THE SET OF PERIODS FOR A CLASS OF SKEW–PRODUCTS

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ABSTRACT. In this paper we give a characterization for the set of periods for a class of skew–products that we can see as deterministic systems driven by some stochastic process. This class coincides with a set of skew product maps from $\Sigma_N \times \mathbb{S}^1$ into itself, where Σ_N is the space of the bi–infinite sequences on N symbols and $\mathbb{S}^1$ is the unit circle.

1. INTRODUCTION

In 1970's Afraimovich and Shilnikov [1] described the semi–hyperbolic invariant set generated by a bifurcation of several homoclinic surfaces of a saddle–node cycle. The invariant set in the last bifurcation is homeomorphic to the product space $\Sigma_N \times \mathbb{S}^1$, where $\Sigma_N = \{0, 1, \ldots, N-1\}^{\mathbb{Z}}$ is the space of all bi–infinite sequences

$$\underline{a} = (\ldots, a_{-n}, \ldots, a_{-1} \cdot a_0, a_1, \ldots, a_n, \ldots) \tag{1.1}$$

of symbols $0, 1, \ldots, N-1$ (we note that in this paper we shall use the same notation as in [9]). The dynamics on the invariant set above, after some rescaling, give rise to a *skew product* as follows. Let $\underline{a} = (\ldots, a_{-1}.a_0, a_1 \ldots) \in \Sigma_N$ then $\sigma : \Sigma_N \to \Sigma_N$, *the shift map*, is given by

$$\sigma(\underline{a}) = (\ldots, a_{-1}, a_0 \cdot a_1, a_2, \ldots). \tag{1.2}$$

Let $\mathrm{Hom}(\mathbb{S}^1)$ be the set of homeomorphisms on $\mathbb{S}^1$ and let $\mathbb{G} \subset \mathrm{Hom}(\mathbb{S}^1)$ be an abelian group. For $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}) \in \mathbb{G}^N$ define the map

$$\Phi_{\mathbf{f}} : \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \times \mathbb{S}^1 \tag{1.3}$$

given by

$$\Phi_{\mathbf{f}}(\underline{a}, x) = (\sigma(\underline{a}), f_{a_0}(x)). \tag{1.4}$$

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$\Phi_{\mathbf{f}}$ is called *the skew product associated to* σ *and* $\mathbf{f}$ (see [8]) or, following the notation of [4], the *crazy map associated to* $\mathbf{f}$. Note that the n–th iterate of this map at the point $(\underline{a}, x) \in \Sigma_N \times \mathbb{S}^1$ is given by

$$\Phi_{\mathbf{f}}^n(\underline{a}, x) = (\sigma^n(\underline{a}), (f_{a_{n-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})(x)). \tag{1.5}$$

On the other hand, the map $\mathbf{f} : \Sigma_N \times \mathbb{S}^1 \to \mathbb{S}^1$ defined by $\mathbf{f}(\underline{a}, x) = f_{a_0}(x)$, is also called a *Random Dynamical System or cocycle* (see [3]). We can consider these maps as particular examples of random dynamical systems. Moreover, this class of maps provides, by using an analogue of Takens Theorem for randomly forced systems (see [8]), a new framework for the analysis of time series arising from nonlinear stochastic systems. To see this, consider $\varrho : \mathbb{S}^1 \to \mathbb{R}$ a measurement function which measures some observable property of the system. The evolution of this quantity with discrete time is then given by

$$\left\{\varrho\left(x\right), \varrho\left(f_{a_0}(x)\right), \ldots, \varrho\left((f_{a_{n-1}} \circ \cdots \circ f_{a_1} \circ f_{a_0})(x)\right), \ldots\right\},$$

where x is considered as the initial condition and $\underline{a}$ represents some of the all the possible states of the world described by Σ_N. Then, the set of periodic points of the skew product $\Phi_{\mathbf{f}}$ provides stationary time series which does not depend on of the measurement function. Thus, the set of periods for a map $\Phi_{\mathbf{f}}$ gives, in some sense, a lower bound to the cyclical behavior and it does not depend on the chosen measurement function. We point out that a characterization of the set of periods also provides a first approach to understand the dynamic behavior of the mapping class under consideration.

