$ is an immersion, and hence $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$, we need to prove that $\mathrm{T}_{\mathbf{v}}\mathfrak{i}\,(\mathbb{T}_{\mathbf{v}}(\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D))) \in \mathbb{G}(\mathbf{V}_{\|\cdot\|_D})$. Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D\setminus\{D\}}$ be a representation of the Banach tensor space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$, in the topological tree-based format and take $\mathbf{V}_D := {}_a\bigotimes_{j\in D} V_j$. A first useful result is the following lemma.

**Lemma 4.32** *Assume that* (4.10) *holds. Let* $\alpha \in T_D \setminus \mathcal{L}(T_D)$ *and take* $\beta \in S(\alpha)$. *If* $W_\beta \in \mathbb{G}(\mathbf{V}_{\beta_{\|\cdot\|_\beta}})$ *satisfies* $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} = U_\beta \oplus W_\beta$ *for some finite-dimensional subspace* $U_\beta$ *in* $\mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, *then* $W_\beta \otimes_a U_{[\beta]} \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ *for every finite-dimensional subspace* $U_{[\beta]} \subset {}_a\bigotimes_{\delta\in S(\alpha)\setminus\beta} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}$.

*Proof.* First, observe that if $W_\beta$ is a finite-dimensional subspace, then $W_\beta \otimes_a U_{[\beta]}$ is also finite dimensional, and hence the lemma follows. Thus, assume that $W_\beta$ is an infinite-dimensional closed subspace of $\mathbf{V}_{\beta_{\|\cdot\|_\beta}}$, and to simplify the notation write

$$\mathbf{X}_\beta := {}_{\|\cdot\|_{\vee(S(\alpha)\setminus\beta)}} \bigotimes_{\delta\in S(\alpha)\setminus\{\beta\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}.$$

If $U_{[\beta]} \subset \mathbf{X}_\beta$ is a finite-dimensional subspace, then there exists $W_{[\beta]} \in \mathbb{G}(\mathbf{X}_\beta)$ such that $\mathbf{X}_\beta = U_{[\beta]} \oplus W_{[\beta]}$. Since the tensor product map

$$\bigotimes : (\mathbf{V}_{\beta_{\|\cdot\|_\beta}}, \|\cdot\|_\beta) \times \left(\mathbf{X}_\beta, \|\cdot\|_{\vee(S(\alpha)\setminus\beta)}\right) \to (\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \|\cdot\|_\alpha)$$

is continuous and by Lemma 3.18 in [11], for each elementary tensor $\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]} \in \mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$ we have

$$\begin{aligned} \|(id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}})(\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]})\|_\alpha &\le C\sqrt{\dim U_{[\beta]}}\, \|\mathbf{v}_\beta\|_\beta \|\mathbf{v}_{[\beta]}\|_{\vee(S(\alpha)\setminus\beta)} \\ &= C\sqrt{\dim U_{[\beta]}}\, \|\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]}\|_{\vee(S(\alpha))} \\ &\le C'\sqrt{\dim U_{[\beta]}}\, \|\mathbf{v}_\beta \otimes \mathbf{v}_{[\beta]}\|_\alpha. \end{aligned}$$

Thus, $(id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}})$ is continuous over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$, and hence in $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$. Now, take into account the fact that

$$id_\beta = P_{U_\beta\oplus W_\beta} + P_{W_\beta\oplus U_\beta},$$

so that

$$id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}} = P_{U_\beta\oplus W_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}} + P_{W_\beta\oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}.$$

Observe that $id_\beta \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ and $P_{U_\beta\oplus W_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ are continuous linear maps over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$, and then $P_{W_\beta\oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}$ is a continuous linear map over $\mathbf{V}_{\beta_{\|\cdot\|_\beta}} \otimes_a \mathbf{X}_\beta$. Thus,

$$\mathcal{P}_\alpha := \overline{P_{W_\beta\oplus U_\beta} \otimes P_{U_{[\beta]}\oplus W_{[\beta]}}} \in \mathcal{L}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}, \mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$$

and $\mathcal{P}_\alpha \circ \mathcal{P}_\alpha = \mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}) = W_\beta \otimes_a U_{[\beta]}$, the lemma follows by Proposition 3.4. ■

**Lemma 4.33** *Let $X$ be a Banach space and assume that $U, V \in \mathbb{G}(X)$. If $U \cap V = \{0\}$, then $U \oplus V \in \mathbb{G}(X)$. Moreover, $U \cap V \in \mathbb{G}(X)$ holds.*

*Proof.* To prove the first statement assume that $U \cap V = \{0\}$. Since $U, V \in \mathbb{G}(X)$ there exist $U', V' \in \mathbb{G}(X)$, such that $X = U \oplus U' = V \oplus V'$. Then $U = X \cap U = (V \oplus V') \cap U = U \cap V'$ and $V = X \cap V = (U \oplus U') \cap V = V \cap U'$. In consequence, we can write

$$U \oplus V \oplus (U' \cap V') = (U \cap V') \oplus (V \cap U') \oplus (U' \cap V') = (U \oplus U') \cap (V \oplus V') = X,$$

and the first statement follows. To prove the second one, observe that $X = (U \cap V) \oplus (U \cap V') \oplus (V \cap U') \oplus (U' \cap V')$. ■

A very useful consequence of the above two lemmas is the following Theorem.

