Bifurcations and Symbolic Dynamics for bimodal degree one circle maps: The Arnol'd Tongues and the Devil's Staircase

Antonio Falcó
Departament de Matemàtiques,
Facultat de Ciències,
Universitat Autnoma de Barcelona

Preface

The pourpose of the present memory is study the bifurcations and the symbolic dynamics of bimodal degree one circle maps and some related topics. The memory is organiced as follows.

In Chapter 1 we complete the work of Levi [32] in order to explain the transition, in a forced relaxation oscillator of van der Pol type, from the non-chaotic behaviour to the chaotic one. In Chapter 2 we give a characterization of the set of kneading sequences for bimodal degree one circle maps. In Chapter 3 we construct two self–similarity operators in order to study the bifurcations of continuous parametrized families of bimodal degree one circle maps. Lastly, in Chapter 4 we give a formula to compute the topological entropy of a sub–class of bimodal degree one circle maps.

The Chapter 1 was published in:

Alsedá L., Falcó A., *Devil's Staircase Route to Chaos in a Forced Relaxation Oscillator*, Ann. Inst. Fourier, **44**, 1, 109–128, 1994.

Alsedá L., Falcó A., *The bifurcations of a piecewise monotone family of circle maps related to the van der Pol equation*, Proceedings of European Conference on Iteration Theory, Caldes de Malavella, World Scientific, 1987.

The Chapter 4 was published in:

Alsedá L., Falcó A., *An entropy formula for a class of circle maps*, C.R. Acad Sci. Paris, **t. 314**, 667–682, 1992.

Contents

1 A one-dimensional approach **3**

1.1 Introduction . . . 3

1.2 The Levi's model and statement of the main result. . . . 6

1.3 Levi's results . . . 9

1.4 Proof of Theorem 1.2.2(a)-(b) . . . 14

1.5 The rotation interval . . . 19

1.6 Twist orbits . . . 21

1.7 Proof of Theorem 1.2.2(c) . . . 23

1.8 The piecewise-monotone family of circle maps related to the Van der Pol equation 27

1.9 Concluding Remarks . . . 36

2 The kneading pair **37**

2.1 Introduction . . . 37

2.2 A survey on the Kneading Theory for circle maps . . . 39

2.3 The characterization of the kneading pair . . . 42

2.3.1 Proof of the first statement of Theorem 2.3.2 . . . 45

2.3.2 Proof of the second statement of Theorem 2.3.2 . . . 51

2.4 Concluding remarks . . . 58

3 Self-similarity operators **62**

3.1 Introduction . . . 62

3.2 The $\star-$product . . . 66

3.2.1 Introduction . . . 66

3.2.2 A survey on the kneading theory for unimodal maps . . . 66

3.2.3 Definition of the $\star-$products and statement of the main result 68

3.2.4 Preliminary results 70

3.2.5 Proof of Theorem 3.2.1 74

3.2.6 Concluding remarks 77

3.3 The $\odot-$product 78

3.3.1 Introduction and preliminary results 78

3.3.2 Definitions and preliminary results 80

3.3.3 Proof of the Theorem 3.3.3 82

3.4 Bifurcation structure in the Arnol'd tongues 87

3.4.1 Proof of Proposition 3.4.1 90

3.4.2 Proof of Theorems 3.4.1 and 3.4.3 92

3.4.3 Concluding remarks 97

3.5 Appendix 97

4 Topological entropy 98

4.1 Introduction 98

4.2 The topological entropy 99

4.3 Kneading Theory and Topological entropy for maps in $\mathcal{A}$ 101

4.4 The Topological entropy formula for maps in $\mathcal{M}$ 104

4.5 Proof of Theorem 4.4.3 108

Chapter 1

A one- dimensional approach to a forced relaxation oscillator

1.1 Introduction

In this chapter we describe the behavior of certain sets of solutions of an oscillator of the Van der Pol type with sinusoidal forcing term. The original problem was proposed by Van der Pol [33] in the study of an electrical circuit with a triode valve. Later on, Van der Pol and Van der Mark [34] studied the forced relaxation oscillator in a circuit as the one in Figure 1.1.1. They analyzed the frequency of the circuit as a function of the capacitance C. While increasing C from its initial value they observed that the electrical system takes a period being a multiple of the forcing period and that, for certain parameter values, two different subharmonics may coexist. Furthermore, there are regions where no subharmonics are detected. Plotting the frequency of the circuit against the capacitance they obtained a staircase structure as shown in Figure 1.1.2.

Recently, Kennedy, Krieg and Chua [22] working with a modern version of the Van der Pol and Van der Mark's circuit observed the appearance of secondary staircases. These staircases present a well-known geometric structure called "the Devil's staircase" (which, roughly speaking, can be defined as the graph of a non-decreasing continuous map with the property that the preimage of any rational number is a closed interval and the preimage of any irrational number is a point). These secondary staircases give the route from the non-chaotic behavior to the chaotic one in the electrical circuit.

The first mathematical investigation on this model was made by Cartwright and Littlewood

Figure 1.1.1: The circuit studied by Van der Pol and Van der Mark.

Figure 1.1.2: The original staircase.

[8]. They studied the solutions of the following non-linear differential equation

$$\frac{d^2x}{dt} + \nu(x^2-1)\frac{dx}{dt} + x = \nu b(\nu)k\cos kt, \tag{1.1.1}$$

where $\nu >> 1$ and discovered a family of solutions with chaotic behavior. Later on Levinson [32] proposed the following version of (1.1.1)

$$\epsilon\ddot{x} + \Psi_0(x)\dot{x} + \epsilon x = bp_0(t), \tag{1.1.2}$$

where $\Psi_0 = \text{sgn}(x^2-1)$, $p_0(t) = \text{sgn}(\sin(\frac{2\pi t}{T}))$, $\epsilon > 0$ is a small parameter and b varies in some finite interval $[b_1, b_2]$. In this new model the solutions could be analyzed explicitly by piecing together solutions at different linearity intervals.

Afterwards, Levi [30] modified the Levinson's model by replacing the functions $\Psi_0(x)$ and $p_0(t)$ by two differentiable $\mathcal{C}^0-$ close functions. Namely, $\Psi(x)$ negative for $|x| < 1$ and positive for $|x| < 1$ and $p(t)$ periodic of period T. In a very complicated process Levi reduced the study of the qualitative behavior of the solutions of this model to the study of the dynamics of a dissipative diffeomorphism in a region of $\mathbf{R}^2$ that, after identifying the upper boundary with the lower one, can be considered as a dissipative diffeomorphism of an annulus into itself. Moreover this diffeomorphism can be approximated (in some sense) by a circle map.

By using these techniques Levi showed that, for ϵ small enough, the interval $[b_1, b_2]$ can be decomposed into union of alternating closed, proper disjoint intervals A_k and B_k separated by thin gaps g_k and $\hat{g}_k$ as follows:

$$[b_1, b_2] = A_1 \cup g_1 \cup B_1 \cup \hat{g}_1 \cup A_2 \cup g_2 \cup, \ldots, \cup \hat{g}_{n-1} \cup A_n \cup g_n \cup B_n.$$

When b belongs to one of the intervals A_k a periodic solution of period $(2q-1)T$ appears, where T is the period of the forcing term $p(t)$ and $q = q(k) > 0$ is an integer number. As b increases it crosses one of the small gaps g_k to fall down in one of the intervals B_k. Then, the above periodic solution is preserved and another one of period $(2q+1)T$ is created. Moreover, it is shown that in the intervals B_k the system exhibits chaotic motion. Afterwards, the parameter b crosses another small gap of type $\hat{g}_k$ to arrive to an interval A_{k+1} where only the periodic solution of period $(2q+1)T$ remains and the chaotic motion disappears. Thus, as b moves trough the intervals A_k, g_k, B_k, $\hat{g}_k$ and A_{k+1} one observes a hysteresis phenomenon (frequency demultiplication). However, Levi did not study in detail the evolution of the system as b crosses the intervals g_k and $\hat{g}_k$ but he predicted the existence of orbits of very high period.

The purpose of the present chapter is to analyze the bifurcations occurring when the parameter b crosses the gaps of type g_k and $\hat{g}_k$ in the Levi's model of the forced relaxation oscillator. Before stating the main result of this chapter we have to introduce some notation and explain Levi's results with more detail.

1.2 The Levi's model and statement of the main result.

Levi's model can be conveniently rewritten as

$$\dot{x} = \frac{1}{\epsilon}(y - \Phi(x)), \quad \dot{y} = \epsilon x + bp(t), \tag{1.2.3}$$

where $y = \epsilon\dot{x} + \Phi(x)$ is the modified velocity and $\Phi(x) = \int_0^x \Psi(u)du$.

We shall denote by P_b be the Poincaré map associated to (1.2.3), defined as $P_b(z) = Z(T, 0, z)$, where $Z(t, t_0, z)$ denotes the solution of the system at time t which starts at z at time t_0. For ϵ small enough and for all $b \in [b_1, b_2]$, the map P_b has the following geometrical properties:

(1) It has exactly two fixed points. One at infinity, and z_0 which is close to the branch of $y = \Phi(x)$ with negative slope.

(2) There exists an annular region $\mathcal{R}$ surrounding z_0 with thickness less than $\sqrt{\epsilon}$ such that any point $z \neq z_0$ enters in $\mathcal{R}$ after sufficiently many iterations of P_b and stays there. In particular $\mathcal{R}$ is P_b-invariant.

(3) The points of $\mathcal{R}$ "circulate clockwise" with respect the point z_0 under the iterates of P_b.

Let $\Pi : \mathbf{R} \times [0, 1] \longrightarrow \mathcal{R}$ be the natural projection. That is, $\Pi|_{[0,2\pi)\times[0,1]} : [0, 2\pi) \times [0, 1] \longrightarrow \mathcal{R}$ is a homeomorphism, Π is periodic of period 2π with respect to the first component and $\Pi(x, y)$ moves "clockwise" as x increases. Moreover, Π can be taken in such a way that if $\Pi(x, y) = z$ with $x \in [0, 2\pi)$ and $y \in [0, 1]$ then x is the "clockwise" angle of the vector $z - z_0$ with respect the horizontal line passing through z_0. In what follows, we shall fix a lifting $\widetilde{P}_b : \mathbf{R} \times [0, 1] \longrightarrow \mathbf{R} \times [0, 1]$ of the map $P_b|_{\mathcal{R}}$. That is, $\widetilde{P}_b$ is a diffeomorphism such that $P_b \circ \Pi = \Pi \circ \widetilde{P}_b$. Let

$\pi_1 : \mathbf{R} \times [0,1] \longrightarrow \mathbf{R}$ denote the projection map with respect to the first component. Take $z \in \mathcal{R}$ and $\widetilde{z} \in \Pi^{-1}(z)$. Then, the real number

$$\rho_{\widetilde{P}_b}(z) = \lim_{i \to \infty} \frac{\pi_1(\widetilde{P}_b^i(\widetilde{z})) - \pi_1(\widetilde{z})}{i}$$

will be called the *rotation number of* z *with respect to* $\widetilde{P}_b$ if it exists. We note that this limit is the average angle by which the point z rotates under iteration of the map P_b with respect to the fixed point z_0 (see (3) above). Let $\Omega \subset \mathcal{R}$ be a P_b-invariant set. The *rotation set of* Ω *with respect to* $\widetilde{P}_b$ is defined to be the set of all rotation numbers of all points from Ω with respect to $\widetilde{P}_b$.

The following theorem summarizes Levi's results on the system (1.2.3) (see [30]).

Theorem 1.2.1 *The interval* $[b_1, b_2]$ *can be decomposed into union of alternating closed, proper disjoint intervals* A_k *and* B_k *separated by gaps* g_k *and* $\hat{g}_k$ *as follows:*

$$[b_1, b_2] = A_1 \cup g_1 \cup B_1 \cup \hat{g}_1 \cup A_2 \cup g_2 \cup, \ldots, \cup \hat{g}_{n-1} \cup A_n \cup g_n \cup B_n.$$

Moreover,

(a) For b *in* A_k *we have:*

(a.1) P_b *has one pair of periodic points of period* $2q-1$ *where* $q = q(\epsilon, k) \sim 1/\epsilon$ *remains constant through the interval* A_k*, and* $q(\epsilon, k+1) = q(\epsilon, k) - 1$*, (i.e. the period of the these points decreases as* b *increases).*

(a.2) One of the two points is a sink and the other a saddle. Moreover, any point which lies off the stable manifold of the saddle (except for the unstable fixed point of P_b*) tends to the sink under forward iterations.*

(a.3) The rotation set of $\mathcal{R}$ *is* $\{2\pi/(2q-1)\}$.

(b) For b *in* B_k*, we have:*

(b.1) The minimal attractor set of P_b *is the union of a hyperbolic Cantor set and two pairs of periodic points, one of these pairs has period* $2q+1$ *and the other one has period* $2q-1$*. Each of these pairs consists on a sink and a saddle. Moreover, the two saddles belong to the Cantor set.*

(b.2) The rotation set of $\mathcal{R}$ is $[2\pi/(2q+1), 2\pi/(2q-1)]$.

(c) There exists b^* in g_k (respectively in $\hat{g}_k$) such that P_{b^*} has a nondegenerate homoclinic tangency. Moreover, there exists a small $\xi > 0$ and an open subset B_ξ in $B_\xi^* = [b^*, b^* + \xi)$ (respectively $B_\xi^* = (b^* - \xi, b^*]$)such that for $b \in B_\xi^* \setminus B_\xi$, P_b is structurally stable. The set $B_\xi^* \setminus B_\xi$ consists on infinitely many components, to which there correspond infinitely many different (structurally stable) types of P_b.

In order to complete statement (c) of Levi's Theorem we study how the Cantor set appearing in the statement (b) and its rotation set associated are formed when b crosses a bifurcation gap g_k or $\hat{g}_k$. This is achieved in the next theorem. We will only state the theorem in the case of the interval g_k. The situation for an interval $\hat{g}_k$ is symmetric. In the rest of the chapter we will use freely the notation introduced above and, in particular, the one from Theorem 1.2.1

Theorem 1.2.2 *For each $b \in g_k$ the map P_b has one pair of periodic points of period $2q-1$; a sink and a saddle. Moreover, there exist a countable sequence $\{b_n^k\}_{n=0}^\infty \subset g_k$ satisfying the following properties:*

(a) For each b_n^k the minimal attractor set of $P_{b_n^k}$ contains an invariant hyperbolic Cantor set, denoted by C_n^k, to which the saddle point belongs.

(b) For $b \geq b_n^k$, the minimal attractor set of P_b contains an invariant hyperbolic Cantor set, denoted by $C_b^{n,k}$, which contains the saddle point of P_b, such that $P_b|_{C_b^{n,k}}$ is topologically conjugate to $P_{b_n^k}|_{C_n^k}$. Moreover, if $b_s^k < b_n^k$, then $C_{b_n^k}^{s,k} \subset C_n^k$.

(c) For each b_n^k there exists a rational number $\alpha_n^k \in [-1, 1]$ such that for $b \geq b_n^k$ the P_b-rotation set of $C_b^{n,k}$ is the closed interval $[2\pi/(2q+\alpha_n^k), 2\pi/(2q-1)]$. Moreover $\{\alpha_n^k\}_{n=0}^\infty = (-1, 1] \cap \mathbf{Q}$.

In view of the above two theorems, the bifurcations of P_b when the parameter b crosses g_k from A_k to B_k can be explained in the following way. When b is close to A_k the dynamics of the map P_b is the same as when b lies in A_k (see Theorem 1.2.1(a)). This is the situation until b reaches the parameter value b^* from Theorem 1.2.1(c). At this point the map P_b has a non-degenerated homoclinic tangency and, in consequence, there exists a wild hyperbolic set by the well-know result of Newhouse [26]. Therefore, all parameter values b_n^k considered in

Theorem 1.2.2 must be larger than or equals to b^* and accumulate to $b^{**} \geq b^*$. Then, for $b \geq b^{**}$, the minimal attractor set of P_b contains an invariant hyperbolic Cantor set which is enlarged each time that b crosses one of the parameter values from the sequence $\{b_n^k\}_{n=0}^{\infty}$ (see Theorem 1.2.2(a)–(b)). As it will be shown later, the dynamics of the system on each of these Cantor sets can be deduced from a subshift of finite type with a certain transition matrix which can be computed explicitly by using one dimensional techniques (see Corollary 1.4.5 and Remark 1.4.6). Finally, when the parameter b is sufficiently close to B_k the dynamics of the map P_b is the same as when b lies on B_k. Moreover, P_b possesses an invariant set, strictly contained in the minimal attractor, with P_b-rotation interval $[2\pi/(2q+1), 2\pi/(2q-1)]$ (see Theorem 1.2.1(b) and [30]). The transition of the rotation interval of the system from the point $2\pi/(2q+1)$ into the interval $[2\pi/(2q+1), 2\pi/(2q-1)]$ is also described by the rotation intervals of P_b restricted to the Cantor sets $C_b^{n,k}$ (see Theorem 1.2.2(c)). The study of the Van der Pol system will be based on the study of the bifurcations of a two parameter families of degree one circle maps (see [4] and [**?**]). Due to the strongly one dimensional character of the Van der Pol system we can transfer the information on the bifurcations, from the one dimensional models to the two dimensional one.

This chapter is organized as follows. In the next section we shall summarize the Levi's results we are using. Then, in Section 1.4 we prove Theorem 1.2.2(a)-(b). To prove Theorem 1.2.2(c) we shall summarize preliminary results about rotation intervals and twist orbits of circle maps of degree one. This will be done in Sections 1.5 and 1.6. Afterwards, in Section 1.7 we prove Theorem 1.2.2(c). In Section 1.8 we study the bifurcations of a simpler (piecewise differentiable) version of Levi's circle maps family defined in Section 1.3. This model already captures the essential features of the Levi's one and has the advantage that the study of its bifurcations can be done in a more complete way than for the Levi's circle map family. In particular, for these maps we are able to characterize the appearance of Cantor sets when the parameter crosses the interval g_k. Finally, in Section 1.9 we give some cloncluding remarks.

1.3 Levi's results

In this section, for completeness, we give a more precise description of the map P_b. Levi takes a region $\mathcal{W}$ (see Figure 1.3.1) , which will be called "the window", bounded by the boundaries of $\mathcal{R}$, by a horizontal line l joining the boundaries of $\mathcal{R}$ and its image $P(l)$.

Figure 1.3.1: The window region $\mathcal{W}$ and its evolution.

The crucial property of $\mathcal{W}$ is that the iterates of any point $z \neq z_0$ pass thought $\mathcal{W}$, and do so repeatedly. It suffices, therefore, to trace the evolution of $\mathcal{W}$ under P_b-iterations. The description of this evolution plays a major role in the understanding of the dynamics of the system; it is depicted in Figure 1.3.1 where the positions of $\mathcal{W}$ at different times are shown.

To determine the qualitative behavior of the map P_b we must determine how the future iterates of the window $\mathcal{W}$ intersect $\mathcal{W}$. To that end, we consider the window map $N_b : \mathcal{W} \longrightarrow \mathcal{W}$ defined by $z \longrightarrow P_b^j(z)$ where $j = j(z) > 0$ is the smallest integer for which $P_b^j(z) \in \mathcal{W}$. The only piece of information we lose considering N_b instead of P_b is the integer-value function $j(z)$. So we will have to keep track of it. The advantage of looking at the window map N_b instead of P_b lies in its simplicity. This simplicity is further enhanced by the symmetry properties of the damping and forcing functions $\Phi(x)$ and $p(t)$, which imply that the window map N_b is the second iterate of the "antipodal half-period" return map $M_b : \mathcal{W} \longrightarrow \mathcal{W}$ defined as $M_b(z) = -Z(mT + \frac{T}{2}, 0, z)$, where $m = m(z)$ is the smallest integer for which $-Z(mT + \frac{T}{2}, 0, z) \in \mathcal{W}$. To see this we have the following lemma due to Levi (see [30] and Figure 1.3.2).

Figure 1.3.2: The evolution from the map M_b of a vertical line $l \subset \mathcal{W}$.

Lemma 1.3.1 *The map N_b is equal to $M_b \circ M_b$. Moreover, there exists $q > 0$ such that*

$$M_b(z) \in \{-Z((2q+1)\frac{T}{2}, 0, z), -Z((2q-1)\frac{T}{2}, 0, z)\}$$

for all $z \in \mathcal{W}$.

Now, we give some of the notions used by Levi to prove Theorem 1.2.1.

Let $A = \mathbf{S}^1 \times [0,1]$ be the standard annulus. The study of the map P_b can be reduced to the study of the annulus map

$$L_b = L(., b, \delta, \delta_1) : A \longrightarrow A,$$

depending on three parameters, namely, $b \in [b_1, b_2]$, $0 < \delta \leq \delta'$ and $0 < \delta_1 \leq \delta_1'$, which satisfy the following properties.

Let $\Pi_1 : A \longrightarrow \mathbf{S}^1$ denote the vertical projection on the first component. For each $\sigma \in [0,1]$ we denote by $f_{b,\sigma}(x)$ the circle map $f(x, b, \sigma, \delta, \delta_1) = \Pi_1 \circ L_b(x, \sigma)$ (see Figure 1.3.3). Then we have:

(L.1) $|f_{b,\sigma} - f_{b,\sigma'}| < \delta$ in the $\mathcal{C}^0$ norm in x, for all $\sigma,\ \sigma' \in [0,1]$.

Figure 1.3.3: The circle map $f_{b,\sigma}$.

(L.2) There exist $\gamma > 1$, $\vartheta > 32$, $C > 0$ and two intervals $\Delta \subset \Delta_1 \subset \mathbf{S}^1$ whose endpoints depend on b, δ and δ_1 (not on σ) such that $|\Delta_1| < \delta_1$ and for all $\sigma \in [0,1]$ it follows:

(i) $f'_{b,\sigma}(x) > \vartheta\gamma$ for all $x \in \Delta$.

(ii) $-1 + C < f'_{b,\sigma}(x) < -1/\gamma$ for all $x \in \mathbf{S}^1 \setminus \Delta_1$.

(L.3) The oscillation (in x) of $f_{b,\sigma}$ on each of the two components of $\Delta_1 \setminus \Delta$ is less than $\varrho(\delta, \delta_1)$, which is independent on b and $\lim_{\delta,\delta_1 \to 0} \varrho(\delta, \delta_1) = 0$.

(L.4) For some $\sigma \in [0,1]$ we have $-\frac{d}{db}(f(x_i(b), b, \sigma, \delta, \delta_1) - x'_i(b)) > \omega(\delta, \delta_1) > 0$ for $i = 1, 2$, where $x_1(b)$, $x_2(b)$, $x'_1(b)$ and $x'_2(b)$, are the endpoints of Δ and Δ_1 respectively (labelled in such a way that $\Delta = [x_1(b),\ x_2(b)]$ and $\Delta_1 = [x'_1(b),\ x'_2(b)]$), all differentiable in b and $\omega(\delta, \delta_1)$ is independent on b (see Figure 1.3.3).

(L.5) L_b has a inverse on $L_b(A)$.

(L.6) The map L_b^{-1} in $Q = \Delta \times [0,1]$ maps vertical strips into vertical strips.

The relation between L_b and P_b can be described as follows. There exists a homeomorphism h from A into $\mathcal{W}$ such that $L_b = h^{-1} \circ M_b \circ h$. Then Lemma 1.3.1 gives the relation between the map P_b of the phase plane of the system (1.2.3) into itself and the annulus map L_b. Moreover there exists a positive integer $m = m(\epsilon)$ such that, for each L_b-invariant set $\Omega \subset A$, we have that $\cup_{i=0}^{m} P_b^i(h(\Omega))$ also is P_b-invariant. Then, the $\widetilde{P}_b-$rotation number of a point $h(z) \in \mathcal{R}$

with $z \in A$ can be obtained from the L_b-rotation number of z as we shall show next (see Remark 1.3.3). In a similar way as we did for the map P_b we shall fix a lifting $\widetilde{L}_b$ of L_b to the covering space $\mathbf{R} \times [0,1]$. Then as usual, the $\widetilde{L}_b-$*rotation number* of a point $z \in A$ is defined to be the limit

$$\rho_{\widetilde{L}_b}(z) = \lim_{i \to \infty} \frac{\pi_1(\widetilde{L}_b^i(\widetilde{z})) - \pi_1(\widetilde{z})}{i}$$

if it exists, where $\widetilde{z}$ is a point in $\mathbf{R} \times [0,1]$ projecting to z by the standard projection map (e, id) with $e(x) = \exp(2\pi i x)$. Let $E(.)$ denotes the integer part function, then we note that this number can also be computed as $\lim_{i \to \infty}(\sum_{j=1}^{i} d_j^b)/i$, where $d_j^b = E(\pi_1(\widetilde{L}_b^j(\widetilde{z}))) - E(\pi_1(\widetilde{L}_b^{j-1}(\widetilde{z})))$.

In the sequel we denote $\max\{\delta', \delta_1'\}$ by $\overline{\delta}$. The following lemma is due to Levi [30].

Lemma 1.3.2 *There exists a lifting $\widetilde{L}_b$ of L_b such that, if $\overline{\delta}$ is small enough, then for all $\sigma \in [0,1]$ we have $1 + C < \pi_1(\widetilde{z}_1(\sigma)) - \pi_1(\widetilde{z}_2(\sigma)) < 2 - C$ where $\widetilde{z}_i(\sigma) = \widetilde{L}_b((\widetilde{x}_i(b), \sigma))$ for $i = 1,2$, $\widetilde{x}_i(b)$ is such that $e(\widetilde{x}_i(b)) = x_i(b)$ for $i = 1,2$ and $|\widetilde{x}_1(b) - \widetilde{x}_2(b)| < 1$. The constant $0 < C < 1$ is independent on b, δ and δ_1.*

In the sequel we shall assume that the lifting $\widetilde{L}_b$ of L_b we are working with is the one from the statement of Lemma 1.3.2.

Remark 1.3.3 From the above lemma it follows that each map $f_{b,\sigma}$ has degree one and that $d_j^b \in \{0,1\}$ for all $b \in [b_1, b_2]$ and $j \geq 1$. Now set $\tau(t) = 1 - 2t$ for $t \in \{0,1\}$. From Levi [30] it follows that if for some $z \in A$ the $\widetilde{L}_b-$rotation number exists and $\overline{\delta}$ is small enough, then $\rho_{\widetilde{P}_b}(h(z)) = \lim_{i \to \infty} 2\pi/[2q + (\sum_{j=1}^{i} \tau(d_j^b)/i)] = 2\pi/(2q + 1 - 2\rho_{\widetilde{L}_b}(z))$. □

Next we characterize the intervals A_k, B_k, g_k and $\hat{g}_k$ in terms of the circle maps $f_{b,\sigma}$. For $x, y \in \mathbf{S}^1$ we denote by $[x,y]$ (respectively (x,y), $[x,y)$ and $(x,y]$) the closed (respectively open, open from the right and open from the left) arc from x to y counterclockwise. Such an arc will be called a closed (respectively open, open from the right and open from the left) interval of $\mathbf{S}^1$. If A is a proper interval in $\mathbf{S}^1$ we also will use the notations $\inf A$ and $\sup A$ in the obvious way.

Let $\widetilde{\Delta}_1$ denote the open interval $(x_1'(b) - \varrho(\delta, \delta_1), x_2'(b) + \varrho(\delta, \delta_1))$. Then, one and only one of the following three cases occurs for $f_{b,\sigma}$ (see Figure 1.3.4):

Case A. The set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is an interval such that its endpoints map onto the endpoints of $\widetilde{\Delta}_1$ and its image is $\widetilde{\Delta}_1$.

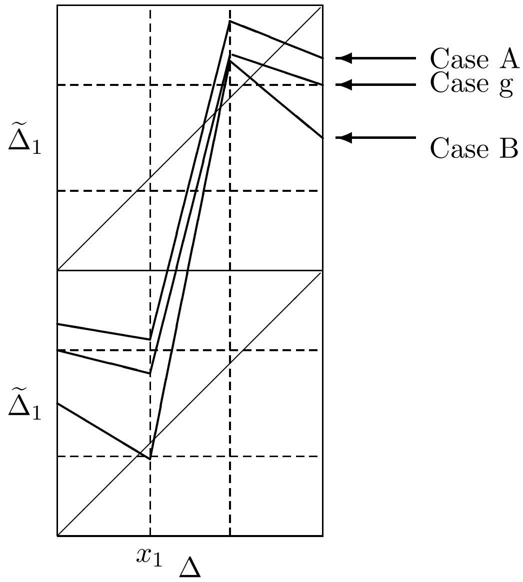

Figure 1.3.4: The three cases for $f_{b,\sigma}(\widetilde{\Delta}_1)$.

Case g. $f_{b,\sigma}(x_i) \in \widetilde{\Delta}_1$, for some $i \in \{1,2\}$ (i.e. the set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is a union of two disjoint intervals such that the endpoints of one of them map onto the endpoints of $\widetilde{\Delta}_1$ and the image of the other one is strictly contained in $\widetilde{\Delta}_1$).

Case B. The set $f_{b,\sigma}^{-1}(\widetilde{\Delta}_1) \cap \Delta$ is a union of two disjoint intervals such that the endpoints of both of them map onto the endpoints of $\widetilde{\Delta}_1$ and their images are $\widetilde{\Delta}_1$.

Let A, g and B be the sets of values of $b \in [b_1, b_2]$ for which the corresponding alternative holds. Then, since the endpoints of $f_{b,\sigma}(\Delta)$ move monotonically (clockwise) with respect to the endpoints of $\widetilde{\Delta}_1$ (see (L4)), the set A (respectively B and g) can be written as $\cup_{k\in I_A} A_k$ (respectively $\cup_{k\in I_B} B_k$ and $(\cup_{k\in I_g} g_k) \cup (\cup_{k\in I_{\hat{g}}} \hat{g}_k)$), where each of the sets A_k (respectively B_k, g_k and $\hat{g}_k$) is a connected component of A (respectively of B and g), in such a way that the intervals A_k, B_k, g_k and $\hat{g}_k$ alternate as stated in Theorem 1.2.1.

1.4 Proof of Theorem 1.2.2(a)-(b)

To prove Theorem 1.2.2(a)–(b) we shall employ the techniques used by Levi in the proof of Theorem 1.2.1(a)–(b) to translate the results concerning the circle maps family to the two

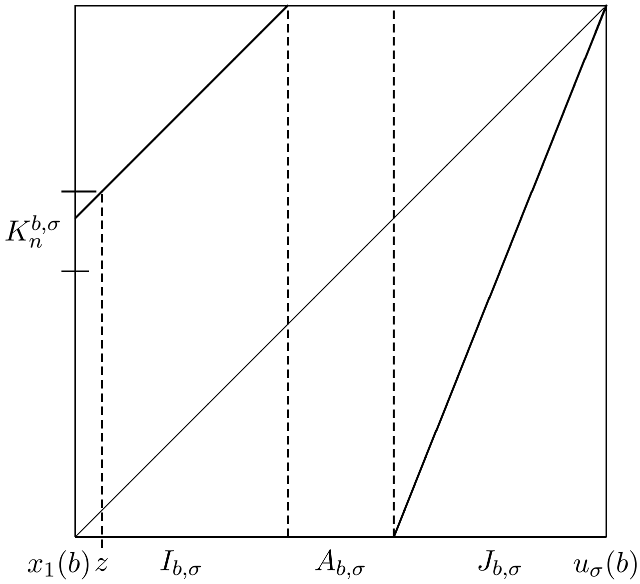

Figure 1.4.5: The map $f_{b,\sigma}$ in Δ_σ^L.

dimensional setting. Thus, we only will prove in detail the results on the family $f_{b,\sigma}$ which are necessary to prove Theorem 1.2.2(a)–(b).