A first work in order to characterize the set of periods of skew product maps was done by Kloeden in [7]. He considered a skew product given by

$$F\left(x, y\right) = \left(f\left(x\right), g\left(x, y\right)\right), \tag{1.6}$$

where $f : I \to I$ and $g : I \times I \to I$ are continuous mappings and $I = [0, 1]$. In this setting, he proved that Ŝarkovskii's Theorem holds for this class of maps. Later, motivated by a previous work of Illiashenko [6], the set of periods for maps $\Phi_{\mathbf{f}}$ with $\mathbf{f} = (x + \alpha_0 \,(\mathrm{mod}\, 1), x + \alpha_1 \,(\mathrm{mod}\, 1))$ and $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ was characterized in [4]. The main goal of this paper is to give a characterization of the set of periods of $\Phi_{\mathbf{f}}$ when each component of $\mathbf{f}$ belongs to a some closed commutative subgroup $\mathbb{G} \subset \mathrm{Hom}(\mathbb{S}^1)$.

The paper is organized as follows. In the next section we give some basic definitions and state the main result of this paper which gives the characterization of the set of periods for random dynamical systems given by N commutative homeomorphisms. In Section 3, the results stated in Section 2 will be proved.

2. Definitions and statement of results

Let $\mathbb{G}$ be a commutative subgroup of homeomorphisms of $\mathbb{S}^1$, that is, $f \circ g = g \circ f$ for all $f, g \in \mathbb{G}$. For any homemorphism f notice that $\mathbb{G}(f) := \{g \in \mathrm{Hom}(\mathbb{S}^1) : g \circ f = f \circ g\}$ contains at least three elements. Moreover, let $\mathrm{Diff}^r(\mathbb{S}^1)$ be the set of $\mathcal{C}^r$–diffeomorphisms of $\mathbb{S}^1$. If $f \in \mathrm{Diff}^r(\mathbb{S}^1)$, then $\mathbb{G}(f)$ contains the closure of $\left\{f^k : k \in \mathbb{Z}\right\}$ in the $\mathcal{C}^r$–topology (see [5]).

Recall that a point $(\underline{a}, x)$ is said *periodic* if there is a positive integer n such that $\Phi_{\mathbf{f}}^n(\underline{a}, x) = (\underline{a}, x)$. The smallest positive integer holding this condition is called the *period* of $(\underline{a}, x)$. Periodic points of period one are called *fixed points*. Denote by $\mathbb{P}(\Phi_{\mathbf{f}})$ and $\mathbb{F}(\Phi_{\mathbf{f}})$ the sets of periodic and fixed points of $\Phi_{\mathbf{f}}$, respectively. By $\mathrm{Per}(\Phi_{\mathbf{f}})$ we will denote the set of periods of $\Phi_{\mathbf{f}}$.