**Theorem 4.34** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D} \bigotimes_{j \in D} V_j$ in the topological tree-based format and assume that (4.10) holds. Then $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$, and hence $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$.*

*Proof.* Since the tensor product map is $T_D$-continuous, Proposition 4.27 gives us the differentiability of $\mathrm{T}_{\mathbf{v}} i$. Assume first that $S(D) = \mathcal{L}(T_D)$. From Corollary 4.28 we have

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left( \bigoplus_{\alpha \in S(D)} W_\alpha^{\min}(\mathbf{v}) \otimes_a U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) \right).$$

For each $\alpha \in S(D)$ we have $W_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ and $U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) \subset {}_a \bigotimes_{\delta \in S(D)\setminus\{\alpha\}} \mathbf{V}_{\delta_{\|\cdot\|_\delta}}$ is a finite-dimensional subspace. From Lemma 4.32 we have $W_\alpha^{\min}(\mathbf{v}) \otimes_a U_{S(D)\setminus\{\alpha\}}^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$ for all $\alpha \in S(D)$. Since ${}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$, by Lemma 4.33, we obtain that $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$.

Now, assume that $S(D) \neq \mathcal{L}(T_D)$. Then

$$\mathbf{Z}^{(D)}(\mathbf{v}) = {}_a \bigotimes_{\alpha \in S(D)} U_\alpha^{\min}(\mathbf{v}) \oplus \left( \bigoplus_{\alpha \in S(D)} f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \right)$$

and

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \begin{cases} \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\alpha=1}^{r_\alpha} W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} & \text{if } \alpha \in \mathcal{L}(T_D) \end{cases}$$

for $\alpha \in S(D)$. For $\alpha \in \mathcal{L}(T_D)$ we have $W_\alpha^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ and $\operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\}$ is a finite-dimensional subspace for $1 \le i_\alpha \le r_\alpha$, and from Lemma 4.32, $W_\alpha^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\} \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$ for $1 \le i_\alpha \le r_\alpha$. By Lemma 4.33, $f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$. Otherwise, if $\alpha \notin \mathcal{L}(T_D)$ then

$$f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = \bigoplus_{i_\alpha=1}^{r_\alpha} \Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha}^{(\alpha)}\},$$

where

$$\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = {}_a \bigotimes_{\beta \in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \oplus \left( \bigoplus_{\beta \in S(\alpha)} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right)$$

for $1 \le i_\alpha \le r_\alpha$. Now,

$$f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) = \begin{cases} \bigoplus_{i_\beta=1}^{r_\beta} \Pi_{i_\beta}(\mathcal{H}_\beta(\mathbf{v})) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}\} & \text{if } \beta \notin \mathcal{L}(T_D) \\ \bigoplus_{i_\beta=1}^{r_\beta} W_\beta^{\min}(\mathbf{v}) \otimes_a \operatorname{span}\{\mathbf{U}_{i_\alpha,i_\beta}^{(\beta)}\} & \text{if } \beta \in \mathcal{L}(T_D) \end{cases}$$

for $1 \leq i_\alpha \leq r_\alpha$. Clearly, if $\beta \in \mathcal{L}(T_D)$ then $f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ for $1 \leq i_\alpha \leq r_\alpha$. Then we can write,

$$\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) = {}_a \bigotimes_{\beta \in S(\alpha)} U_\beta^{\min}(\mathbf{v}) \oplus \left( \bigoplus_{\substack{\beta \in S(\alpha) \\ \beta \in \mathcal{L}(T_D)}} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right) \oplus \left( \bigoplus_{\substack{\beta \in S(\alpha) \\ \beta \notin \mathcal{L}(T_D)}} f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \right)$$