We start by constructing the sequence of parameter values appearing in the statement of the theorem. First we have to fix some notation.

Note that for each $b \in A_k \cup g_k$ there exist $u_{b,\sigma} \in \mathrm{Int}(\Delta)$ depending continuously on b such that $u_{b,\sigma}$ is a unstable fixed point of $f_{b,\sigma}$ (see Case A, Case g and Figure 1.3.4). Then, for $\sigma \in [0,1]$, we define

$$\begin{aligned} \alpha_\sigma^k &= \sup\{b \in g_k : f_{b,\sigma}(x_1(b)) = u_{b,\sigma}\} \text{ and,} \\ \beta_\sigma^k &= \inf\{b \in g_k : \ f_{b,\sigma}(x_1(b)) = x_1'(b) - \varrho(\delta,\delta_1)\}. \end{aligned}$$

In view of (L4) we see that $\alpha_\sigma^k < \beta_\sigma^k$.

In the sequel we shall denote the closed interval $[x_1(b), u_{b,\sigma}] \subset \Delta$ by Δ_σ^L. We note that for $b \in (\alpha_\sigma^k, \beta_\sigma^k)$ we have that $f_{b,\sigma}^{-1}(\Delta_\sigma^L) \cap \Delta_\sigma^L$ is the union of two closed disjoint intervals $I_{b,\sigma}$ and $J_{b,\sigma}$ such that $x_1(b) \in I_{b,\sigma}$, $u_{b,\sigma} \in J_{b,\sigma}$, $f_{b,\sigma}(J_{b,\sigma}) = \Delta_\sigma^L$ and $f_{b,\sigma}(I_{b,\sigma}) \subset \Delta_\sigma^L$ (see Figure 1.4.5). Let $A_{b,\sigma}$ be the open interval $\Delta_\sigma^L \setminus (I_{b,\sigma} \cup J_{b,\sigma})$. Observe that $f_{b,\sigma}(\sup A_{b,\sigma}) = x_1(b)$, $f_{b,\sigma}(\inf A_{b,\sigma}) = u_{b,\sigma}$ and $f_{b,\sigma}(A_{b,\sigma}) = \mathbf{S}^1 \setminus \Delta_\sigma^L$.

Let

$$W_{b,\sigma} = \{x \in \Delta_\sigma^L : f_{b,\sigma}^i(x) \in A_{b,\sigma} \text{ for some } i \in \mathbf{Z}^+\} = \cup_{i=0}^\infty f_{b,\sigma}^{-i}(A_{b,\sigma}) \cap \Delta_\sigma^L.$$

Lemma 1.4.1 *For all $\sigma \in [0,1]$ and for all $b \in (\alpha_\sigma^k, \beta_\sigma^k)$ there exists $\{K_i^{b,\sigma}\}_{i=1}^\infty \subset \Delta_\sigma^L$, a countable sequence of open (in Δ_σ^L) disjoint subintervals of Δ_σ^L, such that $W_{b,\sigma} = \cup_{i=1}^\infty K_i^{b,\sigma}$*

Proof. It uses a standard argument. Clearly, $W_{b,\sigma}$ is open in Δ_σ^L. Then, we only have to prove that $W_{b,\sigma}$ is dense in Δ_σ^L. Suppose not. Then $D = \Delta_\sigma^L \setminus \mathrm{Cl}(W_{b,\sigma})$ is a countable union of open intervals (in Δ_σ^L). Number these intervals and let d_i be the length of the $i-$th one. Each d_i is positive and $\sum_{i=1}^\infty d_i \leq 1$. So $\lim_{i\to\infty} d_i = 0$. Hence there is an i_0 such that $d_i \leq d_{i_0}$ for all i. Now, observe that $f_{b,\sigma}(D) \subset D$ and that the image of the i_0-th interval of D by $f_{b,\sigma}$ is a larger interval because $f'_{b,\sigma}|_\Delta > 1$; a contradiction. ∎

In the sequel we shall assume that the sequence $\{K_i^{b,\sigma}\}_{i=1}^\infty$ is labelled in such a way that if $n < m$, then $\sup K_n^{b,\sigma} \leq \inf K_m^{b,\sigma}$. Note that the whole sequence depends on b and σ.

Now, set $K_0^{b,\sigma} = (x_1'(b) - \varrho(\delta, \delta_1), x_1(b))$. From (L4) we have that for each $n \geq 0$ and for all $\sigma \in [0,1]$ there exists $b_{n,k}^\sigma \in (\alpha_\sigma^k, \beta_\sigma^k)$ such that $f_{b,\sigma}(I_{b,\sigma}) \cap K_n^{b,\sigma} \neq \emptyset$ for all $b \geq b_{n,k}^\sigma$ and $b_{n,k}^\sigma$ is the smallest one having this property.

In view of Lemma 1.4.1 and the definition of $W_{b,\sigma}$, for $n > 0$ there exists $l = l(n) \in \mathbf{Z}^+$ such that $f_{b,\sigma}^l(K_n^{b,\sigma}) = A_{b,\sigma}$. Additionally, we set $l(0) = 0$. The following result will be crucial in the proof of Theorem 1.2.2(a)–(b).

Proposition 1.4.2 *Let $n \geq 0$ and let $b \in (b_{n,k}^\sigma, \beta_\sigma^k)$. Then there exist a set $R_{n,k}^{b,\sigma}$ such that*

(a) $R_{n,k}^{b,\sigma}$ is union of $R_1, \ldots, R_{l(n)+2}$, a finite sequence of closed disjoint intervals in $\Delta_\sigma^L \setminus A_{b,\sigma}$ whose endpoints are preimages of $x_1(b)$ or $u_{b,\sigma}$ by $f_{b,\sigma}^m$ for some $m \geq 0$.

(b) If $f_{b,\sigma}(x_1(b)) \in \mathrm{Int}(K_n^{b,\sigma})$, then the closed $f_{b,\sigma}-$invariant set $\Delta_\sigma^L \setminus W_{b,\sigma}$ is strictly contained in $R_{n,k}^{b,\sigma}$.

Proof. If $n = 0$ then the proposition holds trivially by taking $R_1 = I_{b,\sigma}$ and $R_2 = J_{b,\sigma}$. Assume $n > 0$. Clearly, there exists $z \in (x_1(b), \inf A_{b,\sigma})$ such that $f_{b,\sigma}(z) = \sup K_n^{b,\sigma}$ (see Figure 1.4.5). Observe that for all m such that $0 \leq m < l(n)$, $f_{b,\sigma}^m(K_n^{b,\sigma})$ is an open interval (in Δ_σ^L) whose endpoints map onto the endpoints of $f_{b,\sigma}^{m+1}(K_n^{b,\sigma})$. The complement of $[x_1(b), z) \cup (\cup_{i=0}^{l(n)} f_{b,\sigma}^i(K_n^{b,\sigma}))$

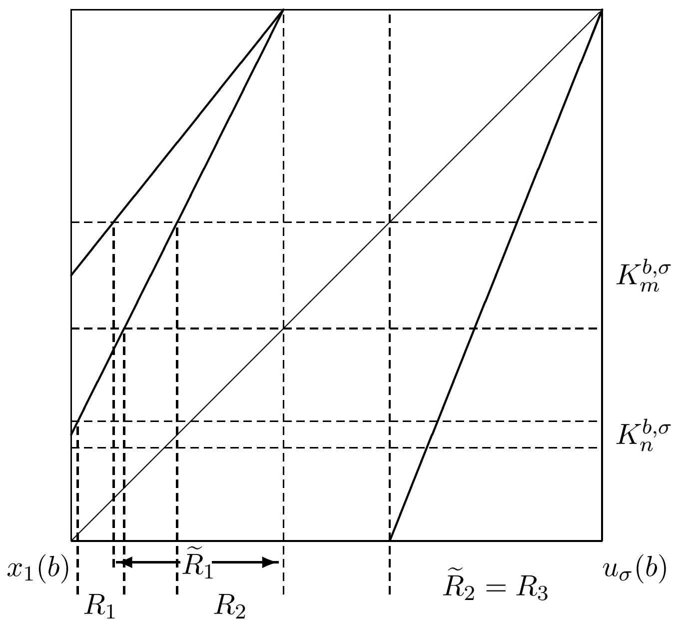

Figure 1.4.6: The sets $R_{n,k}^{b,\sigma}$ and $R_{m,k}^{b,\sigma}$

in Δ_σ^L is union of $l(n)+2$ closed pairwise disjoint intervals. Call them $R_1,\ldots,R_{l(n)+2}$. By construction this sequence satisfies (a). Assume now that $f_{b,\sigma}(x_1(b)) \in \mathrm{Int}(K_n^{b,\sigma})$. Then the complement of $R_{n,k}^{b,\sigma}$ in Δ_σ^L is strictly contained in $W_{b,\sigma}$. From this, statement (b) follows. ∎

Remark 1.4.3 Let $\beta_\sigma^k > b > b_{n,k}^\sigma > b_{m,k}^\sigma$. Then, Proposition 1.4.2 gives us two different sequences of intervals. Namely, $R_{n,k}^{b,\sigma} = \cup_{i=1}^{l(n)+2} R_i$ and $R_{m,k}^{b,\sigma} = \cup_{i=1}^{l(m)+2} \widetilde{R}_i$. From the construction of the sets $R_{n,k}^{b,\sigma}$ and $R_{m,k}^{b,\sigma}$ (see Figure 1.4.6) it is not difficult to see that $l(n) \geq l(m)$ and that there exist $\{k_1,k_2,\ldots,k_{l(m)+2}\} \subset \{1,2,\ldots,l(n)+2\}$ such that $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$ if and only if $\widetilde{R}_{k_i} \cap f_{b,\sigma}(\widetilde{R}_{k_j}) \neq \emptyset$ for $i,j \in \{1,2,\ldots,l(m)+2\}$. □

Now we are ready to define the sequence of parameter values appearing in the statement of Theorem 1.2.2.

In the sequel we shall assume that $\overline{\delta}$ is such that Proposition 1.4.2 holds.

In view of (L4), for $\overline{\delta} > 0$ small enough there exists $\eta_\sigma > 0$ such that for all $b \geq b_{n,k}^\sigma + \eta_\sigma$ we have $f_{b,\sigma}(I_{b,\sigma}) \cap K_n^{b,\sigma} \neq \emptyset$ for all $\sigma \in [0,1]$. Then we define b_n^k as $\sup_\sigma b_{n,k}^\sigma + \eta_\sigma$.

Now, the proof of Theorem 1.2.2(a)-(b) follows directly from the following results.

Proposition 1.4.4 *Let $b \in g_k$ with $b \geq b_n^k$ and let $R_{n,k}^{b,\sigma} = \cup_{i=1}^{l(n)+2} R_i$. Then for $\overline{\delta} > 0$ small enough there exists a finite sequence $V_1^b, \ldots, V_{l(n)+2}^b$ of disjoint vertical strips contained in Q such that $V_i^b \cap L_b(V_j^b) \neq \emptyset$ if and only if $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$.*

Proof. The Implicit Function Theorem implies that $u_{b,\sigma}$ is a smooth function in σ. First we claim that for a fixed b, $(u_{b,\sigma}, \sigma)$ considered as function of σ is a vertical curve in Q. To prove the claim, fix b and σ. From Case g we know that, if $b \in g_k$, then there is a closed interval $V_{b,\sigma}^1 \subset \Delta$ such that $u_{b,\sigma} \in V_{b,\sigma}^1$, $f_{b,\sigma}(V_{b,\sigma}^1) = \Delta$ and the endpoints of $V_{b,\sigma}^1$ map onto the endpoints of Δ. Now we set $V_{b,\sigma}^i = f_{b,\sigma}^{-1}(V_{b,\sigma}^{i-1}) \cap V_{b,\sigma}^1$ for all $i \geq 2$. It easy to see that $V_{b,\sigma}^i \supset V_{b,\sigma}^{i+1}$ and $u_{b,\sigma} \in V_{b,\sigma}^i$ for all $i \geq 1$. From (L2)(i) it follows that the limit of the length of $V_{b,\sigma}^i$ as i tends to infinity is zero. Then, $\cap_{i=1}^{\infty} V_{b,\sigma}^k \times \{\sigma\} = (u_{b,\sigma}, \sigma)$. Now, set $V^i = \cup_{\sigma \in [0,1]} V_{b,\sigma}^i \times \{\sigma\}$ for all $i \geq 1$. Clearly, V^i is a vertical strip and $V^i \supset V^{i+1}$ for all $i \geq 1$. Moreover, the width of V^i tends to zero as i tends to infinity. Then, by a standard result (see for instance Guckenheimer and Holmes [16]; Lemma 5.2.1) we get that

$$V^{\infty} = \cap_{i=1}^{\infty} V^i = \cap_{i=1}^{\infty}[\cup_{\sigma \in [0,1]} V_{b,\sigma}^i \times \{\sigma\}] = \cup_{\sigma \in [0,1]}[\cap_{i=1}^{\infty} V_{b,\sigma}^i \times \{\sigma\}] = \cup_{\sigma \in [0,1]}(u_{b,\sigma}, \sigma)$$

is a vertical curve. This ends the proof of the claim.

Our next step will be the construction of the set of vertical strips. Assume that $b \geq b_n^k$. Then Proposition 1.4.2 holds for all $\sigma \in [0,1]$ and $f'_{b,\sigma} \neq 0$ on each interval R_i. Therefore, from the Implicit Function Theorem we get that the endpoints of R_i are smooth functions in σ. Let $v_i^b = \cup_\sigma(\inf R_i, \sigma)$ and $w_i^b = \cup_\sigma(\sup R_i, \sigma)$. Then, by the construction of the sets R_i, we have that v_i^b and w_i^b are pre-images of the vertical curves $(u_{b,\sigma}, \sigma)$ and $(x_1(b), \sigma)$ under L_b (or L_b^m for some $m \geq 0$). Then by (L6) and by using the same techniques employed by Levi in the proof of Theorem 1.2.1 (see [30] pp.76–86) we obtain the vertical character of v_i^b and w_i^b. Let $V_i^b = [v_i^b, w_i^b] \times [0,1]$. By construction we have $V_i^b \cap L_b(V_j^b) \neq \emptyset$ if and only if $R_i \cap f_{b,\sigma}(R_j) \neq \emptyset$. ∎

Now, for each b_n^k we define the $(l(n)+2) \times (l(n)+2)$-matrix, $T_n^k = (t_{ij})$ by $t_{ij} = 1$ if $V_i^b \cap L_b(V_j^b) \neq \emptyset$ and $t_{ij} = 0$ otherwise. Then, we denote by Σ_n^k the set of infinite sequences $\mathbf{a} = (a_i)_{i=-\infty}^{\infty}$ such that $a_i \in \{1, 2, \ldots, l(n)+2\}$ and $t_{a_i a_{i+1}} = 1$ for all $i \in \mathbf{Z}$. The next corollary follows in the standard way (see Moser [37] pp.76 and Levi [30] pp.78).

Corollary 1.4.5 *For $b \geq b_n^k$ there exists an L_b-invariant hyperbolic Cantor set $S_b^{n,k}$, which contains the saddle point of L_b, such that $L_b|_{S_b^{n,k}}$ is topologically conjugate to the standard shift* *map on* Σ_n^k*. Moreover, for each* $z \in S_b^{n,k}$ *there exists a unique* $\mathbf{a}(z) \in \Sigma_n^k$ *such that* $L_b^i(z) \in V_{a_{-i}}^b$ *for all* $i \in \mathbf{Z}$*.*

Remark 1.4.6 We note that by Proposition 1.4.4 we can compute the transition matrix T_n^k by using the one dimensional map $f_{b,\sigma}$ and the construction of the set $R_{n,k}^{b,\sigma}$ given in Proposition 1.4.2. Moreover, from Remark 1.4.3 we obtain that if $b > b_n^k > b_m^k$ then there exists an injective map $i : \Sigma_m^k \longrightarrow \Sigma_n^k$ which commutes with the standard shift maps on the spaces Σ_m^k and Σ_n^k (i.e. Σ_m^k is a subsystem of Σ_n^k). □

Proof of Theorem 1.2.2(a)-(b). Theorem 1.2.2(a) and the first assertion of Theorem 1.2.2(b) follow immediately from Corollary 1.4.5 and the relation between the maps L_b and P_b. In view of Remark 1.4.3 and the proof of Proposition 1.4.4 we obtain the second assertion of Theorem 1.2.2(b) in a similar way. ∎

1.5 The rotation interval

In the whole memoir we shall deal mainly with continuous circle maps. To study them it is useful to use the equivalent framework of the liftings associated to the given circle map rather than the circle map itself. The main advantages of this choice are that one is able to draw pictures easily and that the points of the space have a total ordering. Then, it is easier to describe where a point lies or which is the image of an interval. Now, we shall introduce the notion of a lifting.

We denote by $e : \mathbf{R} \longrightarrow \mathbf{S}^1 = \{z \in \mathbf{C} : |z| = 1\}$ the natural projection $e(x) = \exp(2\pi i x)$. A continuous map $F : \mathbf{R} \longrightarrow \mathbf{R}$ is called a *lifting* of a continuous map $f : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ if $e \circ F = f \circ e$ (such a map always exists; see Wall [39]). Therefore $F(1) - F(0)$ is an integer independent of x. This integer is called the *degree* of f, and is denoted by deg (f).

In this memoir we concentrate on the circle maps f of degree one. Thus we will denote by $\mathcal{L}$ the class of all liftings of continuous maps of the circle into itself of degree one. That is L is the class of all continuous maps $F : \mathbf{R} \longrightarrow \mathbf{R}$ such that $F(x+1) = F(x) + 1$. It is not difficult to see that $F(x+1) - F(x)$ is an integer independent of x. This integer number is called the *degree of* f.

In the next proposition we describe some of the basic properties of the liftings of circle maps of degree one (see [2]). By $F + k$ we shall denote the map defined by $(F + k)(x) = F(x) + k$.

Proposition 1.5.1 *Let f be a circle map of degree one and let F be a lifting of f. Then the following statements hold.*

(a) The map G is a lifting of f if and only if $G = F + k$ for some integer k.

(b) $F^n(x+k) = F^n(x) + k$ for all $x \in \mathbf{R}, k \in \mathbf{Z}$ and $n \geq 0$. In particular, $F^n \in \mathcal{L}$ for each $n \geq 0$.

(c) $(F+k)^n(x) = F^n(x) + nk$ for all $x \in \mathbf{R}, k \in \mathbf{Z}$ and $n \geq 0$.

We shall say that a point $x \in \mathbf{R}$ is *periodic (mod. 1) of period q with rotation number p/q* for a map $F \in \mathcal{L}$ if $F^q(x) - x = p$ and $F^i(x) - x \notin \mathbf{Z}$ for $i = 1, \ldots, q-1$. A periodic (mod. 1) point of period 1 will be called *fixed (mod. 1)*. Clearly, if F is a lifting of f, then x is periodic (mod. 1) for F if and only if $e(x)$ is periodic for f and their periods are equal.

We advise to the reader that most of the results we are quoting from other authors will be written in terms of class $\mathcal{L}$ unlike the original versions are stated for circle maps of degree one.

The notion of rotation number was introduced by Poincaré [38] for homeomorphisms of the circle of degree one. This notion will be used to characterize the set of periods of circle maps of degree one. The following Theorem is due to Poincaré (see [38]).

Theorem 1.5.2 *Let $F \in \mathcal{L}$ be such that F is increasing. Then*

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}$$

exists and it is independent of x.

Remark 1.5.3 Theorem 1.5.2 holds also for non-decreasing maps from $\mathcal{L}$ (see [37]). □

From Theorem 1.5.2 and Remark 1.5.3 it follows that to every non-decreasing map $F \in \mathcal{L}$ we can associate a real number $\rho(F) = \lim_{n\to\infty} \frac{F^n(x)-x}{n}$, which is called the rotation number of F. Vaguely speaking, $\rho(F)$ is the average angular speed of any point moving around the circle under iteration of the map. We note that $\rho(F)$ is a topological invariant of F. That is, if F and G are topologically conjugate (i.e. there exists an increasing map $h \in \mathcal{L}$ such that $F \circ h = h \circ G$) then $\rho(F) = \rho(G)$. Poincaré also proved that F has a periodic orbit if and only if $\rho(F) \in \mathbf{Q}$.

We remark that for a general map $F \in \mathcal{L}$, $\lim_{n\to\infty} \frac{F^n(x)-x}{n}$ may not exist and if it exists it may depend on the choice of the point x. This motivates the following extension of this notion due to Newhouse, Palis and Takens (see [28]) to each map $F \in \mathcal{L}$. For $F \in \mathcal{L}$ and $x \in \mathbf{R}$ we set

$$\rho_F(x) = \rho(x) = \limsup_{n\to\infty} \frac{F^n(x)-x}{n}.$$

The following Proposition follows from [2].

Proposition 1.5.4 *Let $F \in \mathcal{L}$ and $x \in \mathbf{R}$. Then the following hold.*

(a) $\rho_{F+k}(x) = \rho_F(x) + k$ *for all* $k \in \mathbf{Z}$

(b) $\rho_{F^m}(x) = m\rho_F(x)$.

(c) If $x \in \mathbf{R}$ *and* $k \in \mathbf{Z}$, *then* $\rho_F(x) = \rho_F(x+k)$.

(c) If x is a periodic (mod. 1) point of F with rotation number p/q, then $\rho_F(x) = p/q$

We denote by R_F the set of all rotation numbers of F. Ito (see [18]) proved the following result about the set R_F.

Theorem 1.5.5 *R_F is a closed interval of the real line, perhaps degenerated to a single point.*

In view of Theorem 1.5.5 the set R_F will be called the *rotation interval* of F. Also, for an $F-$invariant set $\Lambda \subset \mathbf{R}$ (i.e. $F(\Lambda) \subset \Lambda$) we set $R_F(\Lambda) = \{\rho_F(x) : x \in \Lambda\}$. Notice that in general $R_F(\Lambda)$ need not be neither closed nor connected.

1.6 Twist orbits

When looking at periodic points of circle maps sometimes it is useful to look at the set of all iterates of the point under consideration. In our framework this means that we have to look at the set of all points projecting on the iterates of the periodic point under consideration. This motivates the following definition.

Let $F \in \mathcal{L}$ and let $x \in \mathbf{R}$. Then the set $\{y \in \mathbf{R} : y = F^n(x)(\text{mod. } 1) \text{ for } n = 0, 1, \ldots\}$ will be called the *(mod. 1) orbit* of x by F. Clearly, if F is a lifting of f, P is a (mod. 1) orbit of F, and $x \in P$ then $P = e^{-1}(\{f^n(e(x)) : n \geq 0\})$. We stress the fact that if P is a (mod. 1) orbit and $x \in P$, then $x + k \in P$ for all $k \in \mathbf{Z}$.

It is not difficult to prove that each point from an orbit (mod. 1) P has the same rotation number. Thus, we can speak about the *rotation number* of P.

If x is a periodic (mod. 1) point of F of period q with rotation number $\frac{p}{q}$ then its (mod. 1) orbit is called a *periodic (mod. 1) orbit of F of period q with rotation number $\frac{p}{q}$*. If P is a (mod. 1) orbit of F we denote by P_i the set $P \cap [i, i+1)$ for all $i \in \mathbf{Z}$. Obviously $P_i = i + P_0$. We note that if P is a periodic (mod. 1) orbit of F with period q, then $\mathrm{Card}(P_i) = q$ for all $i \in \mathbf{Z}$.

Let P be a (mod. 1) orbit of a map $F \in \mathcal{L}$. We say that P is a *twist orbit* if F restricted to P is increasing. If a periodic (mod. 1) orbit is twist then we say that P is a *twist periodic orbit* (from now on TPO). The following result gives a geometrical interpretation of a TPO.

Lemma 1.6.1 *Let $P = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$ be a TPO with period q and rotation number p/q and assume that $x_i < x_j$ if and only if $i < j$. Then $(p, q) = 1$ and $F(x_i) = x_{i+p}$.*

The following result generalizes Theorem 1.5.2 to twist orbits.

Remark 1.6.2 We note that if P is a twist orbit then the rotation number of P can be computed by $\lim_{n\to\infty} \frac{F^n(x)-x}{n}$ for each $x \in P$. □

In [10] the following result is proved.

Lemma 1.6.3 *Let $F \in \mathcal{L}$. For all $a \in \mathbf{R}_F$ there exists a twist orbit P of F with rotation number a. Moreover P is contained in a union of closed intervals in which F is increasing.*

For a map $F \in \mathcal{L}$ we define maps F_l and F_r by (see [29], [3] and [10])

$$F_r(x) = \sup\{F(y) : y \leq x\},$$

$$F_l(x) = \inf\{F(y) : y \geq x\}.$$

Proposition 1.6.4 *The maps F_r, F_l belong to $\mathcal{L}$ and are non-decreasing.*

The map F_r may be characterized as the smallest non-decreasing map in $\mathcal{L}$ greater or equal than F. Similarly F_l may be characterized as the largest non-decreasing map less or equal than F (see Figure 1.6.7). It is easy to see that F_l and F_r coincide if F is non-decreasing. Otherwise there exists intervals on which F_r is constant and strictly greater than F and there exists intervals

Figure 1.6.7: The maps F_l and F_r.

on which F_l is constant and strictly smaller than F. Since F_r and F_l are non-decreasing, from Theorem 1.5.2 they have unique rotation number. Thus, the numbers

$$a^-(F) = \lim_{i\to\infty} \tfrac{1}{i}(F_l^i(X) - X),$$
$$a^+(F) = \lim_{i\to\infty} \tfrac{1}{i}(F_r^i(X) - X),$$

are well defined. The next lemma shows the relation between the rotation numbers of F_r, F_l and the rotation interval of F. The proof is due to Misiurewicz (see [**?**]).

Lemma 1.6.5 *For a map $F \in \mathcal{L}$ we have $R_F = [a^-(F), a^+(F)]$.*

1.7 Proof of Theorem 1.2.2(c)

Prior to start the proof of Theorem 1.2.2(c) we have to introduce some notation and state some preliminary results. In the sequel we shall denote the $f_{b,\sigma}-$invariant set $\Delta_\sigma^L \setminus W_{b,\sigma}$ by $\Lambda_{b,\sigma}$.

Proposition 1.7.1 *Let $b \geq b_n^k$, $b \in g_k$. Then for each $\sigma \in [0,1]$ there exists an open interval $I_\sigma^{n,k} \subset (\alpha_\sigma^k, \beta_\sigma^k)$ satisfying that for all $c \in I_\sigma^{n,k}$ there exists a homeomorphism $\phi_\sigma : S_b^{n,k} \longrightarrow \Lambda_{c,\sigma}$ (here we use the notation from Corollary 1.4.5) such that $\phi_\sigma \circ L_b = f_{c,\sigma} \circ \phi_\sigma$.*

Proof. Recall that for each $\sigma \in [0,1]$ there exists $b_{n,k}^\sigma$ such that $f_{b_{n,k}^\sigma,\sigma}(x_1(b_{n,k}^\sigma)) = \sup(K_n^{b_{n,k}^\sigma,\sigma})$. Then there exists $\kappa_\sigma > 0$ such that for all $c \in I_\sigma^{n,k} = (b_{n,k}^\sigma, b_{n,k}^\sigma + \kappa_\sigma)$ we have that $f_{c,\sigma}(x_1(c)) \in \mathrm{Int}(K_n^{c,\sigma})$. Let $R_{n,k}^{c,\sigma} = \cup_{i=1}^{l(n)+2} R_i$ be the sequence of intervals constructed in Proposition 1.4.2.

Take $z \in S_b^{k,l}$. Then, by Corollary 1.4.5, there exists unique $\mathbf{a}(z) \in \Sigma_n^k$ such that $L_b^i(z) \in V_{a_{-i}}$ for all $i \geq 0$. We recall that $V_{a_{-i}} = \cup_\sigma(R_{a_{-i}} \times \{\sigma\})$ and $L_b(V_{a_{-i}}) \cap V_{a_{-i-1}} \neq \emptyset$ if and only if $f_{b,\sigma}(R_{a_{-i}}) \cap R_{a_{-i-1}} \neq \emptyset$ (see Proposition 1.4.4). Now, for $i > 0$ we define the set $R_{a_{-i}\dots a_0}$ as $R_{a_{-i+1}\dots a_0} \cap f_{b,\sigma}^{-i}(R_{a_{-i}})$. By Proposition 1.4.4 we have that $R_{a_{-i}\dots a_0} \neq \emptyset$ and $R_{a_{-i-1}\dots a_0} \subset R_{a_{-i}\dots a_0}$. Moreover, for each $i > 0$, the set $R_{a_{-i}\dots a_0}$ is a closed interval in Δ_σ^L and the diameter of $R_{a_{-i}\dots a_0}$ is smaller than or equal to $(\vartheta\gamma)^{-i}$ because $f'_{b,\sigma}|_\Delta > \vartheta\gamma > 1$. Therefore, $\cap_{i=0}^\infty R_{a_{-i}\dots a_0}$ contains a unique point $x(z,\sigma) \in R_{a_0}$, such that $f_{b,\sigma}^i(x(z,\sigma)) \in R_{a_{-i}}$ for all $i \geq 0$. Hence, $\{x(z,\sigma) : z \in S_b^{k,l}\} \subset \Lambda_{c,\sigma}$. Moreover, from Proposition 1.4.2(b) it follows that $\{x(z,\sigma) : z \in S_b^{k,l}\} = \Lambda_{c,\sigma}$. Lastly, the map $\phi_\sigma(z) = x(z,\sigma)$ is a homeomorphism. ∎

From the above proposition and its proof we have that the twist periodic orbits of period s and rotation number r/s of the map $f_{c,\sigma}$ in $\Lambda_{c,\sigma}$ for $c \in I_\sigma^{n,k}$ correspond to $(r/s)-$Birkhoff orbits of the annulus map L_b in $S_b^{n,k}$ (see [21]).

In what follows we shall fix a lifting $F_{b,\sigma}$ of the circle map $f_{b,\sigma}$ by setting $F_{b,\sigma} = \pi_1 \circ \widetilde{L}_b$ (where π_1 and $\widetilde{L}_b$ are defined in Section 1.2 and 1.3, respectively). Then from Proposition 1.7.1 we obtain inmediately the following result.

Corollary 1.7.2 *Let $z \in S_b^{n,k}$. Then for all $c \in I_\sigma^{n,k}$ we have that $\rho_{\widetilde{L}_b}(z) = \rho_{F_{c,\sigma}}(Z)$, if it exists, where $Z \in e^{-1}(\phi_\sigma(z))$.*

In view of the above corollary we see that the computation of the rotation set of $S_b^{n,k}$ reduces to the computation of the rotation set of $F_{b,\sigma}|_{e^{-1}(\Lambda_{b,\sigma})}$. Unfortunately, this rotation set is different from the rotation interval of $F_{b,\sigma}$. However, from the family $f_{b,\sigma}$, it is possible to obtain a logistic family of circle maps of degree one such that they still have $\Lambda_{b,\sigma}$ as invariant set and the rotation interval of these maps coincides with the rotation set of $e^{-1}(\Lambda_{b,\sigma})$. This is achieved simply by modifying the maps $f_{b,\sigma}$ in such a way that they loose the differentiability

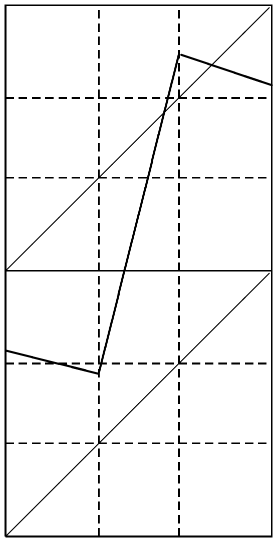

Figure 1.7.8: The logistic family of circle maps

at the endpoints of Δ. To be more precise, we define $h_c = h(., c, \overline{\delta}) : \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ with $c \in [b_1, b_2]$ such that (see Figure 1.7.8):

(ALS1) h_c depends continuously on c.