Let $\mathbf{f} \in \mathbb{G}_1^N$ and $\mathbf{g} \in \mathbb{G}_2^N$, where $\mathbb{G}_1$ and $\mathbb{G}_2$ are two commutative groups of $\mathrm{Hom}(\mathbb{S}^1)$. We will say that the maps $\Phi_{\mathbf{f}}$ and $\Phi_{\mathbf{g}}$ are *topologically conjugated* if there exists an homeomorphism $h : \Sigma_N \times \mathbb{S}^1 \longrightarrow \Sigma_N \times \mathbb{S}^1$ such that $h \circ \Phi_{\mathbf{f}} = \Phi_{\mathbf{g}} \circ h$. Note that if $\Phi_{\mathbf{f}}$ and $\Phi_{\mathbf{g}}$ are topologically conjugate then $\mathrm{Per}(\Phi_{\mathbf{f}}) = \mathrm{Per}(\Phi_{\mathbf{g}})$. A stronger definition than the topological conjugacy condition and weaker than the definition of bundle conjugacy, introduced by Stark in [8], is the following. We will say that $\Phi_{\mathbf{f}}$ and $\Phi_{\mathbf{g}}$ are *weakly bundle conjugate* if $\Phi_{\mathbf{f}}$ and $\Phi_{\mathbf{g}}$ are topologically conjugated by a homeomorphism $h : \Sigma_N \times \mathbb{S}^1 \longrightarrow \Sigma_N \times \mathbb{S}^1$, where $h(\underline{a}, x) = (g(\underline{a}), h_2(\underline{a}, x))$. We remark that, under this condition, g must be a homeomorphism from Σ_N into itself satisfying $g \circ \sigma = \sigma \circ g$. Moreover, if $g = \mathrm{id}$ then we say that $\Phi_{\mathbf{f}}$ and $\Phi_{\mathbf{g}}$ are *bundle conjugate* (see [8]).

Let $\mathcal{S}_N$ denote the group of permutations of N elements, that is, the set of all bijective maps τ from $\{0, 1, \ldots, N-1\}$ into itself. For $\tau \in \mathcal{S}_N$ we define $\tau^* : \mathbb{G}^N \to \mathbb{G}^N$ by

$$\tau^*\left((f_0, f_1, \ldots, f_{N-1})\right) = \left(f_{\tau(0)}, f_{\tau(1)}, \ldots, f_{\tau(N-1)}\right). \tag{2.1}$$

The first main result that we will prove in Section 3 is the following.

Theorem 1. *Let $N \in \mathbb{N}$, $N \geq 2$, and $\mathbf{f} \in \mathbb{G}^N$. Then, for each $\tau \in \mathcal{S}_N$ the maps $\Phi_{\mathbf{f}}$ and $\Phi_{\tau^*(\mathbf{f})}$ are weakly bundle conjugated. In particular, $\Phi_{\mathbf{f}}$ and $\Phi_{\tau^*(\mathbf{f})}$ are conjugated.*

The second main result is concerned with the set of periods of $\Phi_{\mathbf{f}}$ for some $\mathbf{f} \in \mathbb{G}^N$. Before stating the result, we need some additional definitions and notation from [2]. Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous circle map and consider the natural projection $e : \mathbb{R} \to \mathbb{S}^1$ given by $e(x) = \exp(2\pi i x)$. A continuous map $F : \mathbb{R} \to \mathbb{R}$ such that $e \circ F = f \circ e$ is called a *lifting* of f. Let

$$\mathcal{L} = \{F : \mathbb{R} \to \mathbb{R} : F \text{ is a lifting of } f\}.$$

If $F_1, F_2 \in \mathcal{L}$, then $F_1 = F_2 + k$ for some $k \in \mathbb{Z}$. Notice that for all $F \in \mathcal{L}$,

$$F(1) - F(0) = \deg(f) \in \mathbb{Z} \tag{2.2}$$

and it is called the *degree* of f. If $g : \mathbb{S}^1 \to \mathbb{S}^1$ is another continuous circle map, then

$$\deg(f \circ g) = \deg(f) \cdot \deg(g) \tag{2.3}$$

and

$$\deg(f^n) = \deg(f)^n \tag{2.4}$$

for all $n \in \mathbb{N}$.