for $1 \leq i_\alpha \leq r_\alpha$. Starting from the leaves, that is $\gamma \in \mathcal{L}(T_D)$, we have that $\Pi_{i_\gamma}(\mathcal{H}_\gamma(\mathbf{v})) = W_\gamma^{\min}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{\gamma_{\|\cdot\|_\gamma}})$ for $1 \leq i_\gamma \leq r_\gamma$, and hence for $\delta \in T_D$ such that $\gamma \in S(\delta)$ we obtain $f_{\gamma,i_\delta}(\mathcal{H}_\gamma(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\delta_{\|\cdot\|_\delta}})$ for $1 \leq i_\delta \leq r_\delta$. Proceeding inductively from the leaves to the root, we obtain that $f_{\beta,i_\alpha}(\mathcal{H}_\beta(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$, for $\beta \in S(\alpha)$ with $\beta \notin \mathcal{L}(T_D)$ and $1 \leq i_\alpha \leq r_\alpha$. Lemma 4.33 says us that $\Pi_{i_\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}})$ for $1 \leq i_\alpha \leq r_\alpha$. From Lemma 4.32 and Lemma 4.33 we obtain that $f_{D,\alpha}(\mathcal{H}_\alpha(\mathbf{v})) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$. Also by Lemma 4.33, we have $\mathbf{Z}^{(D)}(\mathbf{v}) \in \mathbb{G}(\mathbf{V}_{D_{\|\cdot\|_D}})$, that proves the theorem. ■

**Example 4.35** *Let us recall the topological tensor spaces introduced in Example 4.3. Let $I_j \subset \mathbb{R}$ $(1 \leq j \leq d)$ and $1 \leq p < \infty$. Given tree $T_D$, let $\mathbf{I}_\alpha := \times_{j \in \alpha} I_j$ for $\alpha \in T_D$. Hence $L^p(\mathbf{I}_\alpha)$ is a tensor Banach space for all $\alpha \in T_D$. In this example we denote the usual norm of $L^p(\mathbf{I}_\alpha)$ by $\|\cdot\|_{\alpha,p}$. Since $\|\cdot\|_{\alpha,p}$ is a reasonable crossnorm (see Example 4.72 in [19]), then* (4.10) *holds for all $\alpha \in T_D$. From Theorem 4.34 we obtain that $\mathcal{FT}_{\mathfrak{r}}\left({}_a\bigotimes_{j=1}^d L^p(I_j)\right)$ is an immersed submanifold of $L^p(\mathbf{I}_D)$.*

**Example 4.36** *Now, we return to Example 4.1. From Example 4.42 in [19] we know that the norm $\|\cdot\|_{(0,1),p}$ is a crossnorm on $H^{1,p}(I_1) \otimes_a H^{1,p}(I_2)$, and hence it is not weaker than the injective norm. In consequence, from Theorem 4.34, we obtain that $\mathcal{FT}_{\mathfrak{r}}(H^{1,p}(I_1) \otimes_a H^{1,p}(I_2))$ is an immersed submanifold in $H^{1,p}(I_1) \otimes_{\|\cdot\|_{(0,1),p}} H^{1,p}(I_2)$.*

Since in a reflexive Banach space every closed linear subspace is proximinal (see p. 61 in [13]), we have the following corollary.

**Corollary 4.37** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a reflexive tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$ in the topological tree-based format and assume that* (4.10)*holds. Let $\mathbf{v} \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, then for each $\dot{\mathbf{u}} \in \mathbf{V}_{D_{\|\cdot\|_D}}$ there exists $\dot{\mathbf{v}}_{best} \in \mathbf{Z}^{(D)}(\mathbf{v})$ such that*

$$\|\dot{\mathbf{u}} - \dot{\mathbf{v}}_{best}\| = \min_{\dot{\mathbf{v}} \in \mathbf{Z}^{(D)}(\mathbf{v})} \|\dot{\mathbf{u}} - \dot{\mathbf{v}}\|.$$

Using the standard inclusion map $\mathbf{i} : \mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D) \to \mathbf{V}_{D_{\|\cdot\|_D}}$ the following result can be shown.

**Corollary 4.38** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of a tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$ , in the topological tree-based format and assume that* (4.10) *holds. Then $\mathcal{FT}_{\leq \mathfrak{r}}(\mathbf{V}_D)$ is an immersed submanifold of $\mathbf{V}_{D_{\|\cdot\|_D}}$.*

# 5 On the Dirac–Frenkel variational principle on tensor Banach spaces

## 5.1 Model Reduction in tensor Banach spaces

In this section we consider the abstract ordinary differential equation in a reflexive tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}}$, given by

$$\dot{\mathbf{u}}(t) = \mathbf{F}(t, \mathbf{u}(t)), \text{ for } t \geq 0 \tag{5.1}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \tag{5.2}$$

where we assume $\mathbf{u}_0 \neq \mathbf{0}$ and $\mathbf{F} : [0,\infty) \times \mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbf{V}_{D_{\|\cdot\|_D}}$ satisfying the usual conditions to have existence and unicity of solutions. Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha \in T_D \setminus \{D\}}$ be a representation of $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j \in D} V_j$ in the topological tree-based format and assume that (4.10) holds. As usual we will consider $\mathbf{V}_D = {}_a\bigotimes_{j\in D} V_j$ . We want to approximate $\mathbf{u}(t)$, for $t \in I := (0,\varepsilon)$ for some $\varepsilon > 0$, by a differentiable curve $t \mapsto \mathbf{v}_r(t)$ from $I$ to $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$, where $\mathfrak{r} \in \mathbb{N}^{T_D}$ is such that $\mathbf{v}_r(0) = \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$.