(ALS2) The map h_c satisfies (L2) and (L4) with $\Delta = \Delta_1$.

This family of maps was used by Alsedà, Llibre and Serra [4] to study the bifurcations of the Levi's circle maps at the level of the set of periods.

In the rest of this section we shall use, for the family h_c (and their liftings H_c), the notation and definitions introduced in the preceding sections extended in the natural way.

From (ALS2) is easy to see that the unique h_c-invariant set strictly contained in Δ is $\Lambda_{c,\sigma}$. Moreover, if $c \in I_\sigma^{n,k}$, then by Proposition 1.7.1 and Corollary 1.7.2 we have that the $\widetilde{L}_b-$rotation set of $S_b^{n,k}$ coincides with the H_c-rotation set of $e^{-1}(\Lambda_{c,\sigma})$. We note that by a change of variables, if necessary, we may assume that $e(0) = x_1(c)$ for each $c \in [b_1, b_2]$. Then we denote by $X_2(c)$ the unique element of $[0,1) \cap e^{-1}(x_2(c))$. Let $\widetilde{\Lambda}_c$ be the set of all (mod. 1) orbits of H_c contained in $e^{-1}(\Delta)$.

The next result states that the H_c- rotation set of $e^{-1}(\Lambda_{c,\sigma})$ coincides with R_{H_c}, which is the property we are looking for. It follows from Theorem B of [29], the proof of Theorem 2 of

[10] and the Theorem B of [5].

Theorem 1.7.3 *For the maps $H_c \in \mathcal{L}$ we have:*

(a) The maps $c \to a^-(H_c)$ and $c \to a^+(H_c)$ are continuous.

(b) Let $a \in R_{H_c}$. Then there exists a twist orbit of H_c with rotation number a contained in $e^{-1}(\Delta)$. That is, $R_{H_c} = R_{H_c}(\widetilde{\Lambda}_c)$.

(c) If $a^-(H_c) \in \mathbf{R} \setminus \mathbf{Q}$ (respectively $a^+(H_c) \in \mathbf{R} \setminus \mathbf{Q}$) then $\{H_c^n(0) : n \in \mathbf{Z}^+\} \subset e^{-1}(\Delta)$ and

$$\lim_{i\to\infty} \frac{1}{i} H_c^i(0) = a^-(H_c)$$

(respectively $\{H_c^i(X_2(c)) : i \in \mathbf{Z}^+\} \subset e^{-1}(\Delta)$ and

$$\lim_{i\to\infty} \frac{1}{i} (H_c^i(X_2(c)) - X_2(c)) = a^+(H_c)\ ).$$

The following two lemmas allow us to study the $\widetilde{L}_b$ rotation set of $S_b^{n,k}$. Let $U_{c,\sigma}$ be the unique element of $e^{-1}(u_{c,\sigma}) \cap [0,1)$.

Lemma 1.7.4 *Let $c \in I_\sigma^{n,k}$. Then $a^+(H_c) = 1$, $a^-(H_c) \in \mathbf{Q}$ and the $\widetilde{L}_b$−rotation set of $S_b^{n,k}$ is equal to $[a^-(H_c), 1]$.*

Proof. Without loss of generality we may assume that $H_c(0) \in [0,1)$. Since $H_c|_{e^{-1}(\mathbf{S}^1\setminus\Delta)}$ is strictly decreasing we have that $(H_c)_r(X) = H_c(X_2(c))$ for all $X \in [X_2(c), 1]$. By Lemma 1.3.2 and (ALS2) we see that $(H_c)_r(U_{c,\sigma}) = U_{c,\sigma} + 1$. Therefore, $a^+(H_c) = 1$.

We note that in the proof of Proposition 1.7.1 the definition of $I_\sigma^{n,k}$, the set $K_n^{c,\sigma}$ and the point $x_1(c)$ depend only on $f_{c,\sigma}|_\Delta$. Hence, in view of the definition of the family h_c and since $c \in I_\sigma^{n,k}$, it follows that $h_c(x_1(c)) \in \mathrm{Int}(K_n^{c,\sigma})$. On the other hand, since $h_c(A_{c,\sigma}) = \mathbf{S}^1 \setminus \Delta_\sigma^L$ there exists $j \geq 0$ such that $H_c^j(0) \in e^{-1}(\mathbf{S}^1 \setminus \Delta_\sigma^L)$. Moreover, for each $X \in e^{-1}(\Delta \setminus \Delta_\sigma^L)$ there exists some $i \geq 0$ such that $H_c^i(X) \in e^{-1}(\mathbf{S}^1 \setminus \Delta)$ because $H_c|_{e^{-1}(\Delta\setminus\Delta_\sigma^L)}$ is strictly increasing and $U_{c,\sigma}$ is a unstable fixed (mod. 1) point of H_c. Therefore, $H_c^j(0) \in e^{-1}(\mathbf{S}^1 \setminus \Delta)$ for some $j \geq 0$. Hence, from Theorem 1.7.3(c) we get that $a^-(H_c) \in \mathbf{Q}$.

From the construction made in Section 1.4 we see that the definition of Δ_σ^L and $A_{c,\sigma}$ depend only on $f_{c,\sigma}|_\Delta$. Thus, $e(\widetilde{\Lambda}_c) \subset \Delta_\sigma^L$. Since $f_{b,\sigma}(A_{c,\sigma}) = \mathbf{S}^1 \setminus \Delta_\sigma^L$, from (ALS2), we have that

$e(\widetilde{\Lambda}_c) = \Lambda_{c,\sigma}$. So, from Corollary 1.7.2 and Theorem 1.7.3(b) it follows that the $\widetilde{L}_b$−rotation set of $S_b^{n,k}$ is $[a^-(H_c), 1]$. ∎

Lemma 1.7.5 *For each $a \in [0,1)$ there exists $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ such that $a^-(H_c) = a$. Moreover, for each $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ we have that $a^-(H_c) \in [0,1)$.*

Proof. From the definitions of α_σ^k and β_σ^k we have that for $c \in (\alpha_\sigma^k, \beta_\sigma^k]$ we may assume, without loss of generality, that $H_c(0) \in (U_{c,\sigma} - 1, U_{c,\sigma})$.

We recall that, for $c \in (b_{0,k}^\sigma, \beta_\sigma^k]$ we have $h_c(x_1(b)) \in \mathbf{S}^1 \setminus \Delta_\sigma^L$. Then, $H_c(0) < 0$. Thus, there exists $\widetilde{U}_{c,\sigma} \in [0, U_{c,\sigma}]$ such that $H_c(\widetilde{U}_{c,\sigma}) = \widetilde{U}_{c,\sigma}$. We have that $H_c(1) = H_c(0) + 1 > U_{c,\sigma} \geq \widetilde{U}_{c,\sigma}$. Therefore, by the definition of H_c we have $(H_c)_l(\widetilde{U}_{c,\sigma}) = H_c(\widetilde{U}_{c,\sigma})$. So, $a^-(H_c) = 0$.

Let $c = \alpha_\sigma^k$. Then, $H_c(0) = U_{c,\sigma}$. Clearly, $(H_c)_l(U_{c,\sigma}) = H_c(U_{c,\sigma}) = U_{c,\sigma} + 1$. Thus, $a^-(H_c) = 1$. Then, in view of Theorem 1.7.3(a), the first statement of the lemma follows.

Since $H_c(0) < U_{c,\sigma}$ it is not difficult to see that for $c \in \left(\alpha_\sigma^k, \beta_\sigma^k\right]$, $(H_c)_l(X) < X + 1$ for all $X \in \mathbf{R}$ (recall that $H_c(U_{c,\sigma})) \in \{U_{c,\sigma}, U_{c,\sigma} + 1\}$). Hence $a^-(H_c) < 1$. ∎

Proof of Theorem 1.2.2(c). From Lemma 1.7.4 and Remark 1.3.3, we get that the $\widetilde{P}_b$−rotation set of $C_b^{n,k}$ for $c \in I_\sigma^{n,k}$ is the closed interval $[2\pi/(2q + 1 - 2a^-(H_c)), 2\pi/(2q - 1)]$. Then Theorem 1.2.2(c) follows from Lemma 1.7.5. ∎

1.8 The piecewise-monotone family of circle maps related to the Van der Pol equation

In this section we shall study the bifurcations of a family h_c of circle maps satisfying (ALS1)-(ALS2) defined in the previous section. This is interesting because this model already captures the essential features of the Levi's one and has the advantage that the study of its bifurcations can be done in a more complete way than for the Levi's circle map family considered in Section 1.3. In particular, for the maps h_c we shall characterize the appearance of Cantor sets when the parameter c crosses the interval g_k. Moreover we shall see that these Cantor sets contain the invariant sets $\Lambda_{c,\sigma}$.

We start this section by recalling the definition of the family h_c. Indeed $h_c = h(., c, \delta)$ be a three parameter family of $\mathcal{C}^0$ maps of the circle into itself of degree one, with the parameters ranging in $b_1 \leq c \leq b_2, 0 < \delta \leq \overline{\delta}$, and satisfying that there exist $\gamma > 1,\ \vartheta > 1/\gamma,\ c > 0$ and an interval $\Delta = [x_1(c), x_2(c)] \subset \mathbf{S}^1$ such that $x_1(c), x_2(c)$ depend on c and δ (differentiably on c), $|\Delta| < \delta$ and

$$h_c'(x) > \vartheta\gamma \text{ for all } x \in \Delta \tag{1.8.4}$$

$$-1 + c < h_c'(x) < -1/\gamma \text{ for all } x \in \mathbf{S}^1 \setminus \Delta \tag{1.8.5}$$

$$-d/db[h_c(x_i(c), c, \delta) - x_i(c)] > \omega > 0,\ i = 1, 2 \tag{1.8.6}$$

where $\omega = \omega(\delta)$ is independent of c (see Figure 1.7.8).

In [4] the following result is given. It characterizes the dynamics of h_c for certain values of c (compare with Theorem 1.2.1(a)–(b)).

Theorem 1.8.1 *If the map h_c satisfies (1.8.4)-(1.8.6) then for $\overline{\delta}$ small enough the interval $[b_1, b_2]$ consist of two alternating types of intervals A_k, B_k separated by (short) gaps g_k :*

$$[b_1, b_2] = A_1 \cup g_1 \cup B_1 \cup g_2 \cup A_2 \cup g_3 \cup, \ldots, \cup A_n \cup g_{2n-1} \cup B_n,$$

such that:

(A) *For $c \in A_k$ the map h_c has exactly two fixed points, one stable and another unstable. Moreover, the basin of attraction of the stable fixed point is the whole circle except the unstable fixed point.*

(B) *For $c \in B_k$ the map h_c has four fixed points, two stable and two unstable. Moreover, these two unstable fixed points belong to a Cantor set C such that $h_c|_C$ is topologically conjugated to a certain subshift of finite type.*

The goal of this section is to give a complete characterization of the bifurcations of this circle maps family. The main result of this section is the following:

Theorem 1.8.2 *Let $g_k = (g_{k,1}, g_{k,2})$. For every gap g_k there exist $\alpha_k,\ \beta_k$ such that $g_{k,1} < \alpha_k \leq \beta_k < g_{k,2}$ and*

(a) *If* $c \in g_k$*, then* h_c *has exactly two fixed points, one stable and another unstable.*

(b) *If* $c \in (g_{k,1}, \alpha_k]$ *then the basin of attraction of the stable fixed point is either the whole circle except the unstable fixed point or the whole circle except the unstable fixed point union* $x_i(b)$ *with* $i = 1$ *or* 2.

(c) *If* $c \in (\beta_k, g_{k,2})$ *then there exist a Cantor set* C *(which depend on* c*), containing the unstable fixed point and such that* $h_c|_C$ *is topologically conjugate to a subshift of finite type. Moreover, the basin of attraction of the stable fixed point is either the complementary of the Cantor set* C *or the complementary of the Cantor set* C *union* $\cup_{n=0}^{\infty} h_c^{-n}(x_i(c))$ *with* $i = 1$ *or* 2.

(d) *If* $\alpha_k \neq \beta_k$ *then the interval* $(\alpha_k, \ \beta_k]$ *consists of two sets* $D_k, \ E_k$ *such that* $(\alpha_k, \beta_k] = D_k \cup E_k$*,* E_k *(resp.* D_k*) is closed (resp. open) in* $(\alpha_k, \beta_k]$ *and if* c *belongs to* E_k *(resp.* D_k*) then the dynamics of* h_c *is analogous to the case* $c \in (g_{k,1}, \alpha_k]$ *(resp.* $c \in (\beta_k, g_{k,2})$*).*

We note that the above theorem characterize completely the minimal invariant sets for all values of parameter c and gives the full picture of the bifurcations occurring in g_k. This characterization could not achieved in Theorem 1.2.2 for the Levi's circle maps family because of the differentiability of Levi's circle maps family in $\Delta_1 \setminus \Delta$.

The rest of the section will be devoted to prove Theorem 1.8.2.

From now one we will use lower case letters to denote points in $\mathbf{S}^1$ and for the corresponding point in the covering space $\mathbf{R}$ we will use the corresponding upper case letter.

We recall that for the family h_c only one of the following three cases can occur:

Case $\tilde{\mathbf{A}}$. The set $I = h_c^{-1}(\Delta) \cap \Delta$ is an interval, such that $h_c(I) = \Delta$ and the endpoints of I map onto the endpoints of Δ.

Case $\tilde{\mathbf{g}}$. $h_c(x_i) \in \mathrm{Int}\Delta$, for $i = 1$ or 2 (i.e. the set I is a union of two disjoint intervals I_1 and I_2 so that the endpoints I_1 map onto the endpoints of Δ and $h_c(I_2)$ is strictly contained in Δ).

Case $\tilde{\mathbf{B}}$. The set I is a union of two disjoint intervals I_1 and I_2 so that the endpoints of each I_i map onto the endpoints of Δ.

Call $A_k, \ B_k, \ g_k$ the maximal intervals of c for which the corresponding alternative holds.

Since the endpoints of $h_c(\Delta)$ move monotonically (clockwise) with respect to the endpoints of Δ (see (1.8.6)) the intervals alternate as stated in Theorem 1.8.1.

We shall study the bifurcations when c crosses a gap g_{2k-1} from A_k to B_k (e.g. $h_c(x_1(c)) \in \text{Int}\,(\Delta)$). In a similar way, we may study them when b crosses a gap g_{2k} from B_{k-1} to A_k. We describe these bifurcations in terms of symbolic dynamics. So we use the following definitions.

Let $S = \{1, 2, \ldots, m\}$ and $T = (t_{i,j})$ an $m \times m$ matrix such that $t_{i,j} \in \{0, 1\}$. We denote by Σ_T the set of infinite sequences $\mathbf{a} = (a_i)_{i=0}^{\infty}$ such that $a_i \in S$ and $t_{a_i a_{i+1}} = 1$ for all $i \in \mathbf{Z},\ i \geq 0$. We define the shift map $\sigma : \Sigma_T \longrightarrow \Sigma_T$ by $\sigma(\mathbf{a}) = (a_i)_{i=1}^{\infty}$. Then the set Σ_T with the shift map σ is called a *subshift of finite type with transition matrix T*. If $t_{i,j} = 1$ for all i, j, then we call it *full shift on m symbols.* The set Σ_T has a metric defined by $d(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{\infty} \gamma(a_i, b_i) 2^{-i}$ where

$$\gamma(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } a \neq b. \end{cases}$$

Then Σ_T is a Hausdorff compact space.

Let $h_c \in C(\mathbf{S}^1, \mathbf{S}^1)$ and let $\Sigma \subset \mathbf{S}^1$ be an invariant set (i.e. $h_c(\Sigma) \subset \Sigma$) we say that $h_c|_\Sigma$ is topologically conjugated to a subshift of finite type $\sigma|_{\Sigma_T}$ if there is a homeomorphism $h_c : \Sigma_T \longrightarrow \Sigma$ such that $h_c \circ h = h \circ \sigma$.

Let h_c be a continuous map of the circle into itself which satisfies (1.8.4)-(1.8.6). Assume that $c \in g_k$. We note that for $c \in A_k \cup g_k$, then f has exactly two fixed points one stable and the other unstable (see Case $\tilde{\text{A}}$ and Case $\tilde{\text{g}}$). From now one we denote by $u(c)$ the unstable fixed point of h_c and by $s(c)$ the stable fixed one. By the definition of the intervals A_k and g_k we have that $s(c) \in \mathbf{S}^1 \setminus \Delta$ and $u(c) \in \text{Int}\,(\Delta)$. Let $W = \{x \in \mathbf{S}^1 : \lim_{n\to\infty} h_c^n(x) = s(c)\}$ (i.e. W is the basin of attraction of the stable fixed point).

Now, we will use a lifting H_c of the map h_c, and so we have to fix our notation. Without loss of generality we may assume that $0 \in e^{-1}(x_2(c))$ (that is 0 is a local maximum of H_c). Then $\overline{\Delta}$ denotes the interval $e^{-1}(\Delta) \cap [0, 1]$. Also, $U(c)$ (resp. $X_1(c)$) denotes the only element of $e^{-1}(u(c)) \cap \overline{\Delta}$ (resp. $e^{-1}(x_1(c)) \cap \overline{\Delta}$). Lastly, we choose the lifting H_c such that $H_c(U(c)) = U(c) + 1$ (see Figure 1.8.9).

Also, we recall that if $c \in g_k$ then $h_c(x_1(c)) \in \text{Int}\,(\Delta)$. The following lemma is not difficult to prove (see Figure 1.8.10)

Lemma 1.8.3 *The following statements hold.*

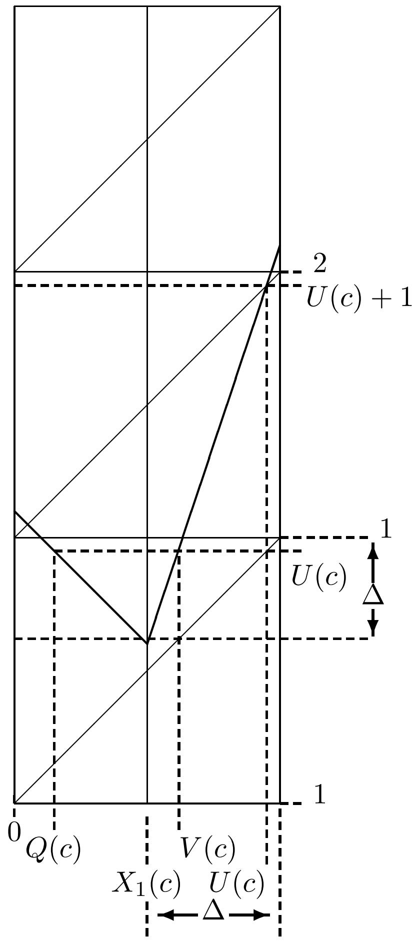

Figure 1.8.9: The lifting H_c.

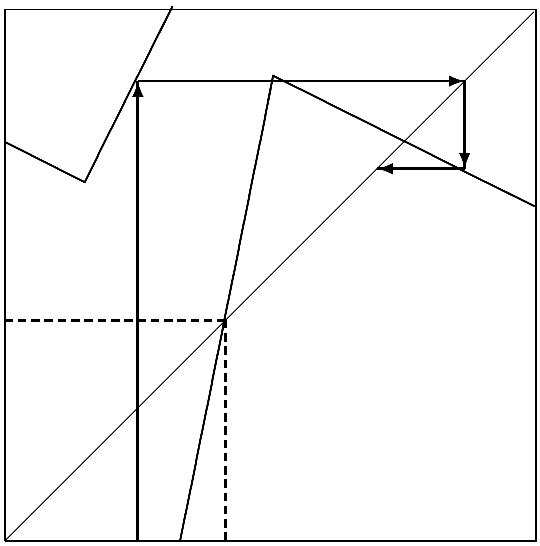

Figure 1.8.10: The map h_c for $c \in A_k$.

(a) If $H_c(X_1(c)) > U(c)$ *then,* $W = \mathbf{S}^1 \setminus \{u(c)\}$.

(b) If $H_c(X_1(c)) = U(c)$ *then,* $W = \mathbf{S}^1 \setminus \{u(c), x_1(c)\}$.

Remark 1.8.4 We note that the situation described in the Lemma 1.8.3(a) is similar to the case when $c \in A_k$ and persists in a small open interval contained in g_k. □

Lemma 1.8.5 *If* $H_c(X_1(c)) < U(c)$ *then there are two points* $V(c),\ Q(c)$ *such that* $0 < Q(c) < X_1(c) < V(c) < U(c)$ *and* $H_c(Q(c)) = H_c(V(c)) = U(c)$.

Proof. Observe that $H_c(X_1(c)) = \inf_{x\in[0,1]} H_c(X)$. Since $H_c|_{\overline{\Delta}}$ is strictly increasing, $u(c)$ is the only fixed point in Δ and $H_c(U(c)) = U(c) + 1$ we have $H_c(1) > 2$ (see Figure 1.8.9). Hence $H_c(0) > 1$. By using the intermediate value theorem we find two points $V(c) > X_1(c)$ and $Q(c) < X_1(c)$ such that $H_c(V(c) = H_c(Q(c)) = U(c)$. Also, $V(c) < U(c)$ because $H_c|_{\overline{\Delta}}$ is strictly increasing. ∎

Let $q(c) = e(Q(c))$, $v(c) = e(V(c))$ and $I = [q(c), u(c)]$ (see Figure 1.8.11). Clearly $H_c([V(c), U(c)]) = [U(c), U(c) + 1]$. Then there is a unique point $R(c) \in (V(c), U(c))$ such that $H_c(R(c)) = Q(c) + 1$. Let $r(c) = e(R(c))$. So $h_c([r(c), u(c)]) = I$.

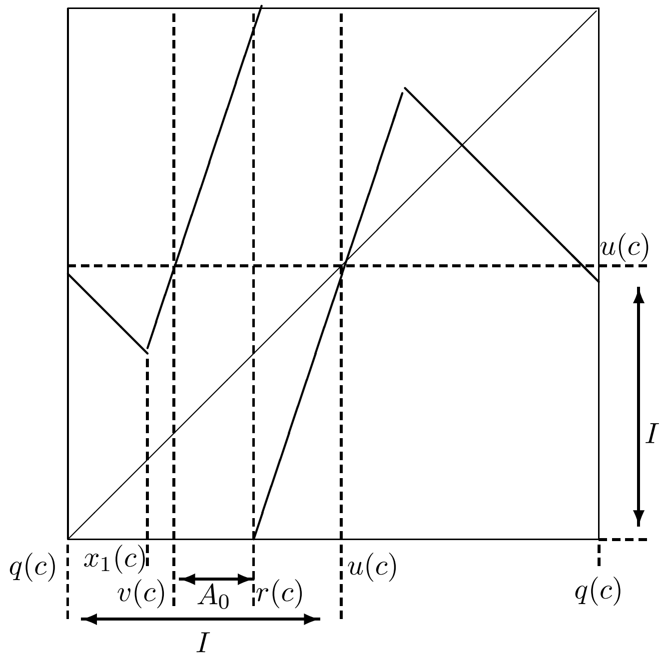

Figure 1.8.11: The interval $[q(c), u(c)]$.

Observe that $\mathbf{S}^1 \setminus I$ is contained in W and let A_0 denote the interval $(v(c), r(c))$. Then the following lemma follows from the fact that $h_c(A_0) = \mathbf{S}^1 \setminus I$ and $h_c(I \setminus A_0) = I$ (see Figure 1.8.11).

Lemma 1.8.6 *Let $H_c(X_1(c)) < U(c)$. Then, $(\mathbf{S}^1 \setminus I) \cup A_0$ is contained in W. Moreover, $W \cap I = \cup_{i=0}^{\infty} h_c^{-i}(A_0)$.*

We denote by W_I the open (in I) set $W \cap I$.

Proposition 1.8.7 *W is a open dense set in $\mathbf{S}^1$.*

Proof. From Lemma 1.8.6 we have that W_I is open. Then the proposition will follow by showing that W_I is dense in $I \setminus x_1(c)$ (which is a minor variation of the proof of Lemma 1.4.1). Suppose not. Then $D = (I \setminus x_1(c)) \setminus \mathrm{Cl}(W)$ is a countable union of open (in I) intervals. Number these intervals and let d_i be the length of the $i - th$ one. Then $\sum_{i=1}^{\infty} d_i \leq 1$ and each d_i is positive. So $\lim_{i\to\infty} d_i = 0$. Hence there is a i_0 with the property that $d_i \leq d_{i_0}$ for all i. By using that $h_c(x_1(c)) \in \Delta$ we have that if $x \in (q(c), v(c))$, then $h_c(x) \in (x_1(c), u(c))$. From (1.8.4) and (1.8.5) we obtain that $(h_c^2)'|_D > \vartheta > 1$. Now observe that $h_c^2(D) \subset D$ and that h_c^2 restricted to the $i_0 - th$ interval of D maps this interval to a larger interval because $(h_c^2)' > 1$. But such an interval can not be in D. This is the required contradiction. ∎

Let $\Sigma = \mathbf{S}^1 \setminus W$. Clearly, $\Sigma = I \setminus W_I$.

Corollary 1.8.8 *The set Σ is a closed totally disconnected invariant set.*

Remark 1.8.9 For the family h_c we can define the sets $\Delta_\sigma^L = [x_1(c), u(c)]$ and the h_c-invariant set $\Lambda^h_{c,\sigma} \subset \Delta$, in a similar way as they were defined for the family $f_{b,\sigma}$. To prove Theorem 1.2.2 we have used the dynamics of the family h_c restricted to $\Lambda^h_{c,\sigma}$. However, already for h_c which is a one-dimensional model simpler than $f_{b,\sigma}$ we know that the dynamics is more rich. Indeed, since $\Delta_\sigma^L \subset I$ we have that $\Lambda^h_{c,\sigma}$ is strictly contained in Σ which is an invariant set for h_c. The fact that we did not use the dynamics of h_c in $\Sigma \setminus \Lambda^h_{c,\sigma}$ tells us that still there are some features of the general model that we have not been able to capture by using the one dimensional approximation. □

Next we use symbolic dynamics to describe the behavior of f in Σ. To do this we define $K_1(c) = \cup_{n=0}^{\infty} h_c^{-n}(x_1(c))$.

Theorem 1.8.10 *Let $c \in g_k$ such that $H_c(X_1(c)) < U(c)$. Then there is a sequence $R_1, \ldots, R_m$ with $m = m(x_1(c))$ of closed pairwise disjoint intervals in I such that*

(a) $\Sigma \subset (\cup_{i=1}^m R_i)) \cup \{x_1(c)\}$

(b) $h_c|_{\Sigma\setminus K_1(c)}$ *is topologically conjugate to* $\sigma|_{\Sigma_T}$*, a subshift of finite type.*

Proof. First of all we construct the sequence $R_1, \ldots, R_m$. By Proposition 1.8.7 there exists $\mathcal{V}$, a connected component of W_I such that $h_c(x_1(c)) \in \mathrm{Cl}(\mathcal{V})$ and $h_c((q(c), v(c)) \cap \mathcal{V} \neq \emptyset$ and let $\mathcal{V}' = (y, z)$ be such that $h_c(y) = h_c(z) = \sup \mathcal{V}$ (see Figure 1.8.12). Note that $x_1(c) \in \mathcal{V}'$ and, if $x_1(c) \in W_I$, then $\mathcal{V}'$ is a connected component of W_I. Otherwise $\mathcal{V}'$ is $x_1(c)$ union two connected components of W_I. Clearly, there exists a non-negative integer l such that $h_c^l(\mathcal{V}) = A_0$ (recall that $W_I = \cup_{i=0}^{\infty} h_c^{-i}(A_0)$. Observe that for all n such that $0 \leq n \leq l$, $h_c^n(\mathcal{V})$ is an open interval and the endpoints of $h_c^n(\mathcal{V})$ map onto the endpoints of $h_c^{n+1}(\mathcal{V})$. Moreover, $\mathcal{V}' \cup \mathcal{V} \cup h_c(\mathcal{V}) \cup, \ldots, \cup h_c^l(\mathcal{V}) \subset W_I \cup \{x_1(c)\}$.

The complement of $\mathcal{V}' \cup \mathcal{V} \cup h_c(\mathcal{V}) \cup, \ldots, \cup h_c^l(\mathcal{V})$ in I is union of a finite sequence of closed pairwise disjoint intervals. Call them $R_1, \ldots, R_m$. Let $R = \cup_{i=1}^m R_i$. Clearly, $\Sigma \subset R \cup \{x_1(c)\}$ and statement (a) is proved.

Figure 1.8.12: The intervals $\mathcal{V}$ and $\mathcal{V}'$.

The map f is monotonic on each of the closed intervals R_i and we have that $h_c^{-1}(R) \subset R$. Moreover, for all i, j the set $R_i \cap h_c^{-1}(R_j)$ has at most one connected component. Define the $m \times m$ matrix $T = (t_{i,j})$ by $t_{i,j} = 1$ if $R_i \cap h_c^{-1}(R_j) \neq \emptyset$ and $t_{i,j} = 0$ if $R_i \cap h_c^{-1}(R_j) = \emptyset$. Let (Σ_T, σ) be the subshift of finite type with transition matrix T. Now, statement (c) follows in the standard way (see [**?**]). ∎

Corollary 1.8.11 *Let $x_1(c) \in W_I$. Then there is a sequence $R_1, \ldots, R_m$ with $m = m(x_1(c))$ of closed pairwise disjoint intervals in I such that*

(a) $\Sigma \subset (\cup_{i=1}^{m} R_i)$

(b) $h_c|_\Sigma$ is topologically conjugate to $\sigma|_{\Sigma_T}$, a subshift of finite type.

Proof. It follows from the fact that $K_1(c) \subset W_I$ and, hence, $\Sigma \cap K_1(c) = \emptyset$. ∎

Finally, we are ready to prove Theorem 1.8.2.

Proof of Theorem 1.8.2. From Case $\tilde{\text{g}}$ it follows immediately statement (a) (see also Figure 1.7.8). Let $\alpha_k = \inf\{b \in A_k \cup g_k : H_c(X_1(c) = U(c)\}$. Since $H_c(X_1(c)) > U(c)$ for $b \in A_k$ and $u(c) \in \mathrm{Int}\Delta$

we have $\alpha_k \in g_k$. From Lemma 1.8.3 it follows statement (c). Let $\beta_k = \sup\{b \in A_k \cup g_k : H_c(X_1(c)) = U(c)\}$. Clearly, $\alpha_k \leq \beta_k$. From Theorem 1.8.10 and Corollary 1.8.11 it follows statement (b). If $\alpha_k \neq \beta_k$, set $E_k = \{b \in (\alpha_k, \beta_k] : H_c(X_1(c)) \geq U(c)\}$. Since $H_c(X_1(c)) - U(c)$ depends continuously on b we have that E_k is closed in $(\alpha_k, \beta_k]$. From statements (b) and (c) it follows (d). ∎

1.9 Concluding Remarks

In view of the results given in the previous sections we can conclude that only a small part of the information gotten in the study of the one dimensional system can be taken satisfactorily to the two dimensional one.