Notice that if $f \in \mathrm{Hom}(\mathbb{S}^1)$, then F is always strictly increasing or strictly decreasing. Moreover, since $f \circ f^{-1} = \mathrm{id}$, by (2.3) it holds that

$$1 = \deg(\mathrm{id}) = \deg(f \circ f^{-1}) = \deg(f) \cdot \deg(f^{-1}). \tag{2.5}$$

Then, either $\deg(f) = \deg(f^{-1}) = 1$ or $\deg(f) = \deg(f^{-1}) = -1$. In the case of $\deg(f) = 1$, a useful tool to describe the set of periods of f is the *rotation number* of their liftings. For all $F \in \mathcal{L}$ let

$$\rho(F) := \lim_{n\to\infty} \frac{F^n(x) - x}{n} \tag{2.6}$$

for some $x \in \mathbb{S}^1$. It can be shown (see [2]) that (2.6) does not depend on the choice of $x \in \mathbb{S}^1$ and $\rho(F) - \rho(F') \in \mathbb{Z}$ for all $F', F \in \mathcal{L}$. Roughly speaking, $\rho(F)$ is the average of angular speed of any point moving around the circle under iteration of the map. Then, we have the following result (see [2]).

Proposition 1. *Let $f \in \mathrm{Hom}(\mathbb{S}^1)$. If $\mathbb{F}(f)$ denotes the set of fixed points of f the following statements hold.*

(a) *Let $\deg(f) = -1$. Then $\mathbb{F}(f) \neq \emptyset$.*
(b) *Let $\deg(f) = 1$. Then $\mathbb{F}(f) \neq \emptyset$ if and only if $\rho(F) \in \mathbb{Z}$ for all $F \in \mathcal{L}$.*

Now, we can state our second main result. From now one, we will denote by F_i the lifting of f_i for $i = 0, 1, \ldots, n-1$.

Theorem 2. *Let $N \in \mathbb{N}$, $N \geq 2$, and $\mathbf{f} = (f_0, f_1, ..., f_{N-1}) \in \mathbb{G}^N$. Then*

(a) *If there are* $i, j \in \{0, 1, ..., N-1\}$, $i \neq j$, *such that* $\deg(f_j) \cdot \deg(f_i) = -1$, *then* $\mathrm{Per}(\Phi_{\mathbf{f}}) = \mathbb{N}$.

(b) *If* $\deg(f_i) = -1$ *for all* $i \in \{0, 1, ..., N-1\}$, *then* $\mathbb{N} \setminus \{2\} \subseteq \mathrm{Per}(\Phi_{\mathbf{f}})$. *Moreover,* $2 \in \mathrm{Per}(\Phi_{\mathbf{f}})$ *if and only if either* $2 \in \mathrm{Per}(f_i)$ *for some* $i \in \{0, 1, ..., N-1\}$ *or* $\rho(F_i \circ F_j) \in \mathbb{Z}$ *for some* $i, j \in \{0, 1, ..., N-1\}$, $i \neq j$.

(c) *If* $\deg(f_i) = 1$ *for all* $i \in \{0, 1, ..., N-1\}$, *then* $n \in \mathrm{Per}(\Phi_{\mathbf{f}})$ *if and only if either* $n \in \mathrm{Per}(f_i)$ *for some* $i \in \{0, 1, ..., N-1\}$ *or* $n \notin \mathrm{Per}(f_i)$ *for all* $i \in \{0, 1, ..., N-1\}$ *and there exists* $(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \in \mathbb{N}^N$, *with* $\mathrm{Card}\{i : \lambda_i \neq 0\} \geq 2$, *such that* $\sum_{i=0}^{N-1} \lambda_i = n$ *and* $\sum_{i=0}^{N-1} \lambda_i \rho(F_i) \in \mathbb{Z}$.

This theorem gives a characterization of the set of periods for skew products given by N commuting homeomorphisms. The rest of the paper is devoted to the proofs of Theorems 1 and 2.

3. Proofs of results

We start this section by proving Theorem 1.