Our main goal is to construct a Reduced Order Model of (5.1)–(5.2) over the Banach manifold $\mathcal{FT}_{\mathfrak{r}}(\mathbf{V}_D)$. Since $\mathbf{F}(t,\mathbf{v}_r(t)) \in \mathbf{V}_{D_{\|\cdot\|_D}}$, for each $t \in I$, and $\mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ is a closed linear subspace in $\mathbf{V}_{D_{\|\cdot\|_D}}$, we have the existence of a $\dot{\mathbf{v}}_r(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ such that

$$\|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D.$$

It is well known that, if $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a Hilbert space, then $\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$, where

$$\mathcal{P}_{\mathbf{v}_r(t)} = \mathcal{P}_{\mathbf{Z}^{(D)}(\mathbf{v}_r(t))\oplus\mathbf{Z}^{(D)}(\mathbf{v}_r(t))^{\perp}}$$

is called the *metric projection.* It has the following important property: $\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$ if and only if

$$\langle \dot{\mathbf{v}}_r(t) - \mathbf{F}(t,\mathbf{v}_r(t)), \dot{\mathbf{v}}(t)\rangle_D = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

The concept of a metric projection can be extended to the Banach space setting. To this end we recall the following definitions. A Banach space $X$ with norm $\|\cdot\|$ is said to be *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|x_n+y_n\|/2 = 1$. It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space $X$ is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in U := \{z \in X : \|z\| = 1\}$. Finally, a Banach space $X$ is said to be *uniformly smooth* if its modulus of smoothness

$$\rho(\tau) = \sup_{x,y\in U}\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1\right\},\ \tau > 0,$$

satisfies the condition $\lim_{\tau\to 0}\rho(\tau) = 0$. In uniformly smooth spaces, and only in such spaces, the norm is uniformly Fréchet differentiable. A uniformly smooth Banach space is smooth. The converse is true if the Banach space is finite dimensional. It is known that the space $L^p$ $(1 < p < \infty)$ is a uniformly convex and uniformly smooth Banach space.

Let $\langle\cdot,\cdot\rangle : X \times X^* \longrightarrow \mathbb{R}$ denote the duality pairing, i.e.,

$$\langle x, f\rangle := f(x).$$

The normalised duality mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) := \{f \in X^* : \langle x, f\rangle = \|x\|^2 = (\|f\|^*)^2\},$$

and it has the following properties (see [2]):

(a) If $X$ is smooth, the map $J$ is single-valued;

(b) if $X$ is smooth, then $J$ is norm–to–weak$^*$ continuous;

(c) if $X$ is uniformly smooth, then $J$ is uniformly norm–to–norm continuous on each bounded subset of $X$.

**Remark 5.1** *Notice that, in a Hilbert space and after identifying $X$ with $X^*$, it can be shown (see Proposition 4.8(i) in [7]) that the normalised duality mapping is the identity operator.*

Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of a reflexive and strictly convex tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ , in the topological tree-based format and assume that (4.10) holds. For $\mathbf{F}(t,\mathbf{v}_r(t))$ in $\mathbf{V}_{D_{\|\cdot\|_D}}$, with a fixed $t \in I$, it is known that the set

$$\left\{ \dot{\mathbf{v}}_r(t) : \|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D \right\}$$

is always a singleton. Let $\mathcal{P}_{\mathbf{v}_r(t)}$ be the mapping from $\mathbf{V}_{D_{\|\cdot\|_D}}$ onto $\mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ defined by $\dot{\mathbf{v}}_r(t) := \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$ if and only if

$$\|\dot{\mathbf{v}}_r(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \|\dot{\mathbf{v}}(t) - \mathbf{F}(t,\mathbf{v}_r(t))\|_D.$$

It is also called *the metric projection.* The classical characterisation of the metric projection allows us to state the next result.