On the other hand we would have desired to get the following result: *For each b_n^k the map $P_{b_n^k}$ has one pair of periodic points of period $2q + \alpha_n^k$ a sink and a saddle.* We note that if the above result were true then, from Theorem 1.2.2 the map $P_{b_n^k}$ would have two pairs of periodic points, one of these pairs has period $2q-1$ and the other has period $2q+\alpha_n^k$. Each of these pairs consist on a sink and a saddle.

From Theorem 1.2.2 we get the existence of the saddle point but, unfortunately, we cannot guarantee the existence of the corresponding sink. Namely, we used the piecewise linear model in the proof of Theorem 1.2.2 but Theorem 1.8.2 tells us that for $c \in g_k$ the family h_c does not have any sink in Σ. We believe that if one wants to prove the existence of these attractors one has to use, essentially, two dimensional techniques.

Chapter 2

The characterization of the kneading pair for a class of circle maps

2.1 Introduction

The goal of this chapter is to characterize a set of symbolic sequences which is the equivalent at a symbolic level of the class $\mathcal{A}$ of maps which are liftings of degree one circle maps with a single maximum and a single minimum. The study of these maps arise naturally in different contexts in dynamical systems. For instance, a three parameter family of maps from $\mathcal{A}$ has been introduced by Levi [30] and used in Chapter 1 to study the Van der Pol equation. On the other hand, the standard maps family defined as $F_{b,w}(x) = x + w + b\dfrac{\sin(2\pi x)}{2\pi}$ where $x, w \in \mathbf{R}$ and $b \in (0, \infty)$ belongs to the class $\mathcal{A}$ for all $b > 1$. The study of this two parameter family displays a correspondence with periodically forced chick-heart cells (see [12]) and the plot of the phase-locking zones as a function of b and w gives the Arnold tongues (see [7]). Also, the class $\mathcal{A}$ is relevant in the description of the transition to chaos for contracting annulus maps.

We shall use the extension of the Kneading Theory of Milnor and Thurston [20] given by Alsedà and Mañosas [5] to maps from $\mathcal{A}$. The key point of this Kneading Theory is a suitable definition of itinerary. With this notion they extended some basics results of the kneading theory for unimodal maps to the class $\mathcal{A}$. Moreover, they showed that for a map from class $\mathcal{A}$, the set

Figure 2.1.1: An example of a map F from class $\mathcal{A}$.

of itineraries of all points can be characterized by the *kneading pair*; that is, the itinerary of the maximum and of the minimum. Thus, in the study of bifurcations of parametrized families in $\mathcal{A}$ this two sequences play a crucial role. This is our motivation to characterize the set of kneading pairs of maps from $\mathcal{A}$. This will be done in the main result of this chapter.

Now, we introduce the class $\mathcal{A}$ of maps we study (see Figure 2.1.1). We say that $F \in \mathcal{A}$ if:

(1) $F \in \mathcal{L}$ (that is, $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$),

(2) There exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

We note that every map $F \in \mathcal{A}$ has a unique local maximum and a unique local minimum in $[0,1)$. To define the class $\mathcal{A}$ we restricted ourselves to the case in which F has the minimum at 0. Since each map from $\mathcal{L}$ is conjugated by a translation to a map from $\mathcal{L}$ having the minimum at 0, the fact that in (2) we fix that F has a minimum in 0 is not restrictive.

The chapter is organized as follows. In Section 2.2 we give a survey of the kneading theory developed by Alsedà and Mañosas [5]. In Section 2.3 we state and prove the main result of this chapter. Finally, in Section2.4 we make some concluding remarks.

2.2 A survey on the Kneading Theory for circle maps

We start by introducing some notation. In what follows we shall denote the integer part function by $E(\cdot)$. For $x \in \mathbf{R}$ we set $D(x) = x - E(x)$. For $F \in \mathcal{A}$ we define *the height of* F, as

$$p_{_F} = \begin{cases} E(F(c_{_F})) - E(F(0)) & \text{if } F(c_{_F}) \notin \mathbf{Z}, \\ E(F(c_{_F})) - E(F(0)) - 1 & \text{if } F(c_{_F}) \in \mathbf{Z}. \end{cases}$$

If $A \subset \mathbf{R}$ and $x \in \mathbf{R}$, we shall write $x + A$ or $A + x$ to denote the set $\{x + a : a \in A\}$. Also, if $B \subset \mathbf{R}$ we shall write $A + B$ to denote the set $\{a + b : a \in A,\ b \in B\}$. Let $F \in \mathcal{A}$ be with height p. Then the points of the set $\Delta(F) = \mathbf{Z} \cup F^{-1}(\mathbf{Z}) \cup c_{_F} + \mathbf{Z}$ will be called the *turning points* of F. We note that if $x \in \Delta(F)$ then $x + \mathbf{Z} \subset \Delta(F)$.

Now, we define the notion of address we are going to use. For $F \in \mathcal{A}$ and $x \in \mathbf{R}$ let

$$s(x) = \begin{cases} R & \text{if } D(x) > c_F, \\ C & \text{if } D(x) = c_F, \\ L & \text{if } D(x) \in (0, c_F), \\ M & \text{if } D(x) = 0, \end{cases}$$

and $d(x) = E(F(x)) - E(x)$.

Now, we define the *reduced itinerary of* x, denoted by $\underline{\widehat{I}}_{_F}(x)$, as follows. For $i \in \mathbf{N}$, set $s_i = s(F^i(x))$ and $d_i = d(F^{i-1}(x))$. Then $\underline{\widehat{I}}_{_F}(x)$ is defined by

$$\begin{cases} d_1^{s_1} d_2^{s_2} \dots & \text{if } s_i \in \{L, R\} \text{ for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \dots d_n^{s_n} & \text{if } s_n \in \{M, C\} \text{ and } s_i \in \{L, R\} \text{ for all } i \in \{1, \dots, n-1\}. \end{cases}$$

Since $F \in \mathcal{L}$ we have that $\underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x + k)$ for all $k \in \mathbf{Z}$.

Let $x, y \in \mathbf{R}$ be such that $D(x) \neq D(y)$. We say that x and y are *conjugate* if and only if $F(D(x)) = F(D(y))$. Note that if x and y are conjugate then they have the same reduced itinerary.

Let $\mathcal{S} = \{M, L, C, R\}$ and let $\underline{\alpha} = \alpha_0\alpha_1 \dots$ be a sequence of elements $\alpha_i = d_i^{s_i}$ of $\mathbf{Z} \times \mathcal{S}$. We say that $\underline{\alpha}$ is *admissible* if one of the two following conditions is satisfied:

(1) $\underline{\alpha}$ is infinite, $s_i \in \{L, R\}$ for all $i \geq 1$ and there exists $k \in \mathbf{N}$ such that $|d_i| \leq k$ for all $i \geq 1$.

(2) $\underline{\alpha}$ is finite of length n, $s_n \in \{M, C\}$ and $s_i \in \{L, R\}$ for all $i \in \{1, \ldots, n-1\}$.

Notice that any itinerary is an admissible sequence. Now we shall introduce some notation for admissible sequences (and hence for reduced itineraries).

The cardinality of an admissible sequence $\underline{\alpha}$ will be denoted by $|\underline{\alpha}|$ (if $\underline{\alpha}$ is infinite we write $|\underline{\alpha}| = \infty$).

We denote by S the shift operator which acts on the set of admissible sequences of length greater than one as follows : $S(\underline{\alpha}) = \alpha_2\alpha_3\ldots$ if $\underline{\alpha} = \alpha_1\alpha_2\alpha_3\ldots$. We will write S^k for the k-th iterate of S. Obviously S^k is only defined for admissible sequences of length greater than k. Clearly, for each $x \in \mathbf{R}$ we have $S^n(\widehat{\underline{I}}_F(x)) = \widehat{\underline{I}}_F(F^n(x))$ if $|\widehat{\underline{I}}_F(x)| > n$.

Let $\underline{\alpha} = \alpha_1\alpha_2\ldots\alpha_n$ and $\underline{\beta} = \beta_1\beta_2\ldots$ be two sequences of symbols in $\mathbf{Z} \times \mathcal{S}$. We shall write $\underline{\alpha}\,\underline{\beta}$ to denote the concatenation of $\underline{\alpha}$ and $\underline{\beta}$ (i. e. the sequence $\alpha_1\alpha_2\ldots\alpha_n\beta_1\beta_2\ldots$). We also shall use the symbols $\underline{\alpha}^n$ to denote $\overbrace{\underline{\alpha}\,\underline{\alpha}\ldots\,\underline{\alpha}}^{n\ times}$ and $\underline{\alpha}^\infty$ to denote $\underline{\alpha}\,\underline{\alpha}\ldots$.

Let $\underline{\alpha} = \alpha_1\alpha_2\ldots\alpha_n$, be an admissible sequence. Set $\alpha_i = d_i^{s_i}$ for $i = 1, 2, \ldots, n$. We say that $\underline{\alpha}$ is *even* if $\mathrm{Card}\{i \in \{1, \ldots, n\} | s_i = R\}$ is even. Otherwise we say that $\underline{\alpha}$ is *odd*.

Now we endow the set of admissible sequences with a total ordering. First set $M < L < C < R$. Then we extend this ordering to $\mathbf{Z} \times \mathcal{S}$ lexicographically. That is, we write $d^s < t^m$ if and only if either $d < t$ or $d = t$ and $s < m$. Let now $\underline{\alpha} = \alpha_1\alpha_2\ldots$ and $\underline{\beta} = \beta_1\beta_2\ldots$ be two admissible sequences such that $\underline{\alpha} \neq \underline{\beta}$. Then there exists n such that $\alpha_n \neq \beta_n$ and $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, n-1$. We say that $\underline{\alpha} < \underline{\beta}$ if either $\alpha_1\alpha_2\ldots\alpha_{n-1}$ is even and $\alpha_n < \beta_n$ or $\alpha_1\alpha_2\ldots\alpha_{n-1}$ is odd and $\alpha_n > \beta_n$.

The following result shows that the above ordering of reduced itineraries is, in fact, the ordering of points in $[0, c_F]$.

Proposition 2.2.1 *Let $F \in \mathcal{A}$. Then*

(a) If $x, y \in [0, c_F]$, and $x < y$ then $\widehat{\underline{I}}_F(x) \leq \widehat{\underline{I}}_F(y)$.

(b) If $x, y \in [c_F, 1)$, and $x < y$ then $\widehat{\underline{I}}_F(x) \geq \widehat{\underline{I}}_F(y)$.

Corollary 2.2.2 *Let $F \in \mathcal{A}$. For all $x \in \mathbf{R}$ we have $\widehat{\underline{I}}_F(0) \leq \widehat{\underline{I}}_F(x) \leq \widehat{\underline{I}}_F(c_F)$.*

For a point $x \in \mathbf{R}$ we define the sequences $\underline{\widehat{I}}_F(x^+)$ and $\underline{\widehat{I}}_F(x^-)$ as follows. For each $n \geq 0$ there exists $\delta(n) > 0$ such that $d(F^{n-1}(y))$ and $s(F^n(y))$ take constant values for each $y \in (x, x+\delta(n))$ (resp. $y \in (x-\delta(n), x)$). Denote these values by $d(F^{n-1}(x^+))$ and $s(F^n(x^+))$ (resp. $d(F^{n-1}(x^-))$ and $s(F^n(x^-))$). Then we set $\underline{\widehat{I}}_F(x^+) = d(x^+)^{s(F(x^+))} d(F(x^+))^{s(F^2(x^+))} \dots$ and $\underline{\widehat{I}}_F(x^-) = d(x^-)^{s(F(x^-))} d(F(x^-))^{s(F^2(x^-))} \dots$. Clearly, $\underline{\widehat{I}}_F(x^+)$ and $\underline{\widehat{I}}_F(x^-)$ are infinite admissible sequences and, $\underline{\widehat{I}}_F(x^+) = \underline{\widehat{I}}_F((x+k)^+)$ and $\underline{\widehat{I}}_F(x^-) = \underline{\widehat{I}}_F((x+k)^-)$ for all $k \in \mathbf{Z}$. Moreover, if $x \notin \mathbf{Z}$ and $|\underline{\widehat{I}}_F(x)| = \infty$ then $\underline{\widehat{I}}_F(x^-) = \underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x^+)$.

Let $F \in \mathcal{A}$. The pair $(\underline{\widehat{I}}_F(0^+), \underline{\widehat{I}}_F(c_F^-))$ will be called *the kneading pair of* F and will be denoted by $\mathcal{K}(F)$. Let $\mathcal{AD}$ denote the set of all infinite admissible sequences. Then for each $F \in \mathcal{A}$ we have that $\mathcal{K}(F) \in \mathcal{AD} \times \mathcal{AD}$.

Let $\underline{\alpha} = d_1^{s_1} \alpha_2 \dots$, be an admissible sequence. We will denote by $\underline{\alpha}'$ the sequence $(d_1 + 1)^{s_1} \alpha_2 \dots$.

Let $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ be admissible sequences such that $\underline{\beta} < \underline{\gamma}$. We will say that $\underline{\alpha}$ is *quasidominated* by $\underline{\beta}$ and $\underline{\gamma}$ if and only if the following statements hold:

(1) $\underline{\beta} \leq S^n(\underline{\alpha}) \leq \underline{\gamma}$ for all $n \in \{0, 1, \dots, |\underline{\alpha}| - 1\}$.

(2) If for some $n \in \{1, 2, \dots, |\underline{\alpha}| - 1\}$ we have $S^n(\underline{\alpha}) = d^R$ then $S^{n+1}(\underline{\alpha}) \geq \underline{\beta}'$.

We will say that $\underline{\alpha}$ is *dominated* by $\underline{\beta}$ and $\underline{\gamma}$ if and only if (1) and (2) hold with strict inequalities.

Let $F \in \mathcal{A}$. We say that $\underline{\alpha}$ is *quasidominated* (respectively *dominated*) by F if $\underline{\alpha}$ is quasidominated (respectively dominated) by $\underline{\widehat{I}}_F(0^+)$ and $\underline{\widehat{I}}_F(c_F^-)$.

We note that for $F \in A$ we have $d(F(0^+)) = d(F(0^-)) - 1$. Hence, $(\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(0^-)$.

The next result characterizes the set of reduced itineraries of a map $F \in \mathcal{A}$ in terms of the kneading pair.

Proposition 2.2.3 *Let $F \in \mathcal{A}$. Then the following hold.*

(a) Let $x \in (0,1)$ with $x \neq c_F$. Then $\underline{\widehat{I}}_F(x)$ is quasidominated by F.

(b) Let $\underline{\alpha}$ be an admissible sequence dominated by F. Then there exists $x \in [0, c_F]$ such that $\underline{\widehat{I}}_F(x) = \underline{\alpha}$.

The following result will be used in the study of the kneading pair.

Corollary 2.2.4 *Let $F \in \mathcal{A}$. Then the following hold.*

(a) Let $x \in (0, c_F)$. Then $\underline{\widehat{I}}_F(0^+) \le \underline{\widehat{I}}_F(x) \le \underline{\widehat{I}}_F(c_F^-)$.

(b) Let $x \in (c_F, 1)$. Then $\underline{\widehat{I}}_F(0^-) \le \underline{\widehat{I}}_F(x) \le \underline{\widehat{I}}_F(c_F^-)$.

Let $\underline{\alpha} = \alpha_1\alpha_2\alpha_3\dots$ be an admissible sequence, we say that $\underline{\alpha}$ is *periodic of period* n if $S^n(\underline{\alpha}) = \underline{\alpha}$ and $S^i(\underline{\alpha}) \neq \underline{\alpha}$ for $i = 1, 2, \dots, n-1$. We note that if $\underline{\alpha}$ is a periodic sequence of period n, then $|\underline{\alpha}| = \infty$ and there exist $\alpha_1, \dots, \alpha_n \in \mathbf{Z} \times \mathcal{S}$ such that $\underline{\alpha} = (\alpha_1 \dots \alpha_n)^\infty$. We also note that if x is a periodic (mod 1) point of F such that $|\underline{\widehat{I}}_F(x)| = \infty$, then $\underline{\widehat{I}}_F(x)$ is periodic (recall that $S^n(\underline{\widehat{I}}_F(x)) = \underline{\widehat{I}}_F(F^n(x)))$ but their periods are not necessarily equal.

2.3 The characterization of the kneading pair

In the preceding section, to each map $F \in \mathcal{A}$, we assigned a pair from $\mathcal{AD} \times \mathcal{AD}$; namely the kneading pair. This pair is the symbolic version of the map because it characterizes the set of itineraries that F can have (see Proposition 2.2.3). The aim of this section is to characterize the pairs in $\mathcal{AD} \times \mathcal{AD}$ that can occur as a kneading pair of a map from $\mathcal{A}$. To prove a first result in this direction we need some preliminary definitions and results.

Let $k \in \mathbf{Z}$. We denote by $(\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}}$ the set of sequences $\underline{\alpha} = d_1^{s_1} d_2^{s_2} \dots \in (\mathbf{Z} \times \mathcal{S})^{\mathbf{N}}$ such that $\mid d_i \mid \le k$ for all $i \ge 1$. Let $\underline{\alpha} = d_1^{s_1} d_2^{s_2} \dots$ and $\underline{\beta} = t_1^{r_1} t_2^{r_2} \dots$ be two sequences in $(\mathbf{Z} \times S)_k^{\mathbf{N}}$. We consider in $(\mathbf{Z} \times \mathcal{S})^{\mathbf{N}}$ the topology defined by the metric $d(\underline{\alpha}, \underline{\beta}) = \sum_{i=0}^{\infty} 2^{-i} d(d_i^{s_i} t_i^{r_i})$ where

$$d(d_i^{s_i} t_i^{r_i}) = \begin{cases} 1 & \text{if } d_i^{s_i} \neq t_i^{r_i}, \\ 0 & \text{if } d_i^{s_i} = t_i^{r_i}. \end{cases}$$

With this topology, $(\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}}$ is a compact metric space and the shift transformation $S : (\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}} \longrightarrow (\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}}$ defined by $S(d_1^{s_1} d_2^{s_2} \dots) = d_2^{s_2} d_3^{s_3} \dots$ is continuous. Moreover, we can extend in a natural way the ordering defined for the admissible sequences to the sequences from $(\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}}$.

Let $\underline{\alpha}, \underline{\beta}$ be to admissible sequences such that $\underline{\alpha}' \le \underline{\beta}$. Let $\mathcal{AD}_{\underline{\alpha},\underline{\beta}}$ denote the set of all admissible sequences quasidominated by $\underline{\alpha}$ and $\underline{\beta}$ union $\left\{\underline{\alpha}, \underline{\beta}, \underline{\alpha}'\right\}$. Now, we define $\Gamma_{\underline{\alpha},\underline{\beta}} : \mathcal{AD}_{\underline{\alpha},\underline{\beta}} \longrightarrow (\mathbf{Z} \times \mathcal{S})_k^{\mathbf{N}}$ as follows. If $\mid \underline{\gamma} \mid = \infty$ then $\Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \underline{\gamma}$. If $\underline{\gamma}$ is finite and ends with C, then the sequence associated is the following

$$\Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \begin{cases} \underline{\gamma}\,\underline{\beta} & \text{if } \underline{\beta} \text{ is infinite,} \\ \underline{\gamma}(\underline{\beta})^{\infty} & \text{if } \underline{\beta} \text{ is finite and ends with } C, \\ \underline{\gamma}\,\underline{\beta}\,\underline{\alpha} & \text{if } \underline{\beta} \text{ is finite and ends with } M \text{ and } \underline{\alpha} \text{ is infinite,} \\ \underline{\gamma}\,\underline{\beta}(\underline{\alpha})^{\infty} & \text{if } \underline{\beta} \text{ is finite and ends with } M \text{ and } \underline{\alpha} \text{ is finite} \\ & \text{and ends with } M, \\ \underline{\gamma}(\underline{\beta}\,\underline{\alpha})^{\infty} & \text{otherwise.} \end{cases}$$

If $\underline{\gamma}$ ends with M we proceed similarly with the roles of $\underline{\alpha}$ and $\underline{\beta}$, and M and C interchanged. We note that the map $\Gamma_{\underline{\alpha},\underline{\beta}}$ preserves the ordering of the sequences and that $S^n \circ \Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \Gamma_{\underline{\alpha},\underline{\beta}} \circ S^n(\underline{\gamma})$ for all $n \in \left\{0, 1, \dots \mid \underline{\gamma} \mid -1\right\}$.

The following proposition gives the main properties of the kneading pair.

Proposition 2.3.1 *For each $F \in \mathcal{A}$ we have that $(\widehat{\underline{I}}_F(0^+))' \leq \widehat{\underline{I}}_F(c_F^-)$ and $\widehat{\underline{I}}_F(0^+)$ and $\widehat{\underline{I}}_F(c_F^-)$ are quasidominated by F.*

Proof. The first statement follows from Corollary 2.2.4(b) and the fact $\widehat{\underline{I}}_F(0^+) = (\widehat{\underline{I}}_F(0^-))'$. Now, we prove the second statement. Denote by Γ_F the map $\Gamma_{\widehat{\underline{I}}_F(0^+),\widehat{\underline{I}}_F(c_F^-)}$. From the part of the proposition already proved it is defined. It is not difficult to show that $\widehat{\underline{I}}_F(x^+) = \lim_{\substack{y \to x \\ y > x}} \Gamma_F(\widehat{\underline{I}}_F(y))$ and $\widehat{\underline{I}}_F(x^-) = \lim_{\substack{y \to x \\ y < x}} \Gamma_F(\widehat{\underline{I}}_F(y))$. Now, we consider several cases. Assume first that $S^n(\widehat{\underline{I}}_F(0^+)) = d^L \dots$ (respectively $S^n(\widehat{\underline{I}}_F(c_F^-)) = d^L \dots$) for some $n \geq 0$. Then there exists $x \in (0, c_F)$ close to 0 (respectively c_F) such that $D(F^{n+1}(x)) \in (0, c_F)$ and $\widehat{\underline{I}}_F(x)$ coincides with $\widehat{\underline{I}}_F(0)$ (resp. $\widehat{\underline{I}}_F(c_F)$) in the first $n+1$ symbols. Then from Corollary 2.2.4(a) we have that $\widehat{\underline{I}}_F(0^+) \leq \widehat{\underline{I}}_F(F^{n+1}(x)) \leq \widehat{\underline{I}}_F(c_F^-)$. Thus $\widehat{\underline{I}}_F(0^+) \leq \Gamma_F(\widehat{\underline{I}}_F(F^{n+1}(x))) \leq \widehat{\underline{I}}_F(c_F^-)$. Since $\Gamma_F(\widehat{\underline{I}}_F(F^{n+1}(x))) = \Gamma_F(S^{n+1}(\widehat{\underline{I}}_F(x))) = S^{n+1}(\Gamma_F(\widehat{\underline{I}}_F(x)))$, letting x tend to 0 from the right we get $\widehat{\underline{I}}_F(0^+) \leq S^n(\widehat{\underline{I}}_F(0^+)) \leq \widehat{\underline{I}}_F(c_F^-)$ (respectively letting x tend to c_F from the left we get $\widehat{\underline{I}}_F(0^+) \leq S^n(\widehat{\underline{I}}_F(c_F^-)) \leq \widehat{\underline{I}}_F(c_F^-)$). Now, assume that $S^n(\widehat{\underline{I}}_F(0^+)) = d^R \dots$ (respectively $S^n(\widehat{\underline{I}}_F(c_F^-)) = d^R \dots$) for some $n \geq 0$. There exists $x \in (0, c_F)$ close to 0 (respectively c_F) such that $D(F^{n+1}(x)) \in (c_F, 1)$ and $\widehat{\underline{I}}_F(x)$ coincides with $\widehat{\underline{I}}_F(0)$ (resp. $\widehat{\underline{I}}_F(c_F)$) in the first $n+1$ symbols. From Corollary 2.2.4(b) we have that $\widehat{\underline{I}}_F(0^-) \leq \widehat{\underline{I}}_F(F^{n+1}(x)) \leq \widehat{\underline{I}}_F(c_F^-)$. Then, in a similar way as above we can show that $\widehat{\underline{I}}_F(0^-) \leq S^{n+1}(\widehat{\underline{I}}_F(0^+)) \leq \widehat{\underline{I}}_F(c_F^-)$ (respectively $\widehat{\underline{I}}_F(0^-) \leq S^{n+1}(\widehat{\underline{I}}_F(c_F^-)) \leq \widehat{\underline{I}}_F(c_F^-)$) and the proposition follows. ∎

To deal with the properties of the kneading pair given by the above proposition we introduce the following notions.

Let $\underline{\alpha} \in \mathcal{AD}$. We say that $\underline{\alpha}$ is *minimal* (respectively *maximal*) if and only if $\underline{\alpha} \leq S^n(\underline{\alpha})$ (respectively $\underline{\alpha} \geq S^n(\underline{\alpha})$) for all $n \in \{1, 2, \dots \mid \underline{\alpha} \mid -1\}$.

To characterize the pairs in $\mathcal{AD} \times \mathcal{AD}$ that can occur as a kneading pair of a map from $\mathcal{A}$ we will define a subset $\mathcal{E}$ of $\mathcal{AD} \times \mathcal{AD}$ and, afterwards, we shall prove that this set consists of all kneading pairs of maps from $\mathcal{A}$. To this end we introduce the following notation.

We will denote by $\mathcal{E}^*$ the set of all pairs $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{AD} \times \mathcal{AD}$ such that $\underline{\nu}_1$ is minimal, $\underline{\nu}_2$ is maximal, $\mid \underline{\nu}_1 \mid = \mid \underline{\nu}_2 \mid = \infty$ and the following conditions hold:

(1) $\underline{\nu}_1' < \underline{\nu}_2$.

(2) $\underline{\nu}_1 \leq S^n(\underline{\nu}_2)$ and $S^n(\underline{\nu}_1) \leq \underline{\nu}_2$ for all $n > 0$.

(3) If for some $n \geq 0$, $S^n(\underline{\nu}_i) = d^R \dots$, then $S^{n+1}(\underline{\nu}_i) \geq \underline{\nu}_1'$ for $i \in \{1, 2\}$.

Let $a \in \mathbf{R}$. We set $\epsilon_i(a) = E(ia) - E((i-1)a)$ and $\delta_i(a) = \tilde{E}(ia) - \tilde{E}((i-1)a)$, where $\tilde{E} : \mathbf{R} \longrightarrow \mathbf{Z}$ is defined as follows

$$\tilde{E}(x) = \begin{cases} E(x) & \text{if } x \notin \mathbf{Z}, \\ x - 1 & \text{if } x \in \mathbf{Z}. \end{cases}$$

Also, we set

$$\underline{\widehat{I}}_\epsilon(a) = \epsilon_1(a)^L \epsilon_2(a)^L \dots \epsilon_n(a)^L \dots$$

and

$$\underline{\widehat{I}}_\delta(a) = \delta_1(a)^L \delta_2(a)^L \dots \delta_n(a)^L \dots$$

Let $\underline{\widehat{I}}^*_\epsilon(a) = (\underline{\widehat{I}}_\epsilon(a))'$ and let $\underline{\widehat{I}}^*_\delta(a)$ denote the sequence that satisfies $(\underline{\widehat{I}}^*_\delta(a))' = \underline{\widehat{I}}_\delta(a)$. Let $a = p/q$ with $(p, q) = 1$. We denote by $\underline{\widehat{I}}_R(a)$ the sequence $(\delta_1(a)^L \dots \delta_{q-1}(a)^L \delta_q(a)^R)^\infty$ and let $\underline{\widehat{I}}^*_R(a)$ be the sequence that satisfies $(\underline{\widehat{I}}^*_R(a))' = \underline{\widehat{I}}_R(a)$. Now we set

$$\mathcal{E}_a = \begin{cases} \{(\underline{\widehat{I}}_\epsilon(a), \underline{\widehat{I}}^*_\epsilon(a)), (\underline{\widehat{I}}^*_\delta(a), \underline{\widehat{I}}_\delta(a)), (\underline{\widehat{I}}^*_R(a), \underline{\widehat{I}}_R(a))\} & \text{if } a = p/q \in \mathbf{Q}, \text{ with } (p, q) = 1, \\ \left\{(\underline{\widehat{I}}^*_\delta(a), \underline{\widehat{I}}_\delta(a))\right\} & \text{if } a \notin \mathbf{Q}. \end{cases}$$

Finally we denote by $\mathcal{E}$ the set $\mathcal{E}^* \cup (\cup_{a\in\mathbf{R}}\mathcal{E}_a)$. The set $\cup_{a\in\mathbf{R}}\mathcal{E}_a$ is the boundary of $\mathcal{E}$ while $\mathcal{E}^*$ is its interior (with respect to the topology introduced above).

The following result characterizes the kneading pairs of the maps from class $\mathcal{A}$ and is the main result of this chapter. It will be proved in the next two subsections.

Theorem 2.3.2 *For $F \in \mathcal{A}$ we have that $K(F) \in \mathcal{E}$. Conversely, for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}$ there exists $F \in \mathcal{A}$ such that $K(F) = (\underline{\nu}_1, \underline{\nu}_2)$.*

2.3.1 Proof of the first statement of Theorem 2.3.2

We start by noting that if for $F \in \mathcal{A}$ we have $(\widehat{\underline{I}}_F(0^+))' < \widehat{\underline{I}}_F(c_F^-)$ then, in view of Proposition 2.3.1 and the definition of $\mathcal{E}$, $K(F) \in \mathcal{E}^* \subset \mathcal{E}$. Thus, to prove the first statement of Theorem 2.3.2, we only have to prove that if $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$, then $K(F) \in \mathcal{E}_a$ for some $a \in \mathbf{R}$.

Before starting the proof of this fact we will study the basic properties of the sequences $\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)$ and $\widehat{\underline{I}}^*_\delta(a)$. The following results are due to Alsedà and Mañosas (see [5]).

Lemma 2.3.3 *Let $a \in \mathbf{R}$. If $a \notin \mathbf{Z}$ then $\delta_1(a) = \epsilon_1(a) + 1$. Furthermore, if $a \notin \mathbf{Q}$ then $\delta_i(a) = \epsilon_i(a)$ for all $i > 1$. That is, $\widehat{\underline{I}}^*_\delta(a) = \widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(a) = \widehat{\underline{I}}^*_\epsilon(a)$. If $a = p/q$ with $(p,q) = 1$ and $q > 1$ then $\epsilon_i(a) = \delta_i(a)$ for $i = 2, \ldots, q-1$, $\delta_q(a) = \epsilon_q(a) - 1$ and, $\epsilon_{i+q}(a) = \epsilon_i(a)$ and $\delta_{i+q}(a) = \delta_i(a)$ for all $i \in \mathbf{N}$.*

Theorem 2.3.4 *Let $F \in \mathcal{A}$. Then $R_F = [a,b]$ if and only if $\widehat{\underline{I}}^*_\delta(a) \le \widehat{\underline{I}}_F(0^+) \le \widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(b) \le \widehat{\underline{I}}_F(c_F^-) \le \widehat{\underline{I}}^*_\epsilon(b)$.*

Remark 2.3.5 Since $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(0^-)$, by the definition of the sequences $\widehat{\underline{I}}^*_\delta(a)$, $\widehat{\underline{I}}_\epsilon(a)$, $\widehat{I}_\delta(a)$ and $\widehat{\underline{I}}^*_\epsilon(a)$ we have that $\widehat{\underline{I}}^*_\delta(a) \le \widehat{\underline{I}}_F(0^+) \le \widehat{\underline{I}}_\epsilon(a)$ is equivalent to $\widehat{\underline{I}}_\delta(a) \le \widehat{\underline{I}}_F(0^-) \le \widehat{\underline{I}}^*_\epsilon(a)$. □

In view of the above theorem and remark we get:

Lemma 2.3.6 *Let $F \in \mathcal{A}$ be such that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. Then R_F is degenerate to a point.*

The next lemma characterizes at the symbolic level the maps $F \in \mathcal{A}$ satisfying that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. It follows inmediately from the definitions.