Proof of theorem 1. Let $\tau \in \mathcal{S}_N$ and

$$\tau^*(\mathbf{f}) = \left(f_{\tau(0)}, f_{\tau(1)}, \ldots, f_{\tau(N-1)}\right).$$

Let $g_\tau : \Sigma_N \to \Sigma_N$ be defined by

$$g_\tau(\underline{a}) = (..., \tau^{-1}(a_{-n}), ..., \tau^{-1}(a_{-1}) \cdot \tau^{-1}(a_0), \tau^{-1}(a_1), ..., \tau^{-1}(a_n), ...)$$

for all $\underline{a} = (..., a_{-n}, ..., a_{-1} \cdot a_0, a_1, ..., a_n, ...) \in \Sigma_N$. Clearly, g_τ is a homeomophism and $\sigma \circ g_\tau = g_\tau \circ \sigma$. Let $h : \Sigma_N \times \mathbb{R} \to \Sigma_N \times \mathbb{R}$ be defined by $h(\underline{a}, x) = (g_\tau(\underline{a}), x)$. Obviously, h is also a homeomorphism and

$$\begin{aligned} h(\Phi_{\mathbf{f}}((\underline{a}, x))) &= h(\sigma(\underline{a}), f_{a_0}(x)) \\ &= (g_\tau(\sigma(\underline{a})), f_{a_0}(x)) \\ &= (\sigma(g_\tau(\underline{a})), f_{\tau(\tau^{-1}(a_0))}(x)) \\ &= \Phi_{\tau^*(\mathbf{f})}(g_\tau(\underline{a}), x) \\ &= \Phi_{\tau^*(\mathbf{f})}(h(\underline{a}, x)). \end{aligned}$$

Thus, $\Phi_{\mathbf{f}}$ and $\Phi_{\tau^*(\mathbf{f})}$ are weakly bundle conjugated and the proof concludes.

In order to prove Theorem 2 we need some preliminary work. We will start by recalling some known properties of shift maps (see [9]).

Let $\underline{a} \in \Sigma_N$ be a periodic of period n of σ. Then, it is not difficult to see that $\underline{a}$ is periodic of period n, that is, $a_{i+kn} = a_i$ for all $i \in$

$\{0,1,...,n-1\}$ and all $k \in \mathbb{N}$. Then we write periodic sequence by putting the block of length n starting by a_0, that is

$$\underline{a} = (\overline{a_0, a_1, a_2, \ldots, a_{n-1}}). \tag{3.1}$$

We observe that, for a given $\mathbf{f} \in \mathbb{G}_r^N$, the n–th iterate of $\Phi_{\mathbf{f}}$ at the point $(\underline{a}, x) \in \Sigma_N \times \mathbb{S}^1$ is given by

$$\Phi_{\mathbf{f}}^n(\underline{a}, x) = \left(\sigma^n(\underline{a}), f_0^{\lambda_0^n(\underline{a})} \circ f_1^{\lambda_1^n(\underline{a})} \circ \cdots \circ f_{N-1}^{\lambda_{N-1}^n(\underline{a})}(x)\right) \tag{3.2}$$

where

$$\lambda_i^n(\underline{a}) = \sum_{j=0}^{n-1} \delta_{i,a_j} \tag{3.3}$$

(recall that $\delta_{i,a_j} = 1$ if $a_j = i$ and $\delta_{i,a_j} = 0$ otherwise) for $i = 0, 1, \ldots, N-1$. Note that $\lambda_i^n(\underline{a})$ gives the number of i's in the finite set $\{a_0, \ldots, a_{n-1}\}$.