**Theorem 5.2** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of reflexive and strictly convex tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ in the topological tree-based format and assume that (4.10) holds. Then for each $t \in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$$

*if and only if*

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t,\mathbf{v}_r(t)) - \dot{\mathbf{v}}_r(t))\rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

An alternative approach is the use of the so-called *generalised projection operator* (see also [2]). To formulate this, we will use the following framework. Let $T_D$ a given tree and assume that for each $\alpha \in T_D$ we have a Banach space $\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}$, such that (4.10) holds and where $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a reflexive, strictly convex and smooth tensor Banach space. Following [21], we can define a function $\phi : \mathbf{V}_{D_{\|\cdot\|_D}} \times \mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbb{R}$ by

$$\phi(\mathbf{u},\mathbf{v}) = \|\mathbf{u}\|_D^2 - 2\langle \mathbf{u}, J(\mathbf{v})\rangle + \|\mathbf{v}\|_D^2,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing and $J$ is the normalised duality mapping. It is known that the set

$$\left\{ \dot{\mathbf{v}}_r(t) : \phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t,\mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t,\mathbf{v}_r(t))) \right\}$$

is always a singleton. It allows us to define a map $\Pi_{\mathbf{v}_r(t)} : \mathbf{V}_{D_{\|\cdot\|_D}} \longrightarrow \mathbf{Z}^{(D)}(\mathbf{v}_r(t))$ by $\dot{\mathbf{v}}_r(t) := \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$ if and only if

$$\phi(\dot{\mathbf{v}}_r(t), \mathbf{F}(t,\mathbf{v}_r(t))) = \min_{\dot{\mathbf{v}}(t)\in\mathbf{Z}^{(D)}(\mathbf{v}_r(t))} \phi(\dot{\mathbf{v}}(t), \mathbf{F}(t,\mathbf{v}_r(t))).$$

The map $\Pi_{\mathbf{v}_r(t)}$ is called *the generalised projection.* It coincides with the metric projection when $\mathbf{V}_{D_{\|\cdot\|_D}}$ is a Hilbert space.

**Remark 5.3** *We point out that, in general, the operators $\mathcal{P}_{\mathbf{v}_r(t)}$ and $\Pi_{\mathbf{v}_r(t)}$ are nonlinear in Banach (not Hilbert) spaces.*

Again, a classical characterisation of the generalised projection gives us the following theorem.

**Theorem 5.4** *Let $\{\mathbf{V}_{\alpha_{\|\cdot\|_\alpha}}\}_{\alpha\in T_D\setminus\{D\}}$ be a representation of reflexive, strictly convex and smooth tensor Banach space $\mathbf{V}_{D_{\|\cdot\|_D}} = {}_{\|\cdot\|_D}\bigotimes_{j\in D} V_j$ in the topological tree-based format and assume that (4.10) holds. Then for each $t \in I$ we have*

$$\dot{\mathbf{v}}_r(t) = \Pi_{\mathbf{v}_r(t)}(\mathbf{F}(t,\mathbf{v}_r(t)))$$

*if and only if*

$$\langle \dot{\mathbf{v}}_r(t) - \dot{\mathbf{v}}(t), J(\mathbf{F}(t,\mathbf{v}_r(t))) - J(\dot{\mathbf{v}}_r(t))\rangle \geq 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathbf{Z}^{(D)}(\mathbf{v}_r(t)).$$

## 5.2 The time–dependent Hartree method

Let $\langle\cdot,\cdot\rangle_j$ be a scalar product defined on $V_j$ ($1 \le j \le d$), i.e., $V_j$ is a pre-Hilbert space. Then $\mathbf{V} = {}_a\bigotimes_{j=1}^d V_j$ is again a pre-Hilbert space with a scalar product which is defined for elementary tensors $\mathbf{v} = \bigotimes_{j=1}^d v^{(j)}$ and $\mathbf{w} = \bigotimes_{j=1}^d w^{(j)}$ by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \left\langle \bigotimes_{j=1}^d v^{(j)}, \bigotimes_{j=1}^d w^{(j)} \right\rangle := \prod_{j=1}^d \left\langle v^{(j)}, w^{(j)}\right\rangle_j \qquad \text{for all } v^{(j)}, w^{(j)} \in V_j. \tag{5.3}$$

This bilinear form has a unique extension $\langle\cdot,\cdot\rangle : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$. One verifies that $\langle\cdot,\cdot\rangle$ is a scalar product, called the *induced scalar product.* Let $\mathbf{V}$ be equipped with the norm $\|\cdot\|$ corresponding to the induced scalar product $\langle\cdot,\cdot\rangle$. As usual, the Hilbert tensor space $\mathbf{V}_{\|\cdot\|} = {}_{\|\cdot\|}\bigotimes_{j=1}^d V_j$ is the completion of $\mathbf{V}$ with respect to $\|\cdot\|$. Since the norm $\|\cdot\|$ is derived via (5.3), it is easy to see that $\|\cdot\|$ is a reasonable and even uniform crossnorm.