Lemma 2.3.7 *Assume that $\widehat{\underline{I}}_F(0^+) = d_{1,1}^{s_{1,1}} \ldots$, $\widehat{\underline{I}}_F(c_F^-) = d_{1,2}^{s_{1,2}} \ldots$ and $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. Then, $d_{1,1} + 1 = d_{1,2}$, and $d_{n,1} = d_{n,2}$ and $s_{n-1,1} = s_{n-1,2}$ for all $n > 1$.*

From Lemma 4.4 of [5], the proof of Theorem 2 of [10] and Lemma 1.6.3 we have the following result.

Proposition 2.3.8 *Let $F \in \mathcal{A}$ be such that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$ and $R_F = \{a\}$ with $a \in \mathbf{R}$. Then the map F has a twist orbit P of rotation number a such that $P \cap [0,1) \subset [0, c_F]$ and $F|_P = F_r|_P$. Moreover, if $a = p/q \in \mathbf{Q}$ with $(p,q) = 1$, then P is a twist periodic orbit of period q. Set $\mu_P = \min\ P \cap [0, c_F]$ and $\nu_P = \max\ P \cap [0, c_F]$. Then the following statements hold:*

(a) $\{0, c_F\} \not\subset \{\mu_P, \nu_P\}$.

(b) Assume that $\nu_P \neq c_F$. If $\mu_P \neq 0$ then $\widehat{\underline{I}}_F(\mu_P) = \underline{I}_\epsilon(a)$. Otherwise

$$\widehat{\underline{I}}_F(0) = \epsilon_1(a)^L \ldots \epsilon_{q-1}(a)^L \epsilon_q(a)^M$$

and $\widehat{\underline{I}}_F(0^+) = \widehat{\underline{I}}_\epsilon(a)$.

(c) Assume hat $\mu_P \neq 0$. If $\nu_P \neq c_F$ then $\widehat{\underline{I}}_F(\nu_P) = \underline{I}_\delta(a)$. Otherwise

$$\widehat{\underline{I}}_F(c_F) = \delta_1(a)^L \ldots \delta_{q-1}(a)^L \delta_q(a)^C$$

and $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_\delta(a)$.

Now we are ready to prove the result we are looking for.

Proposition 2.3.9 *Let $F \in \mathcal{A}$ be such that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. Then there exists $a \in \mathbf{R}$ such that $R_F = \{a\}$ and $K(F) \in \mathcal{E}_a$.*

Proof. From Lemma 2.3.6 we have that $R_F = \{a\}$. Assume that $a \notin \mathbf{Q}$. From Lemma 2.3.3 and Theorem 2.3.4 we see that $K(F) \in \mathcal{E}_a$. Now, assume that $a = p/q$ with $(p,q) = 1$. Let P be the twist periodic orbit of period q and rotation number p/q given from Proposition 2.3.8. If $\mu_P = 0$, from Proposition 2.3.8(a) we have $\nu_P \neq c_F$ (here we use the notation from the statement of Proposition 2.3.8). Therefore, from Proposition 2.3.8(b), $\widehat{\underline{I}}_F(0^+) = \widehat{\underline{I}}_\epsilon(a)$. Hence, $\widehat{\underline{I}}_F(c_F^-) = (\widehat{\underline{I}}_\epsilon(a))' = \widehat{\underline{I}}^*_\epsilon(a)$ and so, $K(F) \in \mathcal{E}_a$. If $\nu_P = c_F$ then, as above, $\mu_P \neq 0$. By Proposition 2.3.8(c), $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_\delta(a)$ and, consequently, $\widehat{\underline{I}}_F(0^+) = \widehat{\underline{I}}^*_\delta(a)$. So, $K(F)$ also belongs to $\mathcal{E}_a$. We are left with the case $\mu_P \neq 0$ and $\nu_P \neq 0$.

We recall that $F_r(x) = \sup\{F(y) : y \le x\}$. Hence, for all $y \in P$ and $z \le y$ we have

$$F(z) \le F_r(z) \le F_r(y) = F(y).$$

Let $G = F^q - p$. Then $G(z) \le G(y) = y$ for all $y \in P$ and $z \le y$. Set $P = \{x_i\}_{i\in\mathbf{Z}}$ with $x_i < x_j$ if and only if $i < j$, and $x_0 = \mu_P$. Then, since P has period q we have $x_{q-1} = \nu_P$ and $x_{i+q} = x_i + 1$ for each $i \in \mathbf{Z}$. From Lemma 1.6.1 we get $F(x_i) = x_{i+p}$ for each $i \in \mathbf{Z}$. Thus, since $P \cap [0,1] \subset [0, c_F)$, each interval $[x_i, x_{i+1}]$ is mapped homeomorphically (preserving ordering) into $[x_{i+p}, x_{i+1+p}]$ for $i = 0, 1, \ldots, q-2$. On the other hand $[x_{q-1}, x_q]$ contains $\{c_F, 1\}$ in its interior (recall that $x_{q-1} = \nu_P \ne c_F$ and $x_q = \mu_P + 1 \ne 1$). Since $F|_{[x_{q-1}, c_F]}$ is increasing and $c_F < x_q$ we obtain $x_{q-1+p} = F(x_{q-1}) \le F(z) \le F(c_F) \le F(x_q) = x_{q+p}$ for each $z \in [x_{q-1}, c_F]$. Since $(p,q) = 1$, for each $i \in \{1, 2, \ldots, q-1\}$, we have $ip \not\equiv 0 \pmod{q}$. Therefore, $q - 1 + ip \ne q - 1 + mq$ with $m \in \mathbf{Z}$ and so, $x_{q-1+ip} \ne x_{q-1} + m$. Consequently, $F|_{[x_{q-1+ip}, x_{q+ip}]}$ is strictly increasing for $i = 1, 2, \ldots, q-1$. Therefore, for each $z \in [x_{q-1}, c_F]$, $G(z) \in [x_{q-1+qp} - p, G(c_F)] = [x_{q-1}, G(c_F)] \subset [x_{q-1}, x_q]$. Moreover, $G|_{[x_{q-1}, c_F]}$ is strictly increasing. By Proposition 2.3.8(c) we see that

$$\underline{\widehat{I}}_F(x_{q-1}) = \underline{\widehat{I}}_\delta(a) = (\delta_1(a)^L \ldots \delta_{q-1}(a)^L \delta_q(a)^L)^\infty.$$

So, from above it follows that, for each $z \in [x_{q-1}, c_F]$,

$$\underline{\widehat{I}}_F(z) = \delta_1(a)^L \ldots \delta_{q-1}(a)^L {d_q}^{s(G(z))} \underline{\widehat{I}}_F(G(z))$$

where

$$d_q = \begin{cases} \delta_q(a) & \text{if } G(z) < 1, \\ \delta_q(a) + 1 & \text{otherwise,} \end{cases}$$

(recall that $\underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x+m)$ for each $m \in \mathbf{Z}$). Now we consider three cases.

Case 1: $G(c_F) \in [x_{q-1}, c_F]$ (see Figure 2.3.2). Then $G([x_{q-1}, c_F]) \subset [x_{q-1}, c_F]$ and, if we take $z < c_F$ close enough to c_F, we have

$$\begin{aligned} \underline{\widehat{I}}_F(c_F^-) &= \underline{\widehat{I}}_F(z) = \delta_1(a)^L \ldots \delta_{q-1}(a)^L \delta_q(a)^L \underline{\widehat{I}}_F(G(z)) \\ &= (\delta_1(a)^L \ldots \delta_{q-1}(a)^L \delta_q(a)^L)^2 \underline{\widehat{I}}_F(G^2(z)) \\ &= \ldots \\ &= \underline{\widehat{I}}_\delta(a). \end{aligned}$$

Case 2: $G(c_F) \in (c_F, 1]$ (see Figure 2.3.3). We claim that $G(1) \in (c_F, G(c_F))$. To prove the claim we start showing that $G(1) > c_F$. Otherwise, either $G(1) \in [0, c_F]$ or $G(1) < 0$. In the first case $\underline{\widehat{I}}_F(1^-)$ is of the form $d_1^{s_1} d_2^{s_2} \dots d_q^L \dots$ while $s(G(c_F)^-) = R$. This contradicts the fact that $\underline{\widehat{I}}_F(1^-) = (\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$. In the second case, take $x < 1$ close enough to 1 so that $\underline{\widehat{I}}_F(x)$ and $\underline{\widehat{I}}_F(1^-)$ coincide in the first q symbols and $G(x) < 0$. From above it follows that either $\underline{\widehat{I}}_F(c_F)$ and $\underline{\widehat{I}}_F(c_F^-)$ coincide in the first q symbols when $G(c_F) < 1$ or $\underline{\widehat{I}}_F(c_F)$ and $\underline{\widehat{I}}_F(c_F^-)$ coincide in the first $q-1$ symbols and $d(F^q(c_F)) = d(F^q(c_F^-)) + 1$ when $G(c_F) = 1$. Set

$$s_G = \begin{cases} 1 & \text{if } G(c_F) = 1, \\ 0 & \text{if } G(c_F) < 1. \end{cases}$$

Hence, since $\underline{\widehat{I}}_F(1^-) = (\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$ we have that

$$0 > G(x) \geq E(G(x)) = E(F^q(x)) - p = \left(\sum_{i=1}^q E(F^i(x)) - E(F^{i-1}(x))\right) - p = \\ \left(\sum_{i=1}^q d(F^{i-1}(x))\right) - p = \left(\sum_{i=1}^q d(F^{i-1}(c_F))\right) - p - s_G = E(G(c_F)) - s_G = 0;$$

a contradiction. In short, we have proved that $G(1) > c_F$. Now we prove that $G(1) < G(c_F)$. Note that if $F(1) \leq F(x_{q-1})$ then $G(1) \leq G(x_{q-1}) = x_{q-1} < c_F$. Hence $F(1) > F(x_{q-1})$. So, there exists $z_1 \in [x_{q-1}, c_F)$ such that $F(z_1) = F(1)$. Since $c_F < x_q$ we have $F(1) = F(z_1) \leq F(c_F) < F(x_q)$. Thus, from above it follows that $G(1) < G(c_F)$. This ends the proof of the claim.

From the claim and its proof it follows that $G|_{[c_F,1]}$ is decreasing and $G([c_F, 1]) \subset (c_F, 1]$. We note that from all said above, for each $x \in [c_F, 1]$, there exists $x^* \in [x_{q-1}, c_F]$ such that $G(x^*) = G(x)$. So, $\underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x^*) = \delta_1(a)^L \dots \delta_{q-1}(a)^L \delta_q(a)^R \underline{\widehat{I}}_F(G(x))$. Now take $z < c_F$ close enough to c_F. Since $G^i(z) \in (c_F, 1)$ for each $i \geq 1$, we have

$$\begin{aligned} \underline{\widehat{I}}_F(c_F^-) &= \underline{\widehat{I}}_F(z) = \delta_1(a)^L \dots \delta_{q-1}(a)^L \delta_q(a)^R \underline{\widehat{I}}_F(G(z)) \\ &= (\delta_1(a)^L \dots \delta_{q-1}(a)^L \delta_q(a)^R)^2 \underline{\widehat{I}}_F(G^2(z)) \\ &= \dots \\ &= \underline{\widehat{I}}_R(a). \end{aligned}$$

Case 3: $G(c_F) \in (1, x_q]$ (see Figure 2.3.4). In a similar way as in Case 2 we get that $G(1) \in$

Figure 2.3.2: The map $G|_{[x_{q-1},x_q]}$ in Case 1.

Figure 2.3.3: The map $G|_{[x_{q-1},x_q]}$ in Case 2.

Figure 2.3.4: The map $G|_{[x_{q-1},x_q]}$ in Case 3.

$[1, G(c_F))$. Therefore, $G([1, x_q]) \subset [1, x_q]$. As in Case 2, for each $x \in [1, G(c_F)]$ there exists $x^* \in [x_{q-1}, c_F]$ such that $G(x^*) = G(x)$ and so,

$$\widehat{\underline{I}}_F(x) = \widehat{\underline{I}}_F(x^*) = \delta_1(a)^L \ldots \delta_{q-1}(a)^L(\delta_q(a)+1)^L \widehat{\underline{I}}_F(G(x)).$$

As in the previous two cases, for $z < c_F$ close enough to c_F we have

$$\begin{aligned} \widehat{\underline{I}}_F(c_F^-) &= \widehat{\underline{I}}_F(z) = \delta_1(a)^L \ldots \delta_{q-1}(a)^L(\delta_q(a)+1)^L \widehat{\underline{I}}_F(G(z)) \\ &= (\delta_1(a)^L \ldots \delta_{q-1}(a)^L(\delta_q(a)+1)^L)^2 \widehat{\underline{I}}_F(G^2(z)) \\ &= \ldots \\ &= (\delta_1(a)^L \ldots \delta_{q-1}(a)^L(\delta_q(a)+1)^L)^\infty, \end{aligned}$$

and from Lemma 2.3.3 we get that $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_\epsilon^*(a)$. This ends the proof of the proposition. ∎

Proof of the first statement of Theorem2.3.2. Let $F \in \mathcal{A}$. If $(\widehat{\underline{I}}_F(0^+))' < \widehat{\underline{I}}_F(c_F^-)$ then, as it is been said before, $K(F) \in \mathcal{E}^* \subset \mathcal{E}$ by Proposition 2.3.1. Otherwise, $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$ and, by Proposition 2.3.9, $K(F) \in \mathcal{E}_a$ for some $a \in \mathbf{R}$. ∎

2.3.2 Proof of the second statement of Theorem 2.3.2

The next theorem already proves the second statement of Theorem 2.3.2 in the case $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$.

Theorem 2.3.10 *Let $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$. Then there exists $F \in \mathcal{A}$ such that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$.*

Proof. Set $\underline{\nu}_i = d_{i,1}^{s_{i,1}} d_{i,2}^{s_{i,2}} \ldots d_{i,k}^{s_{i,k}} \ldots$ for $i = 1, 2$. Since $\underline{\nu}_1$ and $\underline{\nu}_2$ are admissible there exist $k_1, k_2 \in \mathbf{Z}$ such that $k_1 \leq |d_{i,j}| \leq k_2$ for all $j \geq 1$ and $i = 1, 2$. Let $F \in \mathcal{A}$ be such that $F(0) = k_1 - 1$ and $F(c_{_F}) = k_2 + 1$. Clearly $K(F) = (((k_1-1)^L)^\infty, (k_2+1)^R((k_1-1)^L)^\infty)$ and $\underline{\nu}_i$ is dominated by F for $i = 1, 2$. From Proposition 2.2.3(b) there exists $x_i \in [0, c_{_F}]$ such that $\underline{\widehat{I}}_{_F}(x_i) = \underline{\nu}_i$ for $i = 1, 2$. By Proposition 2.2.1(a) we have that $0 < x_1 < x_2 < c_{_F}$ because $\underline{\nu}_1 < \underline{\nu}_2$. Let $x_1^*, x_2^* \in [c_{_F}, 1]$ be such that $F(x_1^*) = F(x_1) + 1$ and $F(x_2) = F(x_2^*)$. Thus, $\underline{\widehat{I}}_{_F}(x_1^*) = \underline{\nu}_1'$ and $\underline{\widehat{I}}_{_F}(x_2^*) = \underline{\nu}_2$. Since $\underline{\nu}_1' < \underline{\nu}_2$, from Proposition 2.2.1(b), we obtain that $c_{_F} < x_2^* < x_1^* < 1$. We note that $\underline{\widehat{I}}_{_F}(F^n(x_i)) = S^n(\underline{\widehat{I}}_{_F}(x_i)) = S^n(\underline{\nu}_i)$ for $i = 1, 2$. Therefore, if $F^n(x_i) \in [0, c_{_F}]$ (respectively $F^n(x_i) \in [c_{_F}, 1]$), by Proposition 2.2.1, we see that $F^n(x_i) \in [x_1, x_2]$ (respectively $F^n(x_i) \in [x_2^*, x_1^*]$) because $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$. So,

$$(P_{x_1} \cup P_{x_2}) \cap [0,1] \subset [x_1, x_2] \cup [x_2^*, x_1^*].$$

where P_{x_i} is the (mod. 1) orbit of x_i by F for $i = 1, 2$. Set $K = ((\mathcal{O}(x_1) \cup \mathcal{O}(x_2)) \cap [0,1]) \cup \{x_2^*, x_1^*\}$. Let $\pi : K \longrightarrow K$ be such that $F(x) = \pi(x) + d_x$ for $x \in K$, where $d_x \in \mathbf{Z}$. We note that $\pi(x_i) = \pi(x_i^*)$ for $i = 1, 2$, $d_{x_1} = d_{x_1^*} + 1$ and $d_{x_2} = d_{x_2^*}$.

We choose a map $h : \mathbf{R} \longrightarrow \mathbf{R}$ satisfying the following:

1. $h(x+1) = h(x) + 1$ for all $x \in \mathbf{R}$.

2. $h(0) = 0$.

3. $h|_{\mathbf{R} \setminus (K + \mathbf{Z})}$ is continuous and strictly increasing.

4. If $x \in K$ then $h(x) = \lim\limits_{\substack{y \to x \\ y < x}} h(y) < \lim\limits_{\substack{y \to x \\ y > x}} h(y)$.

Let $g \in \mathcal{L}$ be the nondecreasing map obtained from h^{-1} by extending it to the whole real line. We note g is strictly increasing on $h(\mathbf{R} \setminus (K + \mathbf{Z}))$, for each $x \in K$ there exists a closed interval $[a_x, b_x] \subset (0,1)$ such that $g([a_x, b_x]) = x$ and if $x, x' \in K$ then, $x < x'$ if and only if $b_x < a_{x'}$. In particular, since $(x_2, x_2^*) \cap K = \emptyset$, $h|_{(x_2, x_2^*)}$ is strictly increasing and $g^{-1}(c_{_F}) \in (b_{x_2}, a_{x_2^*})$. Then we define $G \in \mathcal{L}$ as follows:

1. $G|_{[b_{x_1},a_{x_2}]}$ is strictly increasing and $G(a_x) = a_{\pi(x)} + d_x$ and $G(b_x) = b_{\pi(x)} + d_x$ for each $a_x, b_x \in [b_{x_1}, a_{x_2}]$.

2. $G|_{[b_{x_2^*},a_{x_1^*}]}$ is strictly decreasing and $G(a_x) = b_{\pi(x)} + d_x$ and $G(b_x) = a_{\pi(x)} + d_x$ for each $a_x, b_x \in [b_{x_2^*}, a_{x_1^*}]$.

3. $G(g^{-1}(c_F)) \in (a_{\pi(x_2)}+d_{x_2}, b_{\pi(x_2)}+d_{x_2})$, $G|_{[a_{x_2},g^{-1}(c_F)]}$ is strictly increasing and $G|_{[g^{-1}(c_F),b_{x_2^*}]}$ is strictly decreasing.

4. $G(0) \in (a_{\pi(x_1)} + d_{x_1}, b_{\pi(x_1)} + d_{x_1})$, $G|_{[0,b_{x_1}]}$ is strictly increasing and $G|_{[b_{x_1^*},1]}$ is strictly decreasing.

We note that $G \in \mathcal{A}$ and $c_G = g^{-1}(c_F)$. Moreover, for each $x \in K$ we have that $G([a_x, b_x]) \subset [a_{\pi(x)} + d_x, b_{\pi(x)} + d_x]$.

Now, we only have to prove that $\widehat{\underline{I}}_G(0^+) = \widehat{\underline{I}}_G(0) = \widehat{\underline{I}}_F(x_1)$ and $\widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_G(c_G) = \widehat{\underline{I}}_F(x_2)$. From all said above we see that $E(F^n(x_1)) = E(G^n(0))$ and $E(F^n(x_2)) = E(G^n(c_G))$. Since $g(0) = 0$, $g(c_G) = c_F$, g is non-decreasing and $g|_{(b_{x_2},a_{x_2^*})}$ is strictly increasing we have that $g(D(x)) \in (0, c_F)$ (respectively $g(D(x)) \in (c_F, 1)$) if and only if $D(x) \in (0, c_G)$ (respectively $D(x) \in (c_G, 1)$). Therefore, $\widehat{\underline{I}}_G(0) = \widehat{\underline{I}}_F(x_1) = \underline{\nu}_1$ and $\widehat{\underline{I}}_G(c_G) = \widehat{\underline{I}}_F(x_2) = \underline{\nu}_2$. In short, $K(G) = (\underline{\nu}_1, \underline{\nu}_2)$ and the theorem follows. ∎

Another strategy for the proof of the above theorem is the one used by de Melo and van Strien in the proof of Theorem 4.1 of [24]. However our approach, suggested by F. Mañosas, is considerably more simple in the case of maps with two critical points. It seems to us that this approach, which uses strongly the characterization of the itineraries of a map given by Proposition 2.2.3(b), could also simplify the proof in their case and could be used to deal with similar problems for multimodal circle maps of degree one.

To end the proof of the second statement of Theorem 2.3.2 we still have to prove that if $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}_a$ for some $a \in \mathbf{R}$ then there exists $F \in \mathcal{A}$ such that $R_F = \{a\}$ and $K(F) = (\underline{\nu}_1, \underline{\nu}_2)$. We note that the strategy used in the proof of Theorem 2.3.10 also works in this case. However, we prefer a constructive approach which characterizes better the allowed kneading pairs in $\mathcal{E} \backslash \mathcal{E}^*$. We consider separately the rational and the irrational case. To deal with the rational case we need the following two technical lemmas. The first one follows by direct computation.

Lemma 2.3.11 *Let $a \in \mathbf{Z}$ then $\epsilon_i(a) = \delta_i(a) = a$ for all $i > 0$.*

Lemma 2.3.12 *Let $F \in \mathcal{A} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$ and let $p/q \in \mathbf{Q}$ with $(p,q) = 1$. Then the following statements hold.*

(a) Assume that $\widehat{\underline{I}}_F(c_F) = \delta_1(p/q)^L \dots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$. Then there exists U, a neighborhood of F in $\mathcal{A} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$, such that for each $G \in U$, $\widehat{\underline{I}}_G(c_G^-)$ is either $\underline{I}_R(p/q)$ or $\underline{I}_\delta(p/q)$.

(b) Assume that $\widehat{\underline{I}}_F(0) = \epsilon_1(p/q)^L \dots \epsilon_{q-1}(p/q)^L \epsilon_q(p/q)^M$. Then there exists U, a neighborhood of F in $\mathcal{A} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$, such that for each $G \in U$, $\widehat{\underline{I}}_G(0^+)$ is either $\widehat{\underline{I}}_R^*(p/q)$ or $\widehat{\underline{I}}_\epsilon(p/q)$.

Proof. We only prove statement (a). Statement (b) follows in a similar way. Assume that $\widehat{\underline{I}}_F(c_F) = \delta_1(p/q)^L \dots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$. Let $P = \{x_i\}_{i \in \mathbf{Z}}$ be the twist periodic orbit of period q and rotation number p/q such that $x_{q-1} = c_F$. Clearly, we can take $F|_{[x_{q-2}, c_F]}$ (see Figure 2.3.5) in such a way that F has a periodic (mod. 1) point $z \in (x_{q-2}, c_F)$ close to c_F, of period q, such that $(F^q - p)|_{[z, c_F]}$ is strictly increasing, $(F^q - p)(x) > x$ for each $x \in (z, c_F)$ and $\widehat{\underline{I}}_F(z) = \widehat{\underline{I}}_\delta(p/q)$ (in particular $F(z) > 1 = F(1)$). Since $\frac{d}{dx}(F^q - p)(c_F) = 0$ there exists $0 < \epsilon < c_F - z$ such that $|\frac{d}{dx}(F^q - p)(x)| < 1/4$ for each $x \in (c_F - \epsilon, c_F + \epsilon)$. Now we take U, a neighborhood of F in $\mathcal{C}^1(\mathbf{R}, \mathbf{R}) \cap \mathcal{A}$, such that for each $G \in U$ the following conditions hold:

(a) $\widehat{\underline{I}}_G(c_G) = \delta_1(p/q)^L \dots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(c_G))} \dots$,

(b) G has a periodic (mod 1) point $z_G < c_G$ close to z, of period q, such that $G(z_G) > \max\{1, G(1)\}$ and $\widehat{\underline{I}}_G(z_G) = \widehat{\underline{I}}_\delta(p/q)$,

(c) $c_G \in (c_F - \epsilon, c_F + \epsilon)$, $(G^q - p)|_{[z_G, c_G]}$ is strictly increasing and $(G^q - p)|_{[c_G, c_F + \epsilon]}$ is strictly decreasing,

(d) $(G^q - p)(c_G) \in (c_F - \epsilon, c_F + \epsilon)$,

(e) $|\frac{d}{dx}(G^q - p)(x)| < 1/2$ for each $x \in (c_F - \epsilon, c_F + \epsilon)$.

We note that for each $G \in U$ and $x \in [z_G, c_G]$ we have that

$$\widehat{\underline{I}}_G(x) = \delta_1(p/q)^L \dots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \dots .$$

Let $z_G^* \in (c_G, 1)$ be such that $G(z_G) = G(z_G^*)$ (such z_G^* exists because, in view of (b), $G(z_G) > G(1)$). Clearly, for all $x \in \left[z_G, z_G^*\right]$ we also have that

$$\widehat{\underline{I}}_G(x) = \delta_1(p/q)^L \dots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \dots .$$

Figure 2.3.5: The graph of $(F^q - p)$ near c_F.

If $(G^q - p)(c_G) \leq c_G$, then for each $x \in [z_G, c_G]$ we have that $(G^q - p)^i(x) \in [z_G, c_G]$ for each $i \in \mathbf{N}$. Hence, $\widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_\delta(p/q)$. Now, assume that $(G^q - p)(c_G) > c_G$. From (c) and (d) we see that $c_G < (G^q - p)(c_G) \in (c_F - \epsilon, c_F + \epsilon)$ and $(G^q - p)(c_G)$ is the maximum of $G^q - p$ in $(c_F - \epsilon, c_F + \epsilon)$. So $(G^q - p)^2(c_G) < (G^q - p)(c_G)$. On the other hand

$$(G^q - p)(c_G) - (G^q - p)^2(c_G) \quad = |\tfrac{d}{dx}(G^q - p)(\xi)|\,((G^q - p)(c_G) - c_G)$$

with ξ between c_G and $(G^q - p)(c_G)$. In view of (e) we have that $|\frac{d}{dx}(G^q - p)(\xi)| < 1/2$ and hence $(G^q - p)(c_G) - (G^q - p)^2(c_G) < (G^q - p)(c_G) - c_G$. Therefore $c_G < (G^q - p)^2(c_G)$ and, consequently, $(G^q - p)([c_G, (G^q - p)(c_G)]) \subset [c_G, (G^q - p)(c_G)]$. From all said above we see that, in this case, $\widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_R(p/q)$. ∎

Proposition 2.3.13 *Let* $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}_{p/q}$ *with* $p \in \mathbf{Z}$, $q \in \mathbf{N}$ *and* $(p, q) = 1$. *Then there exists* $F \in \mathcal{A}$ *such that* $R_F = \{p/q\}$ *and* $K(F) = (\underline{\nu}_1, \underline{\nu}_2)$.

Figure 2.3.6: The map F.

Proof. We will deal first with the case $p/q \in \mathbf{Z}$. That is, $q = 1$. From Lemma 2.3.11 we have

$$\mathcal{E}_p = \left\{((p^L)^\infty, (p+1)^L(p^L)^\infty), ((p-1)^R(p^R)^\infty, (p^R)^\infty), ((p-1)^L(p^L)^\infty, (p^L)^\infty\right\}.$$

Assume that $(\underline{\nu}_1, \underline{\nu}_2) = ((p-1)^R(p^R)^\infty, (p^R)^\infty)$. Then we take $F \in \mathcal{A}$ such that $a \in (0, c_F)$ is a fixed point of $F - p$ such that $(F-p)|_{[a,1]}$ is a unimodal map satisfying that $c_F < (F-p)(1)$ (see Figure 2.3.6). In consequence $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_F(1^-) = (\widehat{\underline{I}}_F(0^+))'$ and $\widehat{\underline{I}}_F(c_F) = (p^R)^\infty$. Thus $K(F) = ((p-1)^R, (p^R)^\infty, (p^R)^\infty)$. The rest of the cases follow in a similar way.

Now we consider the case $q \neq 1$. Assume first that

$$(\underline{\nu}_1, \underline{\nu}_2) \in \{(\widehat{\underline{I}}^*_\delta(p/q), \widehat{\underline{I}}_\delta(p/q)), (\widehat{\underline{I}}^*_R(p/q), \widehat{\underline{I}}_R(p/q))\}.$$

Set $P = \{x_i\}_{i \in \mathbf{Z}}$ with $x_i = \dfrac{i}{q} + \dfrac{1}{2q}$ for each $i \in \mathbf{Z}$. Let $F \in \mathcal{C}^1(\mathbf{R}, \mathbf{R}) \cap \mathcal{A}$ be such that

(1) $F(0) = 0$ and $c_F = x_{q-1}$,

(2) $F(x) = x + \dfrac{p}{q}$ for each $x \in P$,

(3) F is affine in the interval $[x_0, x_{q-2}]$.

Note that P is $F-$invariant and $F^i(x) = x + i\frac{p}{q}$ for each $x \in P$ and $i \in \mathbf{N}$. Hence, $s(F^i(c_{_F})) = s(F^i(x_{q-1})) = L$ for $i = 1, 2, \ldots, q-1$. Moreover, since $F^q(c_{_F}) = F^q(x_{q-1}) = x_{q-1} + q\frac{p}{q} = c_{_F} + p$ we see that $s(F^q(c_{_F})) = s(F^q(x_{q-1})) = C$. On the other hand,

$$\begin{aligned} d(c_{_F}) = d(x_{q-1}) \quad &= E(F(x_{q-1})) - E(x_{q-1}) \\ &= E(\tfrac{2q-1}{2q} + \tfrac{p}{q}) = E(\tfrac{p}{q}) + 1 \\ &= \epsilon_1(p/q) + 1 = \delta_1(p/q), \end{aligned}$$

and, for $i = 2, \ldots, q-2$,

$$\begin{aligned} d(F^i(c_{_F})) = d(F^i(x_{q-1})) \quad &= E(F^{i+1}(x_{q-1})) - E(F^i(x_{q-1})) \\ &= E(\tfrac{2q-1}{2q} + (i+1)\tfrac{p}{q}) - E(\tfrac{2q-1}{2q} + i\tfrac{p}{q}) \\ &= \Big(E((i+1)\tfrac{p}{q}) + 1\Big) - \Big(E(i\tfrac{p}{q}) + 1\Big) \\ &= \epsilon_i(p/q) = \delta_i(p/q). \end{aligned}$$

Lastly,

$$\begin{aligned} d(F^{q-1}(c_{_F})) = d(F^{q-1}(x_{q-1})) \quad &= E(\tfrac{2q-1}{2q} + p) - E(\tfrac{2q-1}{2q} + (q-1)\tfrac{p}{q}) \\ &= E(p) - \Big(E(\tfrac{(q-1)p}{q}) + 1\Big) \\ &= \epsilon_q(p/q) - 1 = \delta_q(p/q). \end{aligned}$$

In consequence $\underline{\widehat{I}}_{_F}(c_{_F}) = \delta_1(p/q)^L \ldots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$.