Lemma 1. *For each $n \in \mathbb{N}$, the following statements hold.*

(a) *There exists $\underline{a} \in \Sigma_N$ periodic of period n.*

(b) *Let $k_0, k_1, \ldots, k_{N-1} \in \mathbb{N} \cup \{0\}$. If $\sum_{i=0}^{N-1} k_i = n > 0$, then there exists a periodic sequence of period n, $\underline{a} \in \Sigma_N$, such that $\lambda_i^n(\underline{a}) = k_i$ for $l = 0, 1, \ldots, N-1$.*

Proof. Statement (a) follows if we take, for each $n \in \mathbb{N}$, the periodic sequence

$$\underline{a} = \left(\overline{\underbrace{0, ..., 0, 1}_{n}}\right).$$

To prove (b) it is sufficient to take the periodic sequence

$$\underline{a} = \left(\overline{\underbrace{0, ..., 0}_{k_0}, \underbrace{1, ..., 1}_{k_1}, ..., \underbrace{N-1, ..., N-1}_{k_{N-1}}}\right). \quad \blacksquare$$

Lemma 2. *Let $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}) \in \mathbb{G}_r^N$, then* $\mathrm{Per}(f_i) \subseteq \mathrm{Per}(\Phi_{\mathbf{f}})$ *for all $i \in \{0, 1, \ldots, N-1\}$.*

Proof. If $\mathrm{Per}(f_i) = \emptyset$, then there is nothing to prove. So, assume that $n \in \mathrm{Per}(f_i) \neq \emptyset$. Clearly, $\left(\left(\bar{i}\right), x\right)$ is a periodic point of period n of $\Phi_{\mathbf{f}}$. and the proof concludes. $\blacksquare$

Lemma 3. *Assume that $(\underline{a}, x)$ is a periodic point of $\Phi_{\mathbf{f}}$ of period n. If there exists $i \in \{0, 1, \ldots N-1\}$ such that $\lambda_i^n(\underline{a}) = n$, then $n \in \mathrm{Per}(f_i)$.*

Proof. If $\lambda_i^n(\underline{a}) = n$, $\lambda_j^n(\underline{a}) = 0$ for all $j \in \{0, 1, \ldots N-1\}$, $j \neq i$, then it is clear that $\underline{a} = (\bar{i})$. Since $\Phi_{\mathbf{f}}^j((i), x) = ((\bar{i}), f_i^j(x)) \neq ((\bar{i}), x)$, we have that $f_i^j(x) \neq x$ for $0 < j < n$ and $f_i^n(x) = x$. Therefore $n \in \operatorname{Per}(f_i)$. ∎

The following lemma follows from [5, Proposition 2.11], and will be used without citation in the proof of Theorem 2.

Lemma 4. *Let $f, g \in \mathbb{G}$ and let F and G be the liftings of f and g, respectively. Then*

$$\rho(F \circ G) = \rho(F) + \rho(G).$$

Now, we are ready to prove Theorem 2.

Proof of theorem 2 Let $n \in \mathbb{N}$. (a) Assume that $\deg(f_i) = 1$ and $\deg(f_j) = -1$ for some $i, j \in \{0, 1, ..., N-1\}$. By (2.3) and (2.4)

$$\deg(f_i^{n-1} \circ f_j) = \deg(f_i)^n \cdot \deg(f_j) = -1.$$

By Proposition 1(a), there exists $x_0 \in \mathbb{F}\left(f_i^{n-1} \circ f_j\right)$. Then, it is easy to see that

$$\left(\left(\underbrace{\overline{i, i, ..., i, j}}_{n}\right), x_0\right) \in \mathbb{P}\left(\Phi_{\mathbf{f}}\right)$$

and $n \in \operatorname{Per}\left(\Phi_{\mathbf{f}}\right)$. Thus, (a) is proved.

(b) We will distinguish three cases: (b1) n is odd, (b2) $n = 4m$, for some $m \in \mathbb{N}$, and (b3) $n = 4m + 2$ for some $m \in \mathbb{N}$.

(b1) Assume that n is odd. By using (2.3) and (2.4) we obtain that

$$\deg(f_0^{n-1} \circ f_1) = \deg(f_0)^n \deg(f_1) = -1.$$

Then, from Proposition 1(a), there exists $x_0 \in \mathbb{F}\left(f_0^{n-1} \circ f_1\right)$, and hence,

$$\left(\left(\underbrace{\overline{0, 0, ..., 0, 1}}_{n}\right), x_0\right) \in \mathbb{P}\left(\Phi_{\mathbf{f}}\right).$$

So, $n \in \operatorname{Per}(\Phi_{\mathbf{f}})$.