Let us consider in $\mathbf{V}_{\|\cdot\|}$ a flow generated by a densely defined operator $A \in L(\mathbf{V}_{\|\cdot\|}, \mathbf{V}_{\|\cdot\|})$. More precisely, there exists a collection of bijective maps $\boldsymbol{\varphi}_t : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$, here $\mathcal{D}(A)$ denotes the domain of $A$, satisfying

(i) $\boldsymbol{\varphi}_0 = \mathbf{id}$,

(ii) $\boldsymbol{\varphi}_{t+s} = \boldsymbol{\varphi}_t \circ \boldsymbol{\varphi}_s$, and

(iii) for $\mathbf{u}_0 \in \mathcal{D}(A)$, the map $t \mapsto \boldsymbol{\varphi}_t$ is differentiable as a curve in $\mathbf{V}_{\|\cdot\|}$, and $\mathbf{u}(t) := \boldsymbol{\varphi}_t(\mathbf{u}_0)$ satisfies

$$\dot{\mathbf{u}} = A\mathbf{u},$$
$$\mathbf{u}(0) = \mathbf{u}_0.$$

In this framework we want to study the approximation of a solution $\mathbf{u}(t) = \boldsymbol{\varphi}_t(\mathbf{u}_0) \in \mathbf{V}_{\|\cdot\|}$ by a curve $\mathbf{v}_r(t) := \lambda(t) \otimes_{j=1}^d v_j(t)$ in the Hilbert manifold $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$, also called in [25] the *Hartree manifold.* The time–dependent Hartree method consists in the use of the Dirac–Frenkel variational principle on the Hartree manifold. More precisely, we want to solve the following Reduced Order Model:

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I,$$
$$\mathbf{v}_r(0) = \mathbf{v}_0,$$

with $\mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)} \in \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ being an approximation of $\mathbf{u}_0$[6]. By using the characterisation of the metric projection in a Hilbert space, for each $t > 0$ we would like to find $\dot{\mathbf{v}}_r(t) \in \mathrm{T}_{\mathbf{v}_r(t)}\mathbf{i}\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right)$ such that

$$\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\mathbf{v}}(t)\rangle = 0 \text{ for all } \dot{\mathbf{v}}(t) \in \mathrm{T}_{\mathbf{v}_r(t)}\mathbf{i}\left(\mathbb{T}_{\mathbf{v}_r(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))\right), \tag{5.4}$$

$$\mathbf{v}_r(0) = \mathbf{v}_0 = \lambda_0 \otimes_{j=1}^d v_0^{(j)},$$

and where, without loss of generality, we may assume $\|v_0^{(j)}\|_j = 1$ for $1 \le j \le d$. A first result is the following Lemma.

**Lemma 5.5** *Let $\mathbf{v} \in \mathcal{C}^1(I, \mathcal{U}(\mathbf{v}_0))$, where $\mathbf{v}(0) = \mathbf{v}_0 \in \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ and $(\mathcal{U}(\mathbf{v}_0), \Theta_{\mathbf{v}_0})$ is a local chart for $\mathbf{v}_0$ in $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$. Assume that $\mathbf{v}$ is also a $\mathcal{C}^1$-morphism between the manifolds $I \subset \mathbb{R}$ and $\mathcal{U}(\mathbf{v}_0) \subset \mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ such that $\mathbf{v}(t) = \lambda(t)\bigotimes_{j=1}^d v_j(t)$ for some $\lambda \in \mathcal{C}^1(I,\mathbb{R})$ and $v_j \in \mathcal{C}^1(I, V_j)$ for $1 \le j \le d$. Then*

$$\dot{\mathbf{v}}(t) = \dot{\lambda}(t)\bigotimes_{j=1}^d v_j(t) + \lambda(t)\sum_{j=1}^d \dot{v}_j(t) \otimes \bigotimes_{k\neq j} v_k(t) = \mathrm{T}_{\mathbf{v}(t)}\mathbf{i}(\mathrm{T}_t\mathbf{v}(1)). \tag{5.5}$$

*Moreover, if $v_j(t) \in \mathbb{S}_{V_j}$, i.e., $\|v_j(t)\|_j = 1$, for $t \in I$ and $1 \le j \le d$, then $\dot{v}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $t \in I$ and $1 \le j \le d$.*

[6]Indeed, $\mathbf{v}_0$ can be chosen as the best approximation of $\mathbf{u}_0$ in $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})$ because $\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}) = \mathcal{T}_{(1,\ldots,1)}(\mathbf{V}) \setminus \{\mathbf{0}\}$.