Now we are ready to construct maps $H_\delta, H_{_R} \in \mathcal{A}$ such that $R_{H_\delta} = R_{H_R} = \{p/q\}$, $K(H_\delta) = (\underline{\widehat{I}}^*_\delta(p/q), \underline{\widehat{I}}_\delta(p/q))$ and $K(H_{_R}) = (\underline{\widehat{I}}^*_R(p/q), \underline{\widehat{I}}_R(p/q))$. From Lemma 2.3.12(a) we have that there exists U, a neighborhood of F in $\mathcal{A} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$, such that for each $G \in U$, $\underline{\widehat{I}}_{_G}(c_{_G}^-)$ is either $\underline{\widehat{I}}_R(p/q)$ or $\underline{\widehat{I}}_\delta(p/q)$. Moreover, from the proof of Lemma 2.3.12, G has a periodic (mod 1) point $z_{_G} < c_{_G}$ of period q such that $G(z_{_G}) > \max\{1, G(1)\}$ and $\underline{\widehat{I}}_{_G}(z_{_G}) = \underline{\widehat{I}}_\delta(p/q)$. Let $z^*_{_G} \in (c_{_G}, 1)$ be such that $G(z_{_G}) = G(z^*_{_G})$. Clearly, for all $x \in \Big[z_{_G}, z^*_{_G}\Big]$ we also have that

$$\underline{\widehat{I}}_{_G}(x) = \delta_1(p/q)^L \ldots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \ldots \,.$$

To construct H_δ take $G \in U$ such that $(G^q - p)(c_G) \le c_G$ and let $c^* \in (1, c_G + 1)$ be such that $G(c_G) = G(c^*)$. We take $H_\delta \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \cap \mathcal{A}$ such that $c_{H_\delta} = c_G$, $G|_{\left[c^*-1, c_{H_\delta}\right]} = H_\delta|_{\left[c^*-1, c_{H_\delta}\right]}$ and $H_\delta(x) > G(z_G)$ for all $x \in (c_G, c^*)$ (see Figure 2.3.7). We note that $H_\delta([c_G, 1]) \subset H_\delta(\left[z_G, c_{H_\delta}\right]) = G([z_G, c_G])$. Hence, from above we have that $\underline{\widehat{I}}_{H_\delta}(0^-) = \underline{\widehat{I}}_{H_\delta}(c^-_{H_\delta}) = \underline{\widehat{I}}_G(c^-_G) = \underline{\widehat{I}}_\delta(p/q)$. Thus $K(H_\delta) = (\underline{\widehat{I}}^*_\delta(p/q), \underline{\widehat{I}}_\delta(p/q))$. Furthermore, by Lemma 2.3.6 and Theorem 2.3.4 we see that $R_{H_\delta} = \{p/q\}$. To construct H_R we take $G \in U$ such that $(G^q - p)(c_F) > c_F$. Let $a = (G^q - p)(c_G)$ and let $b \in (c_G, z^*_G)$ be such that $(G^q - p)(b) = c_G$. Since $(G^q - p)(b) = c_G < (G^q - p)^2(c_G) = (G^q - p)(a)$ and $a, b \in (c_G, 1)$ we have that $b > a$. Finally, let $c^* \in (1, c_G + 1)$ be such that $G(c_G) = G(c^*)$. Then we take $H_R \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \cap \mathcal{A}$ such that $c_{H_R} = c_G$, $H_R|_{[c^*-1,a]} = G|_{[c^*-1,a]}$ and $H_R(x) > G(b)$ for all $x \in (a, c^*)$ (see Figure 2.3.8). In consequence, since $b < z^*_G$ we have that $G(b) > G(z^*_G) = G(z_G)$ and hence, $H_R(\left[c_{H_R}, 1\right]) \subset H_R(\left[z_G, c_{H_R}\right]) = G([z_G, c_G])$. Therefore, from above we get that $\underline{\widehat{I}}_{H_R}(0^-) = \underline{\widehat{I}}_{H_R}(c^-_{H_R}) = \underline{\widehat{I}}_G(c^-_G) = \underline{\widehat{I}}_R(p/q)$. Thus, $K(H_R) = (\underline{\widehat{I}}^*_R(p/q), \underline{\widehat{I}}_R(p/q))$ and $R_{H_R} = \{p/q\}$.

To end the proof of the proposition it remains to construct a map $H_\epsilon \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \cap \mathcal{A}$ such that $R_{H_\epsilon} = \{p/q\}$ and $K(H_\epsilon) = (\underline{\widehat{I}}_\epsilon(p/q), \underline{\widehat{I}}^*_\epsilon(p/q))$. To do it we proceed as in the above construction of the map H_δ by using Lemma 2.3.12(b) and, instead of the map F, the map $\widetilde{F} \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \cap \mathcal{A}$ defined as follows. Set $\widetilde{P} = \{\widetilde{x}_i\}_{i \in \mathbf{Z}}$ with $\widetilde{x}_i = i/q$ for each $i \in \mathbf{Z}$. Then $\widetilde{F}$ is such that:

(1) $\widetilde{F}$ is affine in the interval $[\widetilde{x}_1, \widetilde{x}_{q-1}]$,

(2) $\widetilde{F}(\widetilde{x}_i) = \widetilde{x}_i + \dfrac{p}{q}$,

(3) $c_{\widetilde{F}} \in (x_{q-1}, 1)$ and $\widetilde{F}(c_{\widetilde{F}}) = c_{\widetilde{F}} + E(p/q) + 1$.

∎

Proof of the second statement Theorem 2.3.2. If $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$ then theorem follows from Theorem 2.3.10. Otherwise, $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}_a$ with $a \in \mathbf{R}$. If $a \in \mathbf{Q}$ then the theorem follows from Proposition 2.3.13. If $a \notin \mathbf{Q}$ then, from the proof of Proposition 1 of [6] it follows that there exists $F \in \mathcal{A}$ such that $R_F = \{a\}$. Now, from Lemma 2.3.3 we see that $\underline{\widehat{I}}_\delta(a) = \underline{\widehat{I}}^*_\epsilon(a)$ and $\underline{\widehat{I}}^*_\delta(a) = \underline{\widehat{I}}_\epsilon(a)$. So, from Theorem 2.3.4 we obtain that $K(F) = (\underline{\widehat{I}}^*_\delta(a), \widehat{I}_\delta(a))$. Hence, by the definition of $\mathcal{E}_a$ we see that $K(F) = (\underline{\nu}_1, \underline{\nu}_2)$. ∎

Figure 2.3.7: The maps H_δ and G.

Remark 2.3.14 As we have said before the set $\mathcal{E} \setminus \mathcal{E}^*$ is the boundary of $\mathcal{E}^*$. It is natural that if for $F \in \mathcal{A}$ we have that $K(F) \notin \mathcal{E}^*$ then the topological entropy of F is zero. Indeed this follows from Proposition 4.3.3. However, there are also maps $F \in \mathcal{A}$ such that $K(F) \in \mathcal{E}^*$ and the topological entropy of F is zero, as the following example shows. Let F be the map shown in Figure 2.3.9. Then, clearly, $\underline{\widehat{I}}_F(c_F^-) = (1^R)^\infty$ and $\underline{\widehat{I}}_F(0^+) = 0^L(1^L)^\infty$. Therefore, $(\underline{\widehat{I}}_F(0^+))' = (1^L)^\infty < (1^R)^\infty = \underline{\widehat{I}}_F(c_F^-)$ and so $K(F) \in \mathcal{E}^*$. On the other hand, the non-wandering set of the circle map which has F as a lifting is just two fixed points: $\exp(2\pi i a)$ and $\exp(2\pi i b)$. Therefore, the topological entropy of F is zero (see for instance [35]). $\square$

2.4 Concluding remarks

In the context of this chapter the following question arises in a natural way: Does there exist a family $F_\mu \in C^1(\mathbf{R}, \mathbf{R}) \cap \mathcal{A}$, depending continuously on μ, such that for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}$ there exists μ_0 in the parameter space such that $K(F_{\mu_0}) = (\underline{\nu}_1, \underline{\nu}_2)$? In the literature, such a

Figure 2.3.8: The maps H_R and G.

parameter family of maps is usually called a *full family* (see [9] and [24]). It is well known that, in the unimodal case, the family $f_\mu(x) = \mu x(1-x)$ with $x \in [0,1]$ and $\mu \in [1,4]$ is full (see [9]).

The simplest non-invertible degree one circle maps are the ones with two critical points. That is, the maps from class $\mathcal{A}$. Among the families of such maps, the standard maps family defined as

$$F_{b,w}(x) = x + w + b\frac{\sin(2\pi x)}{2\pi}$$

where $x \in \mathbf{R}$ and $(b,w) \in (1,\infty) \times \mathbf{R}$ is known to display all dynamical features. Therefore, it is natural to think that this family is full. To discuss this problem we need to state a result due to Malta [23]. First we introduce some notation

Let $F \in \mathcal{L} \cap \mathcal{C}^1(\mathbf{R},\mathbf{R})$ We shall say that $x \in \mathbf{R}$ is a *non-flat critical point* if it is a critical point and there exists an integer $k > 1$ such that F is $\mathcal{C}^k$ in a neighborhood of x and $\frac{d^k}{dx^k}F(x) \neq 0$. We say that $x \in \mathbf{R}$ is a *turning point* if the map F has a local extremum in x.

Theorem 2.4.1 (Malta) *Let $F \in \mathcal{L} \cap \mathcal{C}^2(\mathbf{R},\mathbf{R})$ and suppose that all non-turning critical points*

Figure 2.3.9: A map $F \in \mathcal{A}$ such that $K(F) \in \mathcal{E}^*$ and topological entropy zero.

are non-flat. If F has a turning critical point, then F has periodic points

From the fact that a map $F \in \mathcal{L}$ such that $R_F = \{a\}$ with $a \notin \mathbf{Q}$ has no periodic points we obtain the following simple corollary of Malta's Theorem.

Corollary 2.4.2 *Let $F \in \mathcal{A} \cap \mathcal{C}^2(\mathbf{R}, \mathbf{R})$ be such that $R_F = \{a\}$ with $a \notin \mathbf{Q}$. Then the map F has flat non-turning critical points.*

Therefore, we get

Corollary 2.4.3 *Let $F \in \mathcal{A}$ be analytic. Then $K(F) \notin \cup_{a \notin \mathbf{Q}} \mathcal{E}_a$. That is, R_F is not degenerate to an irrational.*

Proof. Assume that $K(F) \in \mathcal{E}_a$ for some $a \notin \mathbf{Q}$. From Lemma 2.3.3 and Theorem 2.3.4 we have $R_F = \{a\}$. Then, by Corollary 2.4.2, F has flat non-turning critical points. Since F is analytic we have that $F = 0$; a contradiction. ∎

Corollary 2.4.3 tell us that there is no analytic full family in $\mathcal{A}$. In particular, the standard maps family is not full. This suggests that the "good" families from $\mathcal{A}$ will only be *weakly full* in the following sense. We say that a family $F_\mu \in C^1(\mathbf{R},\mathbf{R}) \cap \mathcal{A}$, depending continuously on μ, is *weakly full* if for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^* \cup (\cup_{a\in\mathbf{Q}}\mathcal{E}_a)$ there exists μ_0 in the parameter space such that $K(F_{\mu_0}) = (\underline{\nu}_1, \underline{\nu}_2)$. Following the techniques of the proof of Theorem 4.1 of [24] seems rasonable to be able to prove the following.

Conjeture *The standard maps family is weakly full.*

Chapter 3

Self-similarity operators for maps in $\mathcal{A}$

3.1 Introduction

In this chapter we develop some topological tools in order to describe the bifurcations of parametrized families of maps from $\mathcal{A}$ at the symbolic level. In the literature (see [7], [17], [11], [13], and [19]), certain bifurcations are described in terms of the set of parameter values for which the maps have a determinate rotation interval. More precisely, let $F_\mu : \Delta \times \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous parameter family where $F_\mu \in \mathcal{A}$ for all $\mu \in \Delta$. The bifurcations are then described in terms of the following two sets. For $(a, b) \in \mathbf{R}^2$ with $a \leq b$ we define

$$T_R(a) = \{\mu \in \Delta : \min R_{F_\mu} = a\}$$

and

$$T_L(b) = \{\mu \in \Delta : \max R_{F_\mu} = b\}.$$

The sets $T_L(a)$ and $T_R(b)$ give (in the parameter space Δ) for the standard map family the picture known as an Arnol'd tongue (see [7], [11], [13], and [19]). Indeed, the *Arnol'd tongue* of $a \in \mathbf{R}$ is defined to be $T_L(a) \cup T_R(a)$.

In order to study the bifurcation structure of the Arnol'd tongues at the symbolic level we introduce some notation and preliminary definitions. Let

$$\mathcal{E}_\epsilon = \left\{ \underline{\alpha} \in \mathcal{AD} : \ \exists \underline{\beta} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E} \right\}$$

and

$$\mathcal{E}_\delta = \left\{ \underline{\beta} \in \mathcal{AD} : \ \exists \underline{\alpha} \in \mathcal{AD} \text{ such that } (\underline{\alpha}, \underline{\beta}) \in \mathcal{E} \right\}.$$

We consider in $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ the order topology. Let $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$ be with the product topology. It can be seen that $\mathcal{E}$ is strictly contained in $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$. To see this consider for example the set $A = \{(0^L)^\infty, (1^L)^\infty\}$ of admissible sequences. Since $((-1^L)^\infty, (0^L)^\infty)((0^L)^\infty, (1^L)^\infty), ((1^L)^\infty, (2^L)^\infty) \in \mathcal{E}$, we have that $A \subset \mathcal{E}_\epsilon$ and $A \subset \mathcal{E}_\delta$. In consequence $\{((0^L)^\infty, (1^L)^\infty), ((1^L)^\infty, (0^L)^\infty)\} \subset \mathcal{E}_\epsilon \times \mathcal{E}_\delta$, but $((0^L)^\infty, (1^L)^\infty) \in \mathcal{E}$ and $((1^L)^\infty, (0^L)^\infty) \notin \mathcal{E}$.

We consider $\mathcal{E}$ endowed with the induced topology from $\mathcal{E}_\epsilon \times \mathcal{E}_\delta$. Let $\pi_\epsilon : \mathcal{E}_\epsilon \times \mathcal{E}_\delta \longrightarrow \mathcal{E}_\epsilon$ defined as $\pi_\epsilon((\underline{\alpha}, \underline{\beta})) = \underline{\alpha}$ and $\pi_\delta : \mathcal{E}_\epsilon \times \mathcal{E}_\delta \longrightarrow \mathcal{E}_\delta$ defined as $\pi_\delta((\underline{\alpha}, \underline{\beta})) = \underline{\beta}$. Clearly π_ϵ and π_δ are continuous.

Let $a \in \mathbf{R}$ and set $\mathcal{Q}_\epsilon(a) = [\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\epsilon(a)] \subset \mathcal{E}_\epsilon$ and $\mathcal{Q}_\delta(a) = [\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)] \subset \mathcal{E}_\delta$. With this notation, from Theorems 2.3.4 and 2.3.2, we can write $\mathcal{E}_\epsilon = \cup_{a \in R} \mathcal{Q}_\epsilon(a)$ and $\mathcal{E}_\delta = \cup_{a \in \mathbf{R}} \mathcal{Q}_\delta(a)$. Moreover it can be seen [5] that if $a, b \in \mathbf{R}$ with $a < b$, then for each $\underline{\alpha} \in \mathcal{Q}_\epsilon(a)$ (respectively $\underline{\alpha} \in \mathcal{Q}_\delta(a)$) and $\underline{\beta} \in \mathcal{Q}_\epsilon(b)$ (respectively $\underline{\beta} \in \mathcal{Q}_\delta(b)$) we have $\underline{\alpha} < \underline{\beta}$ (in particular $\mathcal{Q}_\epsilon(a) \cap \mathcal{Q}_\epsilon(b) = \emptyset$ and $\mathcal{Q}_\delta(a) \cap \mathcal{Q}_\delta(b) = \emptyset$).

Now, we define the *symbolic Arnol'd tongues* as follows. For $a \in \mathbf{R}$ we set

$$T_\epsilon(a) = \pi_\epsilon^{-1}(\mathcal{Q}_\epsilon(a)) \cap \mathcal{E}$$

and

$$T_\delta(a) = \pi_\delta^{-1}(\mathcal{Q}_\delta(a)) \cap \mathcal{E}.$$

Then by Theorem 2.3.4 we can write

$$\mathcal{E} = \bigcup_{\substack{(a,b) \in \mathbf{R}^2 \\ a \leq b}} (T_\epsilon(a) \cap T_\delta(b)).$$

Moreover this theorem can be stated in the following way. *Let* $F \in \mathcal{A}$. *Then* $R_F = [a, b]$ *if and only if* $K(F) \in T_\epsilon(a) \cap T_\delta(b)$. To motivate the above definition let $F_\mu : \Delta \times \mathbf{R} \longrightarrow \mathbf{R}$ be a

Figure 3.1.1: An Arnol'd tongue of the standard family in the rational case.

continuous parameter family where $F_\mu \in \mathcal{A}$ for all $\mu \in \Delta$. Let also $\lambda : \Delta \longrightarrow \mathcal{E}$ be the continuous map given by $\lambda(\mu) = K(F_\mu)$. Then, for each $a \in \mathbf{R}$, we have that $T_R(a) = \lambda^{-1}(T_\epsilon(a))$ and $T_L(a) = \lambda^{-1}(T_\delta(a))$. For example consider the standard maps family. It is defined as

$$F_{b,w}(x) = x + w + \frac{b}{2\pi}\sin(2\pi x)$$

where $x \in \mathbf{R}$, $b > 0$, and $w \in \mathbf{R}$. We note that for all $b > 1$ and $w \in \mathbf{R}$, $F_{b,w} \in \mathcal{A}$. Thus, $\Delta = (1, \infty) \times \mathbf{R}$ (see Figure 3.1.1 where a typical picture of the structure of the sets $T_R(a)$ and $T_L(a)$ when $a \in \mathbf{Q}$ is shown).

Since $F_{b,w}$ is a family of analytic maps, from Corollary 2.4.3, we have that if $a \notin \mathbf{Q}$ then $T_R(a) \cap T_L(a) = \emptyset$ because $T_\epsilon(a) \cap T_\delta(a) = \{(\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\delta(a))\} \in \mathcal{E}_a$. Moreover, from Lemma 2.3.3 we have that $T_\epsilon(a) = \pi_\epsilon^{-1}(\widehat{\underline{I}}_\epsilon(a)) \cap \mathcal{E}$ and $T_\delta(a) = \pi_\delta^{-1}(\widehat{\underline{I}}_\delta(a)) \cap \mathcal{E}$. Thus, in the irrational case, we obtain a picture for the Arnol'd tongue as the one shown in Figure 3.1.2.

The aim of this chapter is to study the bifurcation and self–similar structures of the Arnol'd tongues (in the parameter space) by studying the symbolic structure of the symbolic Arnol'd tongues. Also we want to describe at a symbolic level the bifurcations occurring when the left

Figure 3.1.2: The Arnold' tongue in the parameter space in the irrational case.

(respectively right) endpoint of the rotation interval crosses a rational. From all said above, the description arising from this approach will be valid for all "typical" families of maps from $\mathcal{A}$ (i.e., families for which the images of the critical points depend monotonically of the parameters). To do it we will define and study two self–similarity operators in the symbolic spaces $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$. By means of these operators we will be able to describe the internal structure of the "boxes" $\mathcal{Q}_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$ and, hence, to obtain the symbolic structure of the symbolic Arnol'd tongues.

The following result characterizes the sets $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ in a form that will be useful in the rest of the chapter. The proof is given in the appendix 3.5.

Theorem 3.1.1 *The following statements hold.*

(a) $\underline{\alpha} \in \mathcal{E}_\epsilon$ *if and only if it is minimal and satisfies that if for some* $n \geq 0$, $S^n(\underline{\alpha}) = d^R \dots$ *then* $S^{n+1}(\underline{\alpha}) \geq \underline{\alpha}'$.

(b) $\underline{\beta} \in \mathcal{E}_\delta$ *if and only if it is maximal.*

The chapter is organized as follows. In Sections 3.2 and 3.3 we define and study the two symbolic operators. Lastly, in Section 3.4, we use these operators to state and prove the main result of this chapter.

3.2 The $\star-$product

3.2.1 Introduction

The aim of this section is to characterize the sets of sequences which, roughly speaking, correspond to the first (respectively second) component of the kneading pair of maps $F \in \mathcal{A}$ for which there exist $p \in \mathbf{N}$, $q \in \mathbf{Z}$ and a closed interval J containing c_F (respectively 0) such that $(F^q - p)|_J$ is a unimodal map (see definition below). We make this study at a symbolic level by using a $\star-$product which relates the symbolic spaces $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ with the space of kneading sequences of unimodal maps. Moreover, we will show how the "unimodal symbolic space" is embedded into $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$. This section is organized as follows. In Subsection 3.2.2 we introduce the appropriate notation for the symbolic dynamics of unimodal maps. In Subsection 3.2.3 we define the $\star-$product and we state the main result of this section. In Subsection 3.2.4 we give some technical results and finally in Subsection 3.2.5 we prove the main result of this section.

3.2.2 A survey on the kneading theory for unimodal maps

Let I be a closed interval and let $f : I \longrightarrow I$ be a continuous map. We say that f is unimodal if

1. $f(\max I) = f(\min I) \in \partial I$
2. There exists $c \in \mathrm{Int}(I)$ such that the maps $f|_{[\min\ I,c]}$ and $f|_{[c,\max\ I]}$ are homeomorphisms.

The set of all unimodal maps from I to itself will be denoted by $U(I)$. A map $f \in U(I)$ will be called *positive* if $f|_{[\min\ I,c]}$ is increasing. Otherwise, f will be called *negative.*

Let $f \in U(I)$ and let $x \in I$. We associate with x a finite or infinite sequence of the symbols L, C, R called its itinerary. To do it we introduce the following notation. Let $f : I \longrightarrow I$ be continuous. We will say that f is *locally increasing* (respectively *decreasing*) at $x \in I$ if there exists an open neighbourhood V of x in I such that $f|_V$ is increasing (respectively decreasing).

Now, we define the $i-th$ address of a point x, that we denote by $\theta_i(x)$, as follows:

$$\theta_i(x) = \begin{cases} L & \text{if } f^i \text{ is locally increasing at } x. \\ C & \text{if } f^i(x) = c, \\ R & \text{if } f^i \text{ is loacally decreasing at } x. \end{cases}$$

We define the itinerary of x denoted by $\underline{\theta}_f(x)$ as follows

1. $\underline{\theta}_f(x) = \theta_0(x)\theta_1(x)\ldots\theta_n(x)\ldots$ if $\theta_i(x) \in \{L, R\}$ for all $i \geq 0$.

2. $\underline{\theta}_f(x) = \theta_0(x)\theta_1(x)\ldots\theta_n(x)$ if $\theta_n(x) = C$, and $\theta_i(x) \in \{L, R\}$ for all $i \in \{0, 1, \ldots n-1\}$.

Given $n \in \mathbf{N}$ and $y \in I$, there exists $\delta > 0$ such that $\theta_n(y)$ takes constant value L or R in the interval $(x, x+\delta)$. We denote this value by for $\theta_n(x^+)$. In a similar way we can define $\theta_n(x^-)$. With this notation we set $\underline{\theta}_f(x^+) = \theta_1(x^+)\theta_2(x^+)\ldots$ and $\underline{\theta}_f(x^-) = \theta_1(x^-)\theta_2(x^-)\ldots$. We note that if $\underline{\theta}_f(x)$ is infinite then $\underline{\theta}_f(x) = \underline{\theta}_f(x^+) = \underline{\theta}_f(x^-)$.

The sequence $\underline{\theta}_f(f(c)^+)$ is called the *kneading sequence of* f. We will denote it by $k(f)$.

Let $\underline{A} = A_0A_1\ldots$ be a sequence of elements $A_i \in \{L, C, R\}$. We say that $\underline{A}$ is *admissible* if one of the following two conditions is satisfied:

1. $\underline{A} = A_0A_1\ldots A_n\ldots$ if $A_i \in \{L, R\}$ for all $i \geq 0$.

2. $\underline{A} = A_0A_2\ldots A_n$ if $A_n = C$, and $A_i \in \{L, R\}$ for all $i \in \{0, 1, \ldots n-1\}$.

Now, we introduce an ordering in the set of all admissible sequences. We set $L < C < R$ and we extend this ordering lexicographically to the set of all admissible sequences as follows. Let $K_0K_1\ldots K_n$ be a finite (or empty) sequence of symbols L, R. We say that $K_0K_1\ldots K_n$ is even (respectively odd) if it has an even (respectively odd) number of R's. Assume that $\underline{K} = K_0K_1\ldots$ and $\underline{K}' = K_0'K_1'\ldots$ are admissible sequences such that $\underline{K} \neq \underline{K}'$. Let n be such that $K_i = K_i'$ for $i < n$ and $K_n \neq K_n'$. Then we say that $\underline{K} < \underline{K}'$ if either

1. $K_n < K_n'$ and $K_0K_1\ldots K_{n-1}$ is even.

2. $K_n > K_n'$ and $K_0K_1\ldots K_{n-1}$ is odd.

We note that if $x < y$ and $f \in U(I)$ then $\underline{\theta}_f(x) \leq \underline{\theta}_f(y)$ if f is positive and $\underline{\theta}_f(x) \geq \underline{\theta}_f(y)$ if f is negative.

Now, we define the *shift operation* $\mathcal{S}$ on admissible sequences as follows. If $\underline{K} = K_0K_1\dots$ then we set $\mathcal{S}(\underline{K}) = K_1K_2\dots$ which is also an admissible sequence. If $K_0 = C$, then $\mathcal{S}$ is undefined. We write $\mathcal{S}^n$ to denote the $n-$fold iterate of $\mathcal{S}$. Note that for each $x \in I$ and $f \in U(I)$ we have $\mathcal{S}(\underline{\theta}(x)) = (\underline{\theta}(f(x)))$.

An admissible sequence $\underline{K}$ will be called *maximal* if and only if $\mathcal{S}^n(\underline{K}) \leq \underline{K}$ for each $n < |\underline{K}|$ where $|\underline{K}|$ denotes the length of $\underline{K}$. We note that for each $f \in U(I)$ (independently of the fact that f is positive or negative), $k(f)$ is maximal and admissible with length infinite. Given $\underline{K} = K_0K_1\dots$, an admissible sequence, we will write $\widehat{\underline{K}}$ to denote $\widehat{K_0}\widehat{K_1}\dots$ where $\widehat{L} = R$ and $\widehat{R} = L$. We note that $\underline{K}$ is maximal if and only if $\widehat{\underline{K}}$ is minimal, that is $\mathcal{S}^n(\widehat{\underline{K}}) \geq \widehat{\underline{K}}$ for each $n < |\underline{K}|$.

From [9], it follows that for each admissible infinite maximal sequence $\underline{K}$ there exist $f, g \in U(I)$, f positive and g negative, such that $k(f) = k(g) = \underline{K}$. We shall denote by $\mathcal{K}$ the set of all admissible infinite maximal sequences.

3.2.3 Definition of the $\star-$products and statement of the main result

We start by introducing some notation. Let Ξ denote the set of all finite sequences with symbols in $\mathbf{Z}\times\{L, R\}$ (of course we consider the empty sequence as an element of Ξ). Let $\underline{\alpha} = d_1^{s_1}\dots d_n^{s_n} \in \Xi$. We denote by $\underline{\alpha}'$ the sequence $(d_1+1)^{s_1}\dots d_n^{s_n}$ (if $\underline{\alpha}$ is the empty sequence then we set $\underline{\alpha}' = \underline{\alpha}$). We say that $\underline{\alpha}$ is even (respectively odd) if $(s_1, \dots, s_n)$ has an even (respectively odd) number of symbols R. We note that, with this definition, the empty sequence is even.

Now we consider the set of sequences which occur as reduced itineraries of periodic critical points. Indeed we will denote by $\mathcal{P}_\epsilon$ (respectively $\mathcal{P}_\delta$) the set of all minimal sequences of the form $\underline{\beta}d^M$ satisfying that if for some $n \in \{1, \dots, |\underline{\beta}d^M| - 1\}$, $S^{n-1}(\underline{\beta}d^M) = t^R\dots$ then $S^n(\underline{\beta}d^M) > \underline{\beta}'d^M$ (respectively the set of all maximal sequences of the form $\underline{\beta}d^C$) with $\underline{\beta} \in \Xi$ and $d \in \mathbf{Z}$ and such that if $\underline{\beta}$ is not empty then $\left\{(\underline{\beta}d^L)^\infty, \underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty\right\} \subset \mathcal{E}_\epsilon$ (respectively $\left\{(\underline{\beta}d^L)^\infty, (\underline{\beta}d^R)^\infty\right\} \subset \mathcal{E}_\delta$).

We are ready to define the $\star-$products. We start by defining the product $\star_\delta : \mathcal{P}_\delta \times \mathcal{K} \longrightarrow \mathcal{AD}$ as follows. Let $\underline{\gamma} = \underline{\beta}d^C \in \mathcal{P}_\delta$ and $\underline{K} = K_1K_2\dots \in \mathcal{K}$. Then we define

$$\underline{\gamma} \star_\delta \underline{K} = \begin{cases} \underline{\beta}d^{K_1}\underline{\beta}d^{K_2}\underline{\beta}\dots & \text{if } \underline{\beta} \text{ is even,} \\ \underline{\beta}d^{\widehat{K_1}}\underline{\beta}d^{\widehat{K_2}}\underline{\beta}\dots & \text{if } \underline{\beta} \text{ is odd.} \end{cases}$$

Now we define $\star_\epsilon : \mathcal{P}_\epsilon \times \mathcal{K} \longrightarrow \mathcal{AD}$. Let $\underline{\beta} \in \Xi$ and $s \in \{L, R\}$. We set

$$\chi(s, \underline{\beta}) = \begin{cases} \underline{\beta} & \text{if } s = L, \\ \underline{\beta}' & \text{if } s = R. \end{cases}$$

Also, for $d \in \mathbf{Z}$ we set

$$\varphi(s, d) = \begin{cases} d^L & \text{if } s = L, \\ (d-1)^R & \text{if } s = R. \end{cases}$$

Let $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$ and $\underline{K} = K_1 K_2 \ldots \in \mathcal{K}$. Then we define $\underline{\gamma} \star_\epsilon \underline{K}$ as follows. If $\underline{\beta}$ is not empty then

$$\underline{\gamma} \star_\epsilon \underline{K} = \begin{cases} \underline{\beta} \varphi(K_1, d) \chi(K_1, \underline{\beta}) \varphi(K_2, d) \chi(K_2, \underline{\beta}) \ldots & \text{if } \underline{\beta} \text{ is even,} \\ \underline{\beta} \varphi(\widehat{K_1}, d) \chi(K_1, \underline{\beta}) \varphi(\widehat{K_2}, d) \chi(K_2, \underline{\beta}) \ldots & \text{if } \underline{\beta} \text{ is odd.} \end{cases}$$

If $\underline{\beta}$ is empty then $\underline{\gamma} \star_\epsilon \underline{K} = d_1^{K_1} d_2^{K_2} \ldots$ where, if $K_1 = L$ then $d_i = d$ for all $i \geq 1$ and if $K_1 = R$ then $d_1 = d - 1$ and

$$d_i = \begin{cases} d+1 & \text{if } \quad K_{i-1} K_i = RL, \\ d & \text{if } \quad K_{i-1} K_i \in \{LL, RR\}, \\ d-1 & \text{if } \quad K_{i-1} K_i = LR, \end{cases}$$

for $i \geq 2$.