(b2) Now, assume that $n = 4m$, for some $m \in \mathbb{N}$. Notice that from (2.4), we have that $\deg(f_0^2) = \deg(f_1^2) = 1$. Since f_0 and f_1 have degree -1, by Proposition 1(a), both maps have at least a fixed point. Then f_0^2 and f_1^2 have also a fixed point. Again, Proposition 1(b) implies that $\rho(F_0^2)$ and $\rho(F_1^2)$ are integer numbers. By using Lemma 4, we have that

$$\rho(F_0^{n/2} \circ F_1^{n/2}) = \rho(F_0^{2m} \circ F_1^{2m}) = m\rho(F_0^2) + m\rho(F_1^2) \in \mathbb{Z}.$$

Finally, by Proposition 1(b), there exists $x_0 \in \mathbb{F}\left(f_0^{2m} \circ f_1^{2m}\right)$. Therefore,

$$\left(\left(\overline{\underbrace{0,0,...,0}_{\frac{n}{2}},\underbrace{1,1,...,1}_{\frac{n}{2}}}\right),x_0\right) \in \mathbb{P}\left(\Phi_{\mathbf{f}}\right)$$

and $n \in \operatorname{Per}(\Phi_{\mathbf{f}})$.

(b3) Finally, let $n = 4m + 2$, for some $m \in \mathbb{N}$. Following the steps of the proof of (b2) it follows that

$$\rho(F_0^{n-2} \circ F_1^2) = \rho(F_0^{4m} \circ F_1^2) = 2m\rho(F_0^2) + \rho(F_1^2) \in \mathbb{Z}.$$

From Proposition 1(b), there exists $x_0 \in \mathbb{F}\left(f_0^{4m} \circ f_1^2\right)$ and, hence,

$$\left(\left(\overline{\underbrace{0,0,...,0}_{n-2},1,1}\right),x_0\right) \in \mathbb{P}\left(\Phi_{\mathbf{f}}\right)$$

and $n \in \operatorname{Per}(\Phi_{\mathbf{f}})$.

To end the proof of statement (b), if $2 \in \operatorname{Per}(f_i)$ for some $i \in \{0,1,...,N-1\}$, then by applying Lemma 2, $2 \in \operatorname{Per}(\Phi_{\mathbf{f}})$. If $\rho(F_i \circ F_j) \in \mathbb{Z}$, then by Proposition 1, there is an x_0 such that $(f_i \circ f_j)(x_0) = x_0$. We note that $\left(\left(\overline{0,1}\right),x_0\right)$ is a periodic point of $\Phi_{\mathbf{f}}$ of period 2, and therefore $2 \in \operatorname{Per}(\Phi_{\mathbf{f}})$. Conversely, if $2 \in \operatorname{Per}(\Phi_{\mathbf{f}})$, then there is a periodic sequence $\underline{a} \in \Sigma^N$ of period 1 or 2, and an x_0 such that $\Phi_{\mathbf{f}}^2(\underline{a},x_0) = (\underline{a},x_0)$. If $\underline{a}$ has period one, then by Lemma 3, $2 \in \operatorname{Per}(f_i)$ for some $i \in \{0,1,...,N\_1\}$. If $\underline{a}$ has period two, then there are $i,j \in \{0,1,...,N-1\}$, $i \neq j$, such that $(f_i \circ f_j)(x_0) = x_0$. From Proposition 1(b), $\rho(F_i \circ F_j) \in \mathbb{Z}$ and this concludes the proof of (b).