*Proof.* First at all, we recall that by the construction of $\mathcal{U}(\mathbf{v}_0)$ it follows that $W_j^{\min}(\mathbf{v}_0) = W_j^{\min}(\mathbf{v}(t))$ and that $U_j^{\min}(\mathbf{v}_0) = \mathrm{span}\{v_0^{(j)}\}$ is linearly isomorphic to $U_j^{\min}(\mathbf{v}(t))$ for all $t \in I$ and $1 \le j \le d$. Assume $\Theta_{\mathbf{v}_0}(\mathbf{v}(t)) = (\lambda(t), L_1(t), \ldots, L_d(t))$, i.e.,

$$\mathbf{v}(t) := \lambda(t) \bigotimes_{j=1}^{d} (id_j + L_j(t))\,(v_0^{(j)}),$$

where $\lambda \in \mathcal{C}^1(I, \mathbb{R} \setminus \{0\})$, $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ and $(id_j + L_j(t))(v_0^{(j)}) \in U_j^{\min}(\mathbf{v}(t))$ for $1 \le j \le d$. We point out that the linear map $\mathrm{T}_t\mathbf{v} : \mathbb{R} \to \mathbb{T}_{\mathbf{v}(t)}(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V}))$ is characterised by

$$\mathrm{T}_t\mathbf{v}(1) = (\Theta_{\mathbf{v}_0} \circ \mathbf{v})'(t) = (\dot{\lambda}(t), \dot{L}_1(t), \ldots, \dot{L}_d(t)). \tag{5.6}$$

Since $L_j \in \mathcal{C}^1(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ then $\dot{L}_j \in \mathcal{C}^0(I, \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$. Observe that $U_j^{\min}(\mathbf{v}_0)$ and $U_j^{\min}(\mathbf{v}(t))$ have $W_j^{\min}(\mathbf{v}_0)$ as a common complement. From Lemma 3.6 we know that

$$P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} : U_j^{\min}(\mathbf{v}(t)) \longrightarrow U_j^{\min}(\mathbf{v}_0)$$

is a linear isomorphism. We can write

$$L_j(t) = L_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)} \text{ and } \dot{L}_j(t) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)},$$

and then in (5.6) we identify $\dot{L}_j(t) \in \mathcal{L}(U_j^{\min}(\mathbf{v}_0), W_j^{\min}(\mathbf{v}_0)))$ with

$$\dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))} \in \mathcal{L}(U_j^{\min}(\mathbf{v}(t)), W_j^{\min}(\mathbf{v}_0))).$$

Introduce $v_j(t) := (id_j + L_j(t))(v_0^{(j)})$ for $1 \le j \le d$. Then

$$\dot{L}_j(t)(v_j(t)) = \dot{L}_j(t) P_{U_j^{\min}(\mathbf{v}_0) \oplus W_j^{\min}(\mathbf{v}_0)}|_{U_j^{\min}(\mathbf{v}(t))}(v_0^{(j)} + L_j(t)(v_0^{(j)})) = \dot{L}_j(t)(v_0^{(j)})$$

holds for all $t \in I$ and $1 \le j \le d$. Hence

$$\dot{v}_j(t) = \dot{L}_j(t)(v_0^{(j)}) = \dot{L}_j(t)(v_j(t)) \tag{5.7}$$

holds for all $t \in I$ and $1 \le j \le d$. From Lemma 4.27(b) and (5.6), we have

$$\mathrm{T}_{\mathbf{v}(t)}\mathfrak{i}(\mathrm{T}_t\mathbf{v}(1)) = \dot{\lambda}(t) \bigotimes_{j=1}^{d} v_j(t) + \lambda(t) \sum_{j=1}^{d} \dot{L}_j(t)(v_j(t)) \otimes \bigotimes_{k \neq j} v_k(t),$$

and, by using (5.7) for $\mathbf{v}(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$, we obtain (5.5).

To prove the second statement, recall that $U_j^{\min}(\mathbf{v}(t)) = \mathrm{span}\,\{v_j(t)\}$ and $V_j = U_j^{\min}(\mathbf{v}(t)) \oplus W_j^{\min}(\mathbf{v}_0)$ for $1 \le j \le d$. Then we consider

$$W_j^{\min}(\mathbf{v}_0) = \mathrm{span}\,\{v_j(t)\}^{\perp} = \{u_j \in V_j : \langle u_j, v_j(t)\rangle_j = 0\} \text{ for } 1 \le j \le d,$$

and hence $\langle \dot{v}_j(t)), v_j(t)\rangle_j = 0$ holds for $1 \le j \le d$. From Remark 3.20, we have $(\dot{v}_1(t), \ldots, \dot{v}_d(t)) \in \mathcal{C}(I, \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}))$, because $W_j^{\min}(\mathbf{v}_0) = \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$ for $1 \le j \le d$. ■