The main result of this section which studies the properties of the $\star$−products is the following.

Theorem 3.2.1 *Let $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$, $\underline{\alpha} = \underline{\beta} d^C \in \mathcal{P}_\delta$ and $\underline{K} \in \mathcal{K}$. Then $\underline{\gamma} \star_\epsilon \underline{K} \in \mathcal{E}_\epsilon$ and $\underline{\alpha} \star_\delta \underline{K} \in \mathcal{E}_\delta$. If $\underline{\beta}$ is even then $\underline{\gamma} \star_\epsilon$ is order reversing and $\underline{\alpha} \star_\delta$ is order preserving. Otherwise, $\underline{\gamma} \star_\epsilon$ is order preserving and $\underline{\alpha} \star_\delta$ is order reversing. Moreover $\underline{\gamma} \star_\epsilon \mathcal{K}$ is connected in $\mathcal{E}_\epsilon$ and $\underline{\gamma} \star_\delta \mathcal{K}$ in $\mathcal{E}_\delta$.*

The above theorem characterizes at a symbolic level the "unimodal boxes" in the spaces $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$. Indeed, if we consider the set $\mathcal{K}$ endowed with the order topology (that is, $\mathcal{K} = [L^\infty, RL^\infty]$) then, from Theorem 3.2.1, we see that if $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$ (respectively $\underline{\gamma} = \underline{\beta} d^C \in \mathcal{P}_\delta$), then

$$\underline{\gamma} \star_\epsilon \mathcal{K} = \begin{cases} [\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \text{ if } \underline{\beta} \text{ is even,} \\ [\underline{\gamma} \star_\epsilon L^\infty, \underline{\gamma} \star_\epsilon RL^\infty] \text{ if } \underline{\beta} \text{ is odd.} \end{cases}$$

(respectively

$$\underline{\gamma} \star_\delta \mathcal{K} = \begin{cases} [\underline{\gamma} \star_\delta RL^\infty, \underline{\gamma} \star_\delta L^\infty] \text{ if } \underline{\beta} \text{ is odd,} \\ [\underline{\gamma} \star_\delta L^\infty, \underline{\gamma} \star_\delta RL^\infty] \text{ if } \underline{\beta} \text{ is even).} \end{cases}$$

The set $\underline{\gamma} \star_\epsilon \mathcal{K}$ will be called the $\epsilon-$*unimodal box of* $\underline{\gamma}$ and the set $\underline{\gamma} \star_\delta \mathcal{K}$ will be called the $\delta-$*unimodal box of* $\underline{\gamma}$.

3.2.4 Preliminary results

In this subsection we study the itineraries of the critical points when they are periodic and some of the basic properties of the $\star-$products. These results will be used to prove Theorem 3.2.1. We start with the following technical lemmas.

Lemma 3.2.2 *Let $F \in \mathcal{A}$. Then the following statements hold.*

(a) Assume that 0 is a periodic (mod 1) point of period n. Then there exist $\underline{\beta} \in \Xi$ and $d \in \mathbf{Z}$, such that $\widehat{\underline{I}}_F(0^+)$ is either $(\underline{\beta} d^L)^\infty$ with $\underline{\beta}$ even or $\underline{\beta} d^R(\underline{\beta}' d^R)^\infty$ with $\underline{\beta}$ odd. Moreover, if $\widehat{\underline{I}}_F(0^+) = (\underline{\beta} d^L)^\infty$ then $\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \in \mathcal{E}_\epsilon$ and if $\widehat{\underline{I}}_F(0^+) = \underline{\beta} d^R(\underline{\beta}' d^R)^\infty$ then $(\underline{\beta}(d+1)^L)^\infty \in \mathcal{E}_\epsilon$.

(b) Assume that c_F is a periodic (mod 1) point of period n. Then there exist $\underline{\beta} \in \Xi$ and $d \in \mathbf{Z}$, such that $\widehat{\underline{I}}_F(c_F^-)$ is either $(\underline{\beta} d^L)^\infty$ with $\underline{\beta}$ even or $(\underline{\beta} d^R)^\infty$ with $\underline{\beta}$ odd. Moreover, if $\widehat{\underline{I}}_F(c_F^-) = (\underline{\beta} d^L)^\infty$ then $(\underline{\beta} d^R)^\infty \in \mathcal{E}_\delta$ and if $\widehat{\underline{I}}_F(c_F^-) = (\underline{\beta} d^R)^\infty$ then $(\underline{\beta} d^L)^\infty \in \mathcal{E}_\delta$.

Proof. We start proving statement (a). Assume first that $\widehat{\underline{I}}_F(0) = \underline{\beta} t^M$ for some $\underline{\beta} \in \Xi$ of length $n-1$ even. If $x > 0$ is sufficiently close to 0 we have that $F^n \mid_{[0,x]}$ is increasing and $F^n(x)$ is also close to $F^n(0) = 0$. Therefore, $\widehat{\underline{I}}_F(0^+) = \underline{\beta} t^L \widehat{\underline{I}}_F(0^+)$. So $\widehat{\underline{I}}_F(0^+) = (\underline{\beta} t^L)^\infty$. Now, assume that $\underline{\beta}$ is odd. Take $x < 0$ sufficiently close to 0. Then $F^n \mid_{[x,0]}$ is increasing and $F^n(x)$ is also close to $F^n(0)$. Thus $\widehat{\underline{I}}_F(0^-) = \underline{\beta}'(t-1)^R \widehat{\underline{I}}_F(0^-)$. Therefore $\widehat{\underline{I}}_F(0^-) = (\underline{\beta}'(t-1)^R)^\infty$ and, in consequence, $\widehat{\underline{I}}_F(0^+) = \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty$.

To prove the second statement of (a) in this case we only need to show that there exists $G \in \mathcal{A}$ such that $\widehat{\underline{I}}_G(0^+) = \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty$ if $\underline{\beta}$ is even or $\widehat{\underline{I}}_G(0^+) = (\underline{\beta} t^L)^\infty$ if $\underline{\beta}$ is odd. We note that the proof of Lemma 2.3.12 does not depend on the fact that the orbit under consideration is twist. So, if $\widehat{\underline{I}}_F(0) = \underline{\beta} t^M$ the statement follows from Lemma 2.3.12 and the part of (a) already proven.

Now, assume that $\widehat{\underline{I}}_F(0) = \underline{\gamma} k^C$ and $\widehat{\underline{I}}_F(c_F) = \underline{\nu} t^M$ where $\underline{\gamma}, \underline{\nu} \in \Xi$, $\underline{\gamma}$ has length $n_1 - 1$, $\underline{\nu}$ has length $n_2 - 1$ and $n_1 + n_2 = n$. If $x > 0$ is sufficiently close to 0 then $F^{n_1}(x)$ is close to c_F. If $\underline{\gamma}$ is even then $F^{n_1} |_{[0,x]}$ is strictly increasing and, hence, $\widehat{\underline{I}}_F(0^+) = \underline{\gamma} k^R \widehat{\underline{I}}_F(c_F^+)$. Otherwise, if $\underline{\gamma}$ is odd, $F^{n_1} |_{[0,x]}$ is strictly decreasing and $\widehat{\underline{I}}_F(0^+) = \underline{\gamma} k^L \widehat{\underline{I}}_F(c_F^-)$. Let now $x > c_F$ be sufficiently close to c_F. If $\underline{\nu}$ is even, then $F^{n_2} |_{[c_F,x]}$ is strictly decreasing and $\widehat{\underline{I}}_F(c_F^+) = \underline{\nu}(t-1)^R \widehat{\underline{I}}_F(0^-)$. Otherwise, if $\underline{\nu}$ is odd, $F^{n_2} |_{[c_F,x]}$ is strictly increasing and $\widehat{\underline{I}}_F(c_F^+) = \underline{\nu} t^L \widehat{\underline{I}}_F(0^+)$. We recall that $\widehat{\underline{I}}_F(c_F^+) = \widehat{\underline{I}}_F(c_F^-)$ and that if $\widehat{\underline{I}}_F(0^-) = (\widehat{\underline{I}}_F(0^+))'$. Hence, if we set

$$\underline{\beta} = \begin{cases} \underline{\gamma} k^R \underline{\nu} \text{ if } \underline{\gamma} \text{ is even,} \\ \underline{\gamma} k^L \underline{\nu} \text{ if } \underline{\gamma} \text{ is odd,} \end{cases}$$

we get

$$\widehat{\underline{I}}_F(0^+) = \begin{cases} \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty & \text{if } \underline{\nu} \text{ is even,} \\ (\underline{\beta} t^L)^\infty & \text{if } \underline{\nu} \text{ is odd,} \end{cases}$$

This ends the proof of the first part of statement (a).

Now, we prove the second statement of (a) in this case. Let P be the (mod. 1) orbit of 0 by F. Then $0, c_F \in P$. Let $x_0 = \min(P \cap (c_F, 1])$, $x_1 = \max(P \cap (0, c_F))$, $J = (c_F, x_0)$ if $\underline{\gamma}$ is even and $J = (x_1, c_F)$ if $\underline{\gamma}$ is odd. Let $G \in \mathcal{A} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$ be close enough to F such that $c_G \in J$, $G|_{[0,1]\setminus J} = F|_{[0,1]\setminus J}$ and $G(c_G) \in (F(c_F), \min(P \cap (F(c_F), \infty)))$. Thus, clearly $\widehat{\underline{I}}_G(0) = \underline{\beta} k^M$. From the proof of the previous case, since $\underline{\beta}$ has always different parity than $\underline{\nu}$, we get

$$\widehat{\underline{I}}_G(0^+) = \begin{cases} (\underline{\beta} t^L)^\infty & \text{if } \underline{\nu} \text{ is odd } (\underline{\beta} \text{ even}), \\ \underline{\beta}(t-1)^R(\underline{\beta}'(t-1)^R)^\infty & \text{if } \underline{\nu} \text{ is even } (\underline{\beta} \text{ odd}), \end{cases}$$

and the proof of (a) follows by using G instead of F. Statement (b) follows in a similar way. ∎

The next lemma gives some properties of the sequences in $\mathcal{P}_\epsilon$ in $\mathcal{P}_\delta$.

Lemma 3.2.3 *Let $\underline{\beta} = \beta_1 \dots \beta_{n-1} \in \Xi$. The following statements hold.*

(a) If $\underline{\beta} d^M \in \mathcal{P}_\epsilon$. Then $(\underline{\beta} d^L)^\infty$ and $(\underline{\beta}'(d-1)^R)^\infty$ are periodic of period n.

(b) If $\underline{\beta} d^C \in \mathcal{P}_\delta$. Then $(\underline{\beta} d^L)^\infty$ and $(\underline{\beta} d^R)^\infty$ are periodic of period n.

Proof. By the minimality of $\underline{\beta} d^M$ we have that $S^j(\underline{\beta} d^M) > \underline{\beta} d^M$ for $j = 1, 2, \dots, n-1$. Assume that $(\underline{\beta} d^L)^\infty$ is periodic of period $k < n$ and set $m = n/k$. Then $\underline{\beta} d^L = (\beta_1 \dots \beta_{k-1} d^L)^m$ and, hence,

$$(\beta_1 \dots \beta_{k-1} d^L)^{m-1} \beta_1 \dots \beta_{k-1} d^M = \underline{\beta} d^M < S^{n-k}(\underline{\beta} d^M) = \beta_1 \dots \beta_{k-1} d^M.$$

In consequence $\beta_1 \dots \beta_{k-1}$ is even and so

$$\beta_1 \dots \beta_{k-1} d^L > \beta_1 \dots \beta_{k-1} (d-1)^R.$$

Since $\underline{\beta} d^M \in \mathcal{P}_\epsilon$ then $\beta_1 \dots \beta_{k-1} d^L (\underline{\beta}(d-1)^R)(\underline{\beta}'(d-1)^R)^\infty \in \mathcal{E}_\epsilon$. Hence, by Theorem 3.1.1(a), we have that

$$\begin{aligned}&(\beta_1 \dots \beta_{k-1} d^L)^{m-1} \beta_1 \dots \beta_{k-1}(d-1)^R (\underline{\beta}'(d-1)^R)^\infty \leq \beta_1 \dots \beta_{k-1}(d-1)^R \dots = \\ &S^{n-k}(\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty);\end{aligned}$$

a contradiction. The proof of statement (a) in the case $(\underline{\beta}'(d-1)^R)^\infty$ and statement (b) follow in a similar way. ∎

The next lemma studies the relation between the periodic sequences in $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ and their shifts.

Lemma 3.2.4 *The following statements hold.*

(a) Let $\underline{\beta} = (\beta_1 \dots \beta_n)^\infty \in \mathcal{E}_\epsilon$. Then $S^j(\underline{\beta}) > \underline{\beta}^*$ for all $j = 1, 2, \dots, n-1$ where $\underline{\beta}^*$ is either $\underline{\beta}$ if $\beta_j = d^L \dots$ or $\underline{\beta}'$ if $\beta_j = d^R \dots$.

(b) Let $\underline{\beta} = (\beta_1 \dots \beta_n)^\infty \in \mathcal{E}_\delta$. Then $S^j(\underline{\beta}) < \underline{\beta}$ for all $j = 1, 2, \dots, n-1$.

Proof. We prove (a). Statement (b) follows in a similar way. Let $j \in \{2, \dots, n\}$. If $\beta_{j-1} = d^L$ for some $d \in \mathbf{Z}$ then, by Theorem 3.1.1, since $S^{j-1}(\underline{\beta}) \geq \underline{\beta}$ and $S^{j-1}(\underline{\beta}) \neq \underline{\beta}$ the lemma follows in an obvious way. If $\beta_{j-1} = d^R$ for some $d \in \mathbf{Z}$, we have $S^{j-1}(\underline{\beta}) \geq \underline{\beta}'$. Assume that $S^{j-1}(\underline{\beta}) = \underline{\beta}'$. Then

$$\overbrace{\beta_j \beta_{j+1} \dots \beta_n \beta_1 \dots \beta_{j-1}}^{n} \overbrace{\beta_j \dots \beta_n \beta_1 \dots \beta_{j-1}}^{n} \dots = \beta_1' \beta_2 \dots \beta_n \beta_1 \dots \beta_n \dots$$

and, hence, $\beta_1' = \beta_j = \beta_1$; a contradiction. This ends the proof of (a). ∎

The proof of the following lemma follows by direct computation.

Lemma 3.2.5 *The following statements hold.*

(a) *Let $f \in U(I)$ be negative. If $f(c) \geq c$, then $k(f) = L^\infty$. Otherwise $k(f) = R\mathcal{S}(k(f))$ and there exists $c_- < c < c_+$ with $f(c_-) = f(c_+) = c$. Then the following statements hold.*

(a.1) $\underline{\theta}(x) = RL\ldots$ *if and only if* $x \in [\inf I, c_-)$.

(a.2) $\underline{\theta}(x) = RR\ldots$ *if and only if* $x \in (c_-, c)$.

(a.3) $\underline{\theta}(x) = LR\ldots$ *if and only if* $x \in (c, c_+)$.

(a.4) $\underline{\theta}(x) = LL\ldots$ *if and only if* $x \in (c_+, \sup I]$.

(b) *Let $f \in U(I)$ be positive. If $f(c) \leq c$, then $k(f) = L^\infty$. Otherwise $k(f) = R\mathcal{S}(k(f))$ and there exists $c_- < c < c_+$ with $f(c_-) = f(c_+) = c$. Then the following statements hold.*

(b.1) $\underline{\theta}(x) = LR\ldots$ *if and only if* $x \in (c_+, \sup I]$.

(b.2) $\underline{\theta}(x) = RR\ldots$ *if and only if* $x \in (c, c_+)$.

(b.3) $\underline{\theta}(x) = LR\ldots$ *if and only if* $x \in (c_-, c)$.

(b.4) $\underline{\theta}(x) = LL\ldots$ *if and only if* $x \in [\inf I, c_-)$.

Let $I, J \subset \mathbf{R}$ two closed intervals. Let $f : I \longrightarrow I$ and $g : J \longrightarrow J$ two continuous maps. We say that f is *topologically conjugate* to g if there exists a homeomorphism $h : I \longrightarrow J$ such that $h \circ f = g \circ h$. From [9] (see also [24]) we have that if $f \in U(I)$ and $g \in U(J)$ are topologically conjugate then $k(f) = k(g)$.

The next proposition justifies the definition of the $\star$−products in the case $\underline{\beta}$ empty.

Proposition 3.2.6 *Let $\underline{K} \in \mathcal{K}$ and $d \in \mathbf{Z}$. Then the following statements hold.*

(a) *There exist $F \in \mathcal{A}$ and $J \subset \mathbf{R}$, a closed interval containing 0, such that $(F - d)\mid_J$ is unimodal with $k((F - d)\mid_J) = \underline{K}$ and $\widehat{\underline{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$.*

(b) *There exists $F \in \mathcal{A}$ and $J \subset \mathbf{R}$, a closed interval containing c_F, such that $(F - d)\mid_J$ is unimodal with $k((F - d)\mid_J) = \underline{K}$ and $\widehat{\underline{I}}_F(c_F^-) = d^C \star_\delta \underline{K}$.*

Proof. Let $f \in U(I)$ be negative such that $k(f) = \underline{K}$. Take $\epsilon > 0$ and $J = [-\epsilon, \epsilon]$, and let $h : I \longrightarrow J$ be the unique increasing map such that $h(c) = 0$ and h is affine in $[\min I, c]$, $[c, \max I]$. Let $F \in \mathcal{A}$ be such that $F(x) = h \circ f \circ h^{-1}(x) + d$ for each $x \in J$. Clearly, $(F - d)\mid_J$ is topologically conjugate to f. Then $k((F - d)\mid_J) = k(f) = K_1 K_2 \ldots$. We observe that since $(F - d)$ maps J

into itself we have that $F(J) \subset J + d$. Since $F \in \mathcal{L}$ we have that for all $j \geq 1$, $F^j(J) \subset J + jd$. On the other hand, since $s((F-d)^j(0^+)) = s(F^j(0^+))$ we get that for all $j \geq 1$, $s(F^j(0^+)) = K_j$. Assume that $(F-d)(0) \geq 0$, then $f(c) \geq c$ and, from Lemma 3.2.5, we have that $k(f) = L^\infty$. Since $F(0) \geq d$ we have that $F^j(0) \in [0, \epsilon] + jd$ for all $i \geq 0$. Then for all $i \geq 1$ we have $d(F^j(0^+)) = jd - (j-1)d = d$ and $\underline{\widehat{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$. Now, assume that $(F-d)(0) < 0$. Then $f(c) < c$ and, from Lemma 3.2.5(a), we have that $K_1 = R$. Since $F(0) < d$ we obtain that $F(0) \in [-\epsilon, 0] + d$. Then $d(0^+) = d - 1$ and so $\underline{\widehat{I}}_F(0^+) = (d-1)^R \ldots$. Let $j \geq 2$. Assume that $K_{j-1}K_j = RL$. Then $\mathcal{S}^{j-2}(k(f)) = \underline{\theta}(f^{j-2}(x)) = RL \ldots$ for $x > f(c)$, close enough to $f(c)$. From Lemma 3.2.5 (a.1) we have that $f^{j-1}(c) \in [\min I, c_-)$ and, hence, $F^{j-1}(0) \in [-\epsilon, h(c_-)) + (j-1)d$. Moreover $F^j(0) \in (0, \epsilon] + jd$. Then $d(F^{j-1}(0^+)) = jd - ((j-1)d - 1) = d + 1$. If $K_{j-1}K_j = LL$, then, $F^{j-1}(0) \in (h(c_+), \epsilon] + (j-1)d$ and $F^j(0) \in (0, \epsilon] + id$. So $d(F^{j-1}(0^+)) = jd - (j-1)d = d$. If $K_{j-1}K_j = RR$, then $F^{j-1}(0^+) \in (h(c_-), 0) + (j-1)d$ and $F^j(0^+) \in [-\epsilon, 0) + jd$. Thus, $d(F^{j-1}(0^+)) = (jd-1) - ((j-1)d - 1) = d$. Finally, if $K_{j-1}K_j = LR$ then $F^{j-1}(0) \in (0, h(c_+)) + (j-1)d$, $F^j(0) \in [-\epsilon, 0) + jd$. Therefore, $d(F^{j-1}(0^+)) = (jd-1) - (j-1)d = d - 1$. From the definition of $\star_\epsilon$ we have that $\underline{\widehat{I}}_F(0^+) = d^M \star_\epsilon \underline{K}$. Statement (b) follows in a similar way. ∎

3.2.5 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. We only will prove Theorem 3.2.1 for $\star_\epsilon$. The proof for $\star_\delta$ follows in a similar way. Let $\underline{\gamma} = \underline{\beta} d^M \in \mathcal{P}_\epsilon$ and $\underline{K} \in \mathcal{K}$. We only will prove the statement in the case $\underline{\beta}$ even. The case $\underline{\beta}$ odd follows analogously. First we are going to prove that $\underline{\gamma} \star_\epsilon \underline{K} \in \mathcal{E}_\epsilon$. If $\underline{\beta}$ is empty then this follows from Proposition 3.2.6(a), the definition of $\mathcal{E}_\epsilon$ and Theorem 2.3.2. Assume now that $\underline{\beta}$ is not empty. We note that $\underline{\gamma} \star_\epsilon L^\infty = (\underline{\beta} d^L)^\infty$ and $\underline{\gamma} \star_\epsilon R^\infty = \underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty$. Since $\underline{\beta} d^M \in \mathcal{P}_\epsilon$ these two sequences belong to $\mathcal{E}_\epsilon$ and we are done. Thus we can assume that $\underline{K} \notin \{L^\infty, R^\infty\}$. From Collet and Eckmann [9] we have that $\underline{K} = RL \ldots$. Let $\underline{\beta} = \beta_1\beta_2 \ldots \beta_{n-1}$, $\underline{K} = K_1K_2 \ldots$ and $j = nm$ with $m \geq 0$. Then we have $\underline{\gamma} \star_\epsilon \underline{K} = \underline{\beta}\varphi(K_1, d)\chi(K_1, \underline{\beta})\varphi(K_2, d)\chi(K_2, \underline{\beta}) \ldots$. It is not difficult to see that, since $\underline{K}$ is maximal, then $\varphi(K_1, d)\varphi(K_2, d) \ldots \in \mathcal{AD}$ is minimal. Therefore, if $K_{m-1} = L$ then

$$S^j(\underline{\gamma} \star_\epsilon \underline{K}) = \underline{\beta}\varphi(K_m, d)\chi(K_m, \underline{\beta})\varphi(K_{m+1}, d) \ldots \geq \underline{\gamma} \star_\epsilon \underline{K}.$$

Otherwise,

$$S^j(\underline{\gamma} \star_\epsilon \underline{K}) = \underline{\beta}'\varphi(K_m, d)\chi(K_m, \underline{\beta})\varphi(K_{m+1}, d)\ldots \geq (\underline{\gamma} \star_\epsilon \underline{K})'$$

and, by Theorem 3.1.1(a), we are done. So, take now $j = nm + p$ with $m \geq 0$, $1 \leq p < n$. Then we have to compare

$$\begin{aligned} &S^j(\underline{\gamma} \star_\epsilon \underline{K}) = \beta_{p+1}\ldots\beta_{n-1}\varphi(K_m, d)\chi(K_m, \underline{\beta})\varphi(K_{m+1}, d)\ldots = \\ &\underline{\upsilon}\varphi(K_m, d)\chi(K_m, \underline{\beta})\varphi(K_{m+1}, d)\ldots\,, \end{aligned} \tag{3.2.1}$$

with

$$\begin{aligned} \underline{\gamma} \star_\epsilon \underline{K} = \beta_1\ldots\beta_{n-p-1}\beta_{n-p}\ldots\beta_{n-1}\varphi(K_1, d)\ldots = \\ \overline{\underline{\upsilon}}\beta_{n-p}\ldots\beta_{n-1}\varphi(K_1, d)\ldots\,. \end{aligned} \tag{3.2.2}$$

Set

$$\underline{\upsilon}^* = \begin{cases} \overline{\underline{\upsilon}} & \text{if } \beta_p = t^L \\ \overline{\underline{\upsilon}}' & \text{if } \beta_p = t^R \end{cases}$$

where $t \in \mathbf{Z}$ and $(\underline{\gamma} \star_\epsilon \underline{K})^* = \underline{\upsilon}^*\beta_{n-p}\ldots\beta_{n-1}\varphi(K_1, d)\ldots$. By Theorem 3.1.1(a) we have to show that $S^j(\underline{\gamma} \star_\epsilon \underline{K}) \geq (\underline{\gamma} \star_\epsilon \underline{K})^*$. Since $\underline{\beta}d^M \in \mathcal{P}_\epsilon$, $\underline{\beta}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty, (\underline{\beta}d^L)^\infty \in \mathcal{E}_\epsilon$. Therefore, by Theorem 3.1.1(a) and Lemma 3.2.4(a), for all $1 \leq p < n$, we have

$$\underline{\upsilon}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \geq \underline{\upsilon}^*\beta_{n-p}\ldots\beta_{n-1}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty \tag{3.2.3}$$

and

$$\underline{\upsilon}d^L(\underline{\beta}d^L)^\infty > \underline{\upsilon}^*\beta_{n-p}\ldots\beta_{n-1}d^L(\underline{\beta}d^L)^\infty. \tag{3.2.4}$$

Clearly if $\underline{\upsilon} \neq \underline{\upsilon}^*$ then $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ and we are done. So assume that $\underline{\upsilon} = \underline{\upsilon}^*$. First we consider the case $\underline{\upsilon}$ even. If $\varphi(K_m, d) = d^L$ then either $d^L > \beta_{n-p}$ and, from (3.2.1) and (3.2.2), we see that $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ or $d^L = \beta_{n-p}$. In the latter, since $\underline{\upsilon}d^L$ is even, from (3.2.4) we have that

$$(\underline{\beta}d^L)^\infty > \beta_{n-p+1}\ldots\beta_{n-1}d^L(\underline{\beta}d^L)^\infty;$$

a contradiction with Lemma 3.2.4(a). Now, let $\varphi(K_m, d) = (d-1)^R$. From (3.2.3) we have $\beta_{n-p} \leq (d-1)^R$. If $\beta_{n-p} < (d-1)^R$, then $S^j(\underline{\gamma} \star_\epsilon \underline{K}) > (\underline{\gamma} \star_\epsilon \underline{K})^*$ by (3.2.1) and (3.2.2). So, assume that $\beta_{n-p} = (d-1)^R$. Then $\underline{\upsilon}(d-1)^R = \underline{\upsilon}^*(d-1)^R$ is odd and, from (3.2.3), we have that

$$(\underline{\beta}'(d-1)^R)^\infty \leq \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R(\underline{\beta}'(d-1)^R)^\infty.$$

We note that $S^{n-p}((\underline{\beta}'(d-1)^R)^\infty) = (\beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p})^\infty$. Therefore, if

$$\beta_1'\beta_2\dots\beta_{n-1}(d-1)^R = \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p}$$

then, $S^{n-p}((\underline{\beta}'(d-1)^R)^\infty) = (\underline{\beta}'(d-1)^R)^\infty$ which is a contradiction by Lemma 3.2.3(a). In consequence,

$$\beta_1'\beta_2\dots\beta_{n-1}(d-1)^R < \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p} \tag{3.2.5}$$

and, by (3.2.1) and (3.2.2), $S^j(\underline{\gamma}\star_\epsilon\underline{K}) > (\underline{\gamma}\star_\epsilon\underline{K})^*$ if $\varphi(K_{m+1},d) = (d-1)^R$ (recall that $\varphi(K_1,d) = (d-1)^R$). Now, assume that $\varphi(K_{m+1},d) = d^L$. If

$$\beta_1'\beta_2\dots\beta_{n-1} < \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p-1}$$

then we also have $S^j(\underline{\gamma}\star_\epsilon\underline{K}) > (\underline{\gamma}\star_\epsilon\underline{K})^*$. Otherwise, since $\underline{\beta}'$ is even, from (3.2.5) we have that

$$\beta_1'\beta_2\dots\beta_{n-1} = \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p-1}$$

and $\beta_{n-p} \geq d^L$. If $\beta_{n-p} > d^L$ then the statement follows as before. Hence, $\beta_{n-p} = d^L$ and so

$$\beta_1'\beta_2\dots\beta_{n-1}d^L = \beta_{n-p+1}\dots\beta_{n-1}(d-1)^R\beta_1'\dots\beta_{n-p-1}\beta_{n-p}.$$

This is a contradiction because the left hand side of the above equation has different parity that the right hand side. The case $\underline{v}$ odd is handled by analogy. This ends the proof of the first statement of the theorem.

Now, we are going to prove that $\underline{\gamma}\star_\epsilon$ is order reversing. Let $\underline{K},\underline{K}' \in \mathcal{K}$ be such that $\underline{K} < \underline{K}'$. Set $\underline{K} = K_1K_2\dots$ and $\underline{K}' = K_1'K_2'\dots$. Then there exists $n \geq 1$ such that $K_1\dots K_{n-1} = K_1'\dots K_{n-1}'$ and $K_n < K_n'$ if $K_1\dots K_{n-1}$ is even and $K_n > K_n'$ if $K_1\dots K_{n-1}$ is odd. We will only consider the case $K_1\dots K_{n-1}$ even. The proof in the case odd follows similarly. Then we have $K_n = L < R = K_n'$. Assume that $\underline{\beta}$ is not the empty sequence. Then $\underline{\gamma}\star_\epsilon\underline{K} = \underline{\beta}d_1^{K_1}\chi(K_1,\underline{\beta})d_2^{K_2}\dots\chi(K_{n-1},\underline{\beta})d_n^{K_n}\dots$ and $\underline{\gamma}\star_\epsilon\underline{K}' = \underline{\beta}t_1^{K_1'}\chi(K_1,\underline{\beta})t_2^{K_2'}\dots\chi(K_{n-1},\underline{\beta})t_n^{K_n'}\dots$.

Then

$$\underline{\beta} d_1^{K_1} \chi(K_1, \underline{\beta}) d_2^{K_2} \ldots \chi(K_{n-1}, \underline{\beta}) = \underline{\beta} t_1^{K_1'} \chi(K_1, \underline{\beta}) t_2^{K_2'} \ldots \chi(K_{n-1}, \underline{\beta}),$$

$d_n^{K_n} = d^L$, $t_n^{K_n'} = (d-1)^R$ and $\underline{\beta} d_1^{s_1} \chi(K_1, \underline{\beta}) d_2^{s_2} \ldots \chi(K_{n-1}, \underline{\beta})$ is even. Then, clearly, $\underline{\gamma} \star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. Now, assume that $\underline{\beta}$ is the empty sequence. Then $\underline{\gamma} \star_\epsilon \underline{K} = d_1^{K_1} \ldots d_{n-1}^{K_{n-1}} d_n^{K_n} \ldots$ and $\underline{\gamma} \star_\epsilon \underline{K}' = t_1^{K_1'} \ldots t_{n-1}^{K_{n-1}'} t_n^{K_n'} \ldots = d_1^{K_1} \ldots d_{n-1}^{K_{n-1}} t_n^{K_n'} \ldots$ and the result follows as in the case $\underline{\beta}$ not empty. From the assumptions only one of the following two possibilities can occur: either $K_{n-1} K_n = RL$ and $K_{n-1}' K_n' = RR$, or $K_{n-1} K_n = LL$ and $K_{n-1}' K_n' = LR$. Assume that $K_{n-1} K_n = RL$ and $K_{n-1}' K_n' = RR$. Then $d_n^{K_n} = (d+1)^L$ and $t_n^{K_n'} = d^R$ and $\underline{\gamma} \star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. Now, let $K_{n-1} K_n = LL$ and $K_{n-1}' K_n' = LR$. Then $d_n^{K_n} = d^L$ and $t_n^{K_n'} = (d-1)^R$ and also, $\underline{\gamma} \star_\epsilon \underline{K}' < \underline{\gamma} \star_\epsilon \underline{K}$. This concludes the proof of the second statement.