(c) Assume first that $n \in \operatorname{Per}(\Phi_{\mathbf{f}})$. If $n \in \operatorname{Per}(f_i)$ for some $i \in \{0,1,...,N-1\}$, then the proof concludes. So, assume that $n \notin \operatorname{Per}(f_i)$ for all $i \in \{0,1,...,N-1\}$. Let $(\underline{a},x_0)$ be such that

$$\Phi_{\mathbf{f}}^n(\underline{a},x_0) = (\underline{a},x_0).$$

Then $\underline{a}$ is a periodic sequence of period k such that k divides n. Let

$$\underline{a} = (\overline{a_0,a_1,...a_{k-1}}) \in \Sigma_N.$$

Notice that, by Lemma 3, there are at least two distinct symbols in $\underline{a}$. It is easy to see that x_0 is a periodic point of

$$f_{a_{k-1}} \circ ... \circ f_{a_0} = f_0^{\lambda_0^k(\underline{a})} \circ ... \circ f_{N-1}^{\lambda_{N-1}^k(\underline{a})}.$$

of period n/k. Then

$$x_0 \in \mathbb{F}\left(f_0^{\lambda_0^k(\underline{a})\cdot n/k} \circ ... \circ f_{N-1}^{\lambda_{N-1}^k(\underline{a})\cdot n/k}\right).$$

From Proposition 1(b), we have that

$$\rho\left(F_0^{\lambda_0^k(\underline{a})\cdot n/k}\circ\ldots\circ F_{N-1}^{\lambda_{N-1}^k(\underline{a})\cdot n/k}\right)=\sum_{i=0}^{N-1}\frac{\lambda_i^k(\underline{a})\cdot n}{k}\rho(F_i)\in\mathbb{Z}.$$

On the other hand, it is obvious that

$$\sum_{i=0}^{N-1}\frac{\lambda_i^k(\underline{a})\cdot n}{k}=n,$$

and so, taking

$$\frac{\lambda_i^k(\underline{a})\cdot n}{k}=\lambda_i$$

the necessary condition holds.

Conversely, let $n\in\mathbb{N}$ such that either $n\in\operatorname{Per}(f_i)$ for some $i\in\{0,1,...,N-1\}$ or $n\notin\operatorname{Per}(f_i)$ for all $i\in\{0,1,...,N-1\}$ and there exists $(\lambda_0,\lambda_1,\ldots,\lambda_{N-1})\in\mathbb{N}^N$, with $\operatorname{Card}\{i:\lambda_i\neq 0\}\geq 2$ such that

$$\sum_{i=0}^{N-1}\lambda_i=n$$

and

$$\sum_{i=0}^{N-1}\lambda_i\rho(F_i)\in\mathbb{Z}.$$

If $n\in\operatorname{Per}(f_i)$ for some $i\in\{0,1,...,N-1\}$, then by Lemma 2, $n\in\operatorname{Per}(\Phi_{\mathbf{f}})$. Assume now that $n\notin\operatorname{Per}(f_i)$ for all $i\in\{0,1,...,N-1\}$ and the above conditions hold. By (2.3) and (2.4)

$$\rho(F_0^{\lambda_0}\circ\ldots\circ F_{N-1}^{\lambda_{N-1}})=\sum_{i=0}^{N-1}\lambda_i\rho(F_i)\in\mathbb{Z},$$

Then, from Proposition 1(b), there is an $x_0\in\mathbb{S}^1$ such that $(f_0^{\lambda_0}\circ\ldots\circ f_{N-1}^{\lambda_{N-1}})(x_0)=x_0$. Hence,

$$\left(\left(\overline{\underbrace{0,...,0}_{\lambda_0},\underbrace{1,...,1}_{\lambda_1},...,\underbrace{N-1,...,N-1}_{\lambda_{N-1}}}\right),x_0\right)\in\mathbb{P}\left(\Phi_{\mathbf{f}}\right)$$

and therefore $n\in\operatorname{Per}(\Phi_{\mathbf{f}})$. This concludes the proof.

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