Before stating the next result, we introduce for $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ the following time dependent bilinear forms

$$\mathrm{a}_k(t; \cdot, \cdot) : V_k \times V_k \longrightarrow \mathbb{R},$$

by

$$\mathrm{a}_k(t; z_k, y_k) := \left\langle A\left(z_k \otimes \bigotimes_{j \neq k} v_j(t)\right), \left(y_k \otimes \bigotimes_{j \neq k} v_j(t)\right)\right\rangle$$

for each $1 \le k \le d$. Now, we will show the next result (compare with Theorem 3.1 in [25]).

**Theorem 5.6 (Time dependent Hartree method)** *The solution* $\mathbf{v}_r(t) = \lambda(t) \bigotimes_{j=1}^d v_j(t)$ *for* $(v_1(t), \ldots, v_d(t)) \in \times_{j=1}^d \mathbb{S}_{V_j}$ *of*

$$\dot{\mathbf{v}}_r(t) = \mathcal{P}_{\mathbf{v}_r(t)}(A\mathbf{v}_r(t)) \text{ for } t \in I,$$
$$\mathbf{v}_r(0) = \mathbf{v}_0,$$

*satisfies*

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0 \quad \text{for all } \dot{w}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j}), \quad 1 \le j \le d,$$

*and*

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \left\langle A\left(\otimes_{j=1}^d v_j(s)\right), \otimes_{j=1}^d v_j(s)\right\rangle ds\right).$$

*Proof.* From Lemma 5.5 we have $\mathbb{T}_{\mathbf{v}_r(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right) = \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, Thus, for each $\dot{\mathbf{w}}(t) \in \mathrm{T}_{\mathbf{v}(t)} i\left(\mathbb{T}_{\mathbf{v}(t)}\left(\mathcal{M}_{(1,\ldots,1)}(\mathbf{V})\right)\right)$ there exists $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, such that

$$\dot{\mathbf{w}}(t) = \dot{\beta}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t).$$

Then (5.4) holds if and only if

$$\left\langle \dot{\mathbf{v}}_r(t) - A\mathbf{v}_r(t), \dot{\beta}(t) \bigotimes_{j=1}^d v_j(t) + \lambda(t) \sum_{j=1}^d \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t) \right\rangle = 0$$

for all $(\dot{\beta}(t), \dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \mathbb{R} \times \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. Then

$$\dot{\lambda}(t)\dot{\beta}(t) + \lambda(t)^2 \sum_{j=1}^d \left( \langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^d v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t)\rangle \right)$$
$$-\lambda(t)\dot{\beta}(t)\langle A \bigotimes_{j=1}^d v_j(t), \bigotimes_{j=1}^d v_j(t)\rangle = 0,$$

i.e.,

$$\begin{aligned} &\dot{\beta}(t)\left(\dot{\lambda}(t) - \lambda(t)\langle A \textstyle\bigotimes_{j=1}^d v_j(t), \bigotimes_{j=1}^d v_j(t)\rangle\right) \\ &+\lambda(t)^2 \textstyle\sum_{j=1}^d \left(\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^d v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t)\rangle\right) = 0 \end{aligned} \tag{5.8}$$

holds for all $\dot{\beta}(t) \in \mathbb{R}$ and $(\dot{w}_1(t), \ldots, \dot{w}_d(t)) \in \times_{j=1}^d \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$. If $\lambda(t)$ solves the differential equation

$$\dot{\lambda}(t) = \left\langle A\left(\otimes_{j=1}^d v_j(t)\right), \otimes_{j=1}^d v_j(t)\right\rangle \lambda(t)$$
$$\lambda(0) = \lambda_0,$$

i.e.,

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \left\langle A\left(\otimes_{j=1}^d v_j(s)\right), \otimes_{j=1}^d v_j(s)\right\rangle ds\right),$$

then the first term of (5.8) is equal to 0. Therefore, from (5.8) we obtain that for all $j \in D$,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \langle A \bigotimes_{s=1}^d v_s(t), \dot{w}_j(t) \otimes \bigotimes_{k\neq j} v_k(t)\rangle = 0,$$

that is,

$$\langle \dot{v}_j(t), \dot{w}_j(t)\rangle_j - \mathrm{a}_j(t; v_j(t), \dot{w}_j(t)) = 0$$

holds for all $\dot{w}_j(t) \in \mathbb{T}_{v_j(t)}(\mathbb{S}_{V_j})$, and the theorem follows. ∎

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