The third statement follows from Theorem II.2.7 of [9]. ∎

3.2.6 Concluding remarks

In the preceding section we have shown that the unimodal boxes $\underline{\gamma} \star_\epsilon \mathcal{K}$ and $\underline{\gamma} \star_\delta \mathcal{K}$ are connected. However, the topological structure of the spaces

$$\mathcal{E}_\epsilon(\underline{\gamma}) = \underline{\gamma} \star_\epsilon \mathcal{K} \times \mathcal{E}_\delta$$

(respectively

$$\mathcal{E}_\delta(\underline{\gamma}) = \mathcal{E}_\epsilon \times \underline{\gamma} \star_\delta \mathcal{K}$$

is much more complicated. We illustrate this fact with the following examples. Let $\underline{\gamma} = 0^L 1^M$. Then $\underline{\gamma} \star_\epsilon L^\infty = (0^L 1^L)^\infty$ and $\underline{\gamma} \star_\epsilon RL^\infty = 0^L 0^R 1^L 1^L (0^L 1^L)^\infty$. Therefore, $\underline{\gamma} \star_\epsilon \mathcal{K} = [(0^L 1^L)^\infty, 0^L 0^R 1^L 1^L (0^L 1^L)^\infty]$.

Example 1: *the space $\mathcal{E}_\epsilon(\underline{\gamma})$ contains "accumulating" holes in $\mathcal{E}$ consisting of "horizontal lines".* Let $\underline{\alpha} = (3^L)^\infty \in \mathcal{E}_\delta$. Clearly $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times \{\underline{\alpha}\} \subset \mathcal{E}^* \subset \mathcal{E}$. Let now $\underline{\alpha}_n = (3^L)^n (-1^L)^\infty \in \mathcal{E}_\delta$. Then $\underline{\alpha}_n < \underline{\alpha}_{n+1} < \underline{\alpha}$ for all $n \in \mathbf{N}$. Since $S^{n-1}(\underline{\alpha}_n) = (-1^L)^\infty < \underline{\omega}$ for all $\underline{\omega} \in \underline{\gamma} \star_\epsilon \mathcal{K}$ we have that for all $n \in \mathbf{N}$, $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times \{\underline{\alpha}_n\} \notin \mathcal{E}$. We also note that $d(\underline{\alpha}_n, \underline{\alpha})$ tends to 0 as $n \to \infty$.

Example 2: *the 'accumulating" holes in $\mathcal{E}$ consisting of "horizontal lines" are intertwine with "horizontal lines" inside $\mathcal{E}$.* Let $\underline{\beta}_n = (3^L)^n (2^L)^\infty \in \mathcal{E}_\delta$. Then for all $n \in \mathbf{N}$, $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon$

$L^\infty] \times \{\underline{\beta}_n\} \subset \mathcal{E}$ but $d(\underline{\alpha}_n, \underline{\beta}_n) = \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$ which tends to 0 when $n \to \infty$.

Example 3: *there exists "rectangles" in* $\mathcal{E} \cap (\underline{\gamma} \star_\epsilon \mathcal{K} \times \underline{\beta} \star_\delta \mathcal{K})$. Let $\underline{\beta} = 3^M$. Then $\underline{\beta} \star_\delta L^\infty = (3^L)^\infty$ and $\underline{\beta} \star_\delta RL^\infty = 3^R(3^L)^\infty$. It is not difficult to see that $[\underline{\gamma} \star_\epsilon RL^\infty, \underline{\gamma} \star_\epsilon L^\infty] \times [\underline{\beta} \star_\delta L^\infty, \underline{\beta} \star_\delta RL^\infty] \subset \mathcal{E}$.

3.3 The $\odot-$product

3.3.1 Introduction and preliminary results

In this section we shall study the structure of certain subsets of $\mathcal{E}$, the space of kneading pairs, in order to explain the structure of the bifurcations of "canonical" families of maps from $\mathcal{A}$, like the standard maps family.

Let $k \in \mathbf{Z}$. We denote by Σ_k the set of sequences in $\{k^L, (k+1)^L\}^{\mathbf{N}}$. Let $\underline{\alpha} = d_1^L d_2^L \ldots$ and $\underline{\beta} = t_1^L t_2^L \ldots$ be two sequences in Σ_k. We consider in Σ_k the topology defined by the metric $d(\underline{\alpha}, \underline{\beta}) = \sum\limits_{i=0}^{\infty} 2^{-i}|d_i - t_i|$. With this topology, Σ_k is a compact metric space. Let $S_k : \Sigma_k \longrightarrow \Sigma_k$ denote the usual shift transformation restricted to Σ_k. Clearly, S_k is continuous. Let $\pi_k : \Sigma_k \longrightarrow \Sigma_0$ be the order preserving homeomorphism defined by $\pi_k(d_1^L d_2^L \ldots) = (d_1 - k)^L (d_2 - k)^L \ldots$. Clearly, $S_0 \circ \pi_k = \pi_k \circ S_k$.

For $k \in \mathbf{Z}$ we define the sets $\mathcal{B}_\epsilon(k) = \Sigma_k \cap \mathcal{E}_\epsilon$ and $\mathcal{B}_\delta(k) = \Sigma_k \cap \mathcal{E}_\delta$. We note that the sets $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$ are invariant under "translations". That is, if $d_1^{s_1} d_2^{s_2} \ldots$ is a sequence in E_ϵ (respectively in $\mathcal{E}_\delta$) then $(d_1 + k)^{s_1}(d_2 + k)^{s_2} \ldots$ also belongs to $\mathcal{E}_\epsilon$ (respectively $\mathcal{E}_\delta$). Therefore, $\mathcal{B}_\epsilon(k) = \pi_k^{-1}(\mathcal{B}_\epsilon(0))$ and $\mathcal{B}_\delta(k) = \pi_k^{-1}(\mathcal{B}_\delta(0))$. From Theorem 3.1.1 we have that $\mathcal{B}_\epsilon(k)$ (respectively $\mathcal{B}_\delta(k)$) are the minimal (respectively maximal) sequences in Σ_k.

For $a \in \mathbf{R}$ we will denote $a - \widetilde{E}(a)$ by $\widetilde{D}(a)$. Also, $\mathbf{Q} \setminus \mathbf{Z}$ will be denoted by $\mathbf{Q}^*$.

We note that from Lemma 2.3.3, if $a \in \mathbf{Q}^*$ and $a = p/q$ with $(p, q) = 1$ and $q \neq 2$ then the finite sequences $\epsilon_2(a)^L \ldots \epsilon_{q-1}(a)^L$ and $\delta_2(a)^L \ldots \delta_{q-1}(a)^L$ are equal. We will denote this finite sequence by $\underline{r}(a)$ (we take as $r(1/2)$ the empty sequence).

Now we are ready to define the $\odot-$products.

For $\alpha = d^L$ with $d \in \{0, 1\}$ we set $\widehat{\alpha} = (1-d)^L$. Then for $a \in (0, 1]$ and $\underline{\alpha} = \alpha_1 \alpha_2 \ldots \in \mathcal{B}_\epsilon(0)$

we define

$$a \odot_\epsilon \underline{\alpha} = \begin{cases} 0^L \underline{r}(a)\alpha_1\widehat{\alpha}_1\underline{r}(a)\alpha_2\widehat{\alpha}_2 \ldots & \text{if } a \in \mathbf{Q}^*, \\ \widehat{\underline{I}}_\epsilon(a) & \text{if } a \notin \mathbf{Q}^* \text{ and } \underline{\alpha} = (1^L)^\infty, \\ \widehat{\underline{I}}^*_\delta(a) & \text{if } a \notin \mathbf{Q}^* \text{ and } \underline{\alpha} \neq (1^L)^\infty. \end{cases}$$

We extend the above definition to each $a \in \mathbf{R}$ by setting $a \odot_\epsilon \underline{\alpha} = \pi^{-1}_{\widetilde{E}(a)}(\widetilde{D}(a) \odot_\epsilon \underline{\alpha})$.

Now, we define the version $\odot_\delta$ of the $\odot-$product as follows. Let $a \in [0,1)$ and $\underline{\alpha} = \alpha_1\alpha_2 \ldots \in \mathcal{B}_\delta(0)$. Then we set

$$a \odot_\delta \underline{\alpha} = \begin{cases} 1^L \underline{r}(a)\alpha_1\widehat{\alpha}_1\underline{r}(a)\alpha_2\widehat{\alpha}_2 \ldots & \text{if } a \in \mathbf{Q}^*, \\ \widehat{\underline{I}}_\delta(a) & \text{if } a \notin \mathbf{Q}^* \text{ and } \underline{\alpha} = (0^L)^\infty, \\ \widehat{\underline{I}}^*_\epsilon(a) & \text{if } a \notin \mathbf{Q} \text{ and } \underline{\alpha} \neq (0^L)^\infty, \end{cases}$$

and we extend the above definition to each $a \in \mathbf{R}$ by $a \odot_\delta \underline{\alpha} = \pi^{-1}_{E(a)}(D(a) \odot_\delta \underline{\alpha})$.

The next result which we will be proved in Subsection 3.3.2 gives a first motivation to the $\odot-$products.

Proposition 3.3.1 *Let $a \in \mathbf{R}$. Then $a \odot_\epsilon (0^L)^\infty = \widehat{\underline{I}}^*_\delta(a)$, $a \odot_\epsilon (1^L)^\infty = \widehat{\underline{I}}_\epsilon(a)$, $a \odot_\delta (0^L)^\infty = \widehat{\underline{I}}_\delta(a)$ and $a \odot_\delta (1^L)^\infty = \widehat{\underline{I}}^*_\epsilon(a)$.*

From the above proposition we see that Theorem 2.3.4 can be written as.

Theorem 3.3.2 *Let $F \in \mathcal{A}$ and let $a, b \in \mathbf{R}$ with $a \leq b$. Then $L_F = [a,b]$ if and only if $\widehat{\underline{I}}_F(0^+) \in [a \odot_\epsilon (0^L)^\infty, a \odot_\epsilon (1^L)^\infty]$ and $\widehat{\underline{I}}_F(c_F^-) \in [a \odot_\delta (0^L)^\infty, a \odot_\delta (1^L)^\infty]$.*

The next result is the main result of this section. It studies the $\odot-$products and will allow us to describe bifurcations of logistic families of maps from $\mathcal{A}$.

For $\underline{\alpha} \in \Sigma_k$, $\underline{\alpha} = d_1^L d_2^L \ldots$ we define the *symbolic rotation number of* $\underline{\alpha}$ as

$$\rho(\underline{\alpha}) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d_i.$$

Theorem 3.3.3 *Let $a, b \in \mathbf{R}$ with $a \leq b$. Then the following statements hold:*

(a) Let $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\epsilon(0)$ with $\underline{\alpha} < \underline{\beta}$. Then $a \odot_\epsilon \underline{\alpha} \leq b \odot_\epsilon \underline{\beta}$. Moreover if $a \in \mathbf{Q}^*$ then $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$.

(b) Let $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\delta(0)$ with $\underline{\alpha} < \underline{\beta}$. Then $a \odot_\delta \underline{\alpha} \leq b \odot_\delta \underline{\beta}$. Moreover if $a \in \mathbf{Q}^*$ then $a \odot_\delta \underline{\alpha} < a \odot_\delta \underline{\beta}$.

(c) Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$*. Then* $a \odot_\epsilon \underline{\alpha} \in \mathcal{E}_\epsilon$ *and* $\rho(a \odot_\epsilon \underline{\alpha}) = a$*.*

(d) Let $\underline{\alpha} \in \mathcal{B}_\delta(0)$*. Then* $a \odot_\delta \underline{\alpha} \in \mathcal{E}_\delta$ *and* $\rho(a \odot_\epsilon \underline{\alpha}) = a$*.*

(e) Let $a \in \mathbf{Q}^*$ *and* $(\underline{\alpha}, \underline{\beta}) \in \mathcal{B}_\epsilon(0) \times \mathcal{B}_\delta(0)$ *be such that* $\underline{\alpha} \neq (1^L)^\infty$ *and* $\underline{\beta} \neq (0^L)^\infty$*. If* $S^n(\underline{\alpha}) \leq \underline{\beta}$ *and* $S^n(\underline{\beta}) \geq \underline{\alpha}$ *for all* $n \geq 0$*, then* $(a \odot_\epsilon \underline{\alpha}, a \odot_\delta \underline{\beta}) \in \mathcal{E}^* \subset \mathcal{E}$*.*

We note that if $(\underline{\alpha}, \underline{\beta}) \in \mathcal{E}$, by Theorem 2.3.2 and Proposition 2.3.1 we have that $\underline{\alpha}' \leq \underline{\beta}$, $S^n(\underline{\alpha}) \leq \underline{\beta}$ and $S^n(\underline{\beta}) \geq \underline{\alpha}$ for all $n \geq 0$. Thus from Theorem 3.3.3(e) we have the following.

Corollary 3.3.4 *Let* $a \in \mathbf{Q}^*$ *and let* $(\underline{\alpha}, \underline{\beta}) \in (\mathcal{B}_\epsilon(0) \times \mathcal{B}_\delta(0)) \cap \mathcal{E}$ *be such that* $\underline{\alpha} \neq (1^L)^\infty$ *and* $\underline{\beta} \neq (0^L)^\infty$*. Then* $(a \odot_\epsilon \underline{\alpha}, a \odot_\delta \underline{\beta}) \in \mathcal{E}^* \subset \mathcal{E}$*.*

We will prove Theorem 3.3.3 in Subsection 3.3.3.

3.3.2 Definitions and preliminary results

We start by introducing some technical results about the sequences $\underline{\widehat{I}}^*_\delta(a), \underline{\widehat{I}}_\epsilon(a), \underline{\widehat{I}}_\delta(a)$ and $\underline{\widehat{I}}^*_\epsilon(a)$. The following lemma is due to Alsedà and Mañosas [5].

Lemma 3.3.5 *The following statements hold:*

(a) If $a = p/q$ *with* $(p, q) = 1$ *then* $\underline{\widehat{I}}_\epsilon(a)$ *and* $\underline{\widehat{I}}_\delta(a)$ *are periodic with period* q *(i.e.* $S^q(\underline{\widehat{I}}_\epsilon(a)) = \underline{\widehat{I}}_\epsilon(a)$ *and* $S^q(\underline{\widehat{I}}_\delta(a)) = \underline{\widehat{I}}_\delta(a)$*).*

(b) Let $a, b \in \mathbf{R}$ *with* $a < b$*. Then* $\underline{\widehat{I}}_\epsilon(a) < \underline{\widehat{I}}_\epsilon(b)$*,* $\underline{\widehat{I}}_\delta(a) < \underline{\widehat{I}}_\delta(b)$*,* $\underline{\widehat{I}}^*_\epsilon(a) < \underline{\widehat{I}}^*_\epsilon(b)$ *and* $\underline{\widehat{I}}^*_\delta(a) < \underline{\widehat{I}}^*_\delta(b)$*.*

From Theorem 2.3.2 and Theorem 3.1.1 we have the following.

Lemma 3.3.6 *Let* $a \in \mathbf{R}$*. Then* $\underline{\widehat{I}}^*_\delta(a), \underline{\widehat{I}}_\epsilon(a) \in \mathcal{E}_\epsilon$ *are minimal and* $\underline{\widehat{I}}_\delta(a), \underline{\widehat{I}}^*_\epsilon(a) \in \mathcal{E}_\delta$ *are maximal.*

Lemma 3.3.7 *Let* $a \in \mathbf{R}$*. Then* $\epsilon_1(a) \leq \epsilon_i(a) \leq \epsilon_1(a) + 1$ *and* $\delta_1(a) - 1 \leq \delta_i(a) \leq \delta_1(a)$ *for all* $i \geq 1$*.*

Proof. We recall that $\epsilon_i(a) = E(ia) - E((i-1)a) = E(a + (i-1)a) - E((i-1)a)$. Then, from the fact that $E(x) + E(y) \leq E(x+y) \leq E(x) + E(y) + 1$ for all $x, y \in \mathbf{R}$, we have that $\epsilon_1(a) \leq \epsilon_i(a) \leq \epsilon_1(a) + 1$ for all $i \geq 1$. In a similar way we can prove that $\delta_1(a) - 1 \leq \delta_i(a) \leq \delta_1(a)$ for all $i \geq 1$. ∎

Lemma 3.3.8 *Let $a \in \mathbf{R}$. Then $\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}^*_\delta(a) \in \Sigma_{\widetilde{E}(a)}$ and $\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a) \in \Sigma_{E(a)}$.*

Proof. From Lemmas 3.3.5(a) and 2.3.3, the fact that $\epsilon_1(a) = \delta_1(a) - 1 = E(a) = \widetilde{E}(a)$ if $a \notin \mathbf{Z}$ and Lemma 3.3.7 the statement follows when $a \notin \mathbf{Z}$. If $a \in \mathbf{Z}$, then from Lemma 2.3.11 we have that $\widehat{\underline{I}}_\epsilon(a) = \widehat{\underline{I}}_\delta(a) = (a^L)^\infty$, $\widehat{\underline{I}}^*_\epsilon(a) = (a+1)^L(a^L)^\infty$ and $\widehat{\underline{I}}^*_\delta(a) = (a-1)^L(a^L)^\infty$. Since $E(a) = a$ and $\widetilde{E}(a) = a - 1$ the statement follows also in this case. ∎

Then we have the following corollary which will be useful in the next section.

Corollary 3.3.9 *Let $a \in \mathbf{R}$. Then $\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}^*_\delta(a) \in \mathcal{B}_\epsilon(\widetilde{E}(a))$ and $\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a) \in \mathcal{B}_\delta(E(a)))$.*

Proof. It follows from Lemmas 3.3.8 and 3.3.6. ∎

Corollary 3.3.10 *Let $a \in \mathbf{R}$. Then $\widehat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\widetilde{E}(a)}(\widehat{\underline{I}}_\epsilon(\widetilde{D}(a)))$, $\widehat{\underline{I}}^*_\delta(a) = \pi^{-1}_{\widetilde{E}(a)}(\widehat{\underline{I}}^*_\delta(\widetilde{D}(a)))$, $\widehat{\underline{I}}_\delta(a) = \pi^{-1}_{E(a)}(\widehat{\underline{I}}_\delta(D(a)))$ and $\widehat{\underline{I}}^*_\epsilon(a) = \pi^{-1}_{E(a)}(\widehat{\underline{I}}^*_\epsilon(D(a)))$.*

Proof. Let $a \in \mathbf{R}$. Then

$$\begin{aligned}
\epsilon_i(a) &= E(ia) - E((i-1)a) \\
&= E(i(D(a) + E(a))) - E((i-1)(D(a) + E(a)))) \\
&= E(iD(a)) + iE(a) - E((i-1)D(a)) - (i-1)E(a) \\
&= E(iD(a)) - E((i-1)D(a)) + E(a) \\
&= \epsilon_i(D(a)) + E(a).
\end{aligned}$$

If $a \notin \mathbf{Z}$, since $\widetilde{E}(a) = E(a)$ and $\widetilde{D}(a) = D(a)$ we have that $\widehat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\widetilde{E}(a)}(\widehat{\underline{I}}_\epsilon(\widetilde{D}(a)))$. Otherwise, by Lemma 2.3.11, $\widehat{\underline{I}}_\epsilon(a) = (E(a)^L)^\infty$ and since $\widetilde{D}(a) = 1$ and $\widetilde{E}(a) = E(a) - 1$ we get $\widehat{\underline{I}}_\epsilon(a) = \pi^{-1}_{\widetilde{E}(a)}(\widehat{\underline{I}}_\epsilon(\widetilde{D}(a)))$. Also, $\widehat{\underline{I}}^*_\epsilon(a) = \pi^{-1}_{E(a)}(\widehat{\underline{I}}^*_\epsilon(D(a)))$ if $a \notin \mathbf{Z}$. Otherwise, $\widehat{\underline{I}}^*_\epsilon(a) = (E(a)+1)^L(E(a)^L)^\infty = \pi^{-1}_{E(a)}(\widehat{\underline{I}}^*_\epsilon(D(a)))$. The other two cases follow in a similar way. ∎

Lemma 3.3.11 *Let $a \in \mathbf{Q}^*$ be with $(p,q) = 1$. Then $\epsilon_q(a) = \epsilon_1(a) + 1$.*

Proof. If $\epsilon_q(a) \neq \epsilon_1(a) + 1$ then, by Lemma 3.3.7, we can assume that $\epsilon_q(a) = \epsilon_1(a)$. Then by Lemma 3.3.5(a), $\widehat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L)^\infty$. By Lemma 3.3.6, $S^{q-1}(\widehat{\underline{I}}_\epsilon(a)) = (\epsilon_1(a)^L \epsilon_1(a)^L \underline{r}(a))^\infty \geq \widehat{\underline{I}}_\epsilon(a)$. Thus, by Lemma 3.3.7, $\epsilon_2(a) = \epsilon_1(a)$ and, proceeding inductively, we obtain that $\widehat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L)^\infty$; a contradiction by Lemma 3.3.5(a). ∎

Remark 3.3.12 In view of Lemmas 2.3.3 and 3.3.11, for $a \in \mathbf{Q}^*$, we can write

$$\begin{aligned}
\widehat{\underline{I}}^*_\delta(a) &= \epsilon_1(a)^L \underline{r}(a)(\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a))^\infty, \\
\widehat{\underline{I}}_\epsilon(a) &= (\epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)+1)^L)^\infty, \\
\widehat{\underline{I}}_\delta(a) &= ((\epsilon_1(a)+1))^L\underline{r}(a)\epsilon_1(a)^L)^\infty, \\
\widehat{\underline{I}}^*_\epsilon(a) &= (\epsilon_1(a)+1)^L\underline{r}(a)((\epsilon_1(a)+1)^L\epsilon_1(a)^L\underline{r}(a))^\infty.
\end{aligned}$$

□

The above observation already allow us to prove Proposition 3.3.1.

Proof of proposition

Proof. We will only prove that $a \odot_\epsilon (1^L)^\infty = \widehat{\underline{I}}_\epsilon(a)$. The proof of the other three statements follows similarly. From Corollary 3.3.10 and the definition of $\odot_\epsilon$ we can assume that $a \in (0,1]$. Now, the statement follows directly from the definitions if $a \notin \mathbf{Q}^*$. If $a \in \mathbf{Q}^*$ the statement follows from Remark 3.3.12 and the fact that $\epsilon_1(a) = 0$. ∎

3.3.3 Proof of the Theorem 3.3.3

We start with a technical lemma.

Lemma 3.3.13 *Let $a = p/q \in \mathbf{Q}^*$ be with $(p,q) = 1$, Then*

(a) $\epsilon_1(a)^L(\epsilon_1(a)+1)^L\underline{r}(a) > \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L$.

(b) For $1 < j \le q-1$ we have that

$$\epsilon_j(a)^L \ldots \epsilon_{q-1}(a)^L\epsilon_1(a)^L(\epsilon_1(a)+1)^L\epsilon_2(a)^L \ldots \epsilon_{j-1}(a)^L > \epsilon_1(a)^L\underline{r}(a)\epsilon_1(a)^L$$

and

$$\epsilon_j(a)^L \ldots \epsilon_{q-1}(a)^L(\epsilon_1(a)+1)^L\epsilon_1(a)^L\epsilon_2(a)^L \ldots \epsilon_{j-1}(a)^L > \epsilon_1(a)^L\underline{r}(a)(\epsilon_1(a)+1)^L$$

(c) $(\epsilon_1(a)+1)^L\epsilon_1(a)^L\underline{r}(a) < (\epsilon_1(a)+1)^L\underline{r}(a)(\epsilon_1(a)+1)^L$.

(d) For $1 < j \le q-1$ we have that

$$\epsilon_j(a)^L \ldots \epsilon_{q-1}(a)^L(\epsilon_1(a)+1)^L\epsilon_1(a)^L\epsilon_2(a)^L \ldots \epsilon_{j-1}(a)^L < (\epsilon_1(a)+1)^L\underline{r}(a)(\epsilon_1(a)+1)^L$$

and

$$\epsilon_j(a)^L \dots \epsilon_{q-1}(a)^L \epsilon_1(a)^L (\epsilon_1(a)+1)^L \epsilon_2(a)^L \dots \epsilon_{j-1}(a)^L < (\epsilon_1(a)+1)^L \underline{r}(a) \epsilon_1(a)^L$$

Proof. Since, by Remark 3.3.12 and Lemma 3.3.6,

$$\widehat{\underline{I}}_\delta^*(a) = \epsilon_1(a)^L \underline{r}(a) (\epsilon_1(a)^L (\epsilon_1(a)+1)^L \underline{r}(a))^\infty$$

and is a minimal sequence we have

$$\epsilon_1(a)^L (\epsilon_1(a)+1)^L \underline{r}(a) \geq \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L.$$

If

$$\epsilon_1(a)^L (\epsilon_1(a)+1)^L \underline{r}(a) = \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L,$$

then

$$\begin{aligned} &\widehat{\underline{I}}_\delta^*(a) = \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L (\epsilon_1(a)+1)^L \dots > \\ &\epsilon_1(a)^L (\epsilon_1(a)+1)^L \underline{r}(a) \epsilon_1(a)^L \dots = S^{q-1}(\widehat{\underline{I}}_\delta^*(a)); \end{aligned}$$

a contradiction with the minimality of $\widehat{\underline{I}}_\delta^*(a)$. This ends the proof of (a). Now, we prove (b). Again by the minimality of $\widehat{\underline{I}}_\delta^*(a)$, for $1 < j \leq q-1$ we have

$$\epsilon_j(a)^L \dots \epsilon_{q-1}(a)^L \epsilon_1(a)^L (\epsilon_1(a)+1)^L \epsilon_2(a)^L \dots \epsilon_{j-1}(a)^L \geq \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L.$$

If in the above inequality the equality holds, we have

$$\begin{aligned} &S^{j-1}(\widehat{\underline{I}}_\delta^*(a)) = \epsilon_j(a)^L \dots \epsilon_{q-1}(a)^L \epsilon_1(a)^L (\epsilon_1(a)+1)^L \epsilon_2(a)^L \dots \epsilon_{j-1}(a)^L \epsilon_j(a)^L \dots \\ &= \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L \epsilon_1(a)^L \dots < \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L (\epsilon_1(a)+1)^L \underline{r}(a) \dots = \widehat{\underline{I}}_\delta^*(a); \end{aligned}$$

a contradiction. Hence,

$$\epsilon_j(a)^L \dots \epsilon_{q-1}(a)^L \epsilon_1(a)^L (\epsilon_1(a)+1)^L \epsilon_2(a)^L \dots \epsilon_{j-1}(a)^L > \epsilon_1(a)^L \underline{r}(a) \epsilon_1(a)^L.$$

Now, we prove the second part of statement (b). Since by Remark 3.3.12 and Lemma 3.3.6

$$\widehat{\underline{I}}_\epsilon(a) = (\epsilon_1(a)^L \underline{r}(a)(\epsilon_1(a)+1)^L)^\infty$$

is a periodic minimal sequence of period q then for $1 < j \leq q-1$ we have that $S^{j-1}(\widehat{\underline{I}}_\epsilon(a)) > \widehat{\underline{I}}_\epsilon(a)$. Thus

$$\epsilon_j(a)^L \ldots \epsilon_{q-1}(a)^L(\epsilon_1(a)+1)^L \epsilon_1(a)^L \epsilon_2(a)^L \ldots \epsilon_{j-1}(a)^L > \epsilon_1(a)^L \underline{r}(a)(\epsilon_1(a)+1)^L.$$

Otherwise, the equality holds and so $S^{j-1}(\widehat{\underline{I}}_\epsilon(a)) = \widehat{\underline{I}}_\epsilon(a)$ with $j < q$; a contradiction. This concludes the proof of statement (b). By using the sequences $\widehat{\underline{I}}^*_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(a)$ instead of $\widehat{\underline{I}}^*_\delta(a)$ and $\widehat{\underline{I}}_\epsilon(a)$ statements (c) and (d) follow in a similar way. ∎

Proof of Theorem 3.3.3. We start by proving (a). Assume that $\widetilde{E}(a) = k < \widetilde{E}(b)$. From the definition of $\odot_\epsilon$ it follows that $a \odot_\epsilon \underline{\alpha} \in \Sigma_k$ and $a \odot_\epsilon \underline{\beta} \in \Sigma_{\widetilde{E}(b)}$. Then, if $a \odot_\epsilon \underline{\alpha} = k^L \ldots$, clearly, $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$. If $a \odot_\epsilon \underline{\alpha} = (k+1)^L \ldots$ then, from the definition of $\odot_\epsilon$ it follows that $a \notin \mathbf{Q}^*$. Moreover, from the definition of $\widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}^*_\delta(a)$ (see also Lemma 2.3.11) it follows that $a = k+1$ and $a \odot_\epsilon \underline{\alpha} = \widehat{\underline{I}}_\epsilon(k+1) = ((k+1)^L)^\infty$. Clearly, $((k+1)^L)^\infty < \underline{\gamma}$ for each $\underline{\gamma} \in \Sigma_m$ with $m > k$. This proves statement (a) in this case. So, assume that $\widetilde{E}(a) = \widetilde{E}(b)$. By the definition of $\odot_\epsilon$, Corollary 3.3.10 and the fact that $\pi_{\widetilde{E}(a)}$ is order preserving we may assume that $\widetilde{E}(a) = \widetilde{E}(b) = 0$ (that is, $a, b \in (0,1]$). We consider first the case $a = b$. If $a \notin \mathbf{Q}^*$ then, from Theorem 2.3.4 we have that $\widehat{\underline{I}}^*_\delta(a) \leq \widehat{\underline{I}}_\epsilon(a)$. Hence, for each $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \setminus \{(1^L)^\infty\}$, $a \odot_\epsilon \underline{\alpha} = \widehat{\underline{I}}^*_\delta(a) \leq \widehat{\underline{I}}_\epsilon(a) = a \odot_\epsilon (1^L)^\infty$. Therefore, $a \odot_\epsilon \underline{\alpha} \leq a \odot_\epsilon \underline{\beta}$ for each $\underline{\alpha}, \underline{\beta} \in \mathcal{B}_\epsilon(0)$. Take now $a \in \mathbf{Q}^*$and set $\underline{\alpha} = \alpha_1\alpha_2 \ldots$ and $\underline{\beta} = \beta_1\beta_2 \ldots$. Since $\underline{\alpha} < \underline{\beta}$, there exists $k \geq 1$ such that $\alpha_1 \ldots \alpha_{k-1} = \beta_1 \ldots \beta_{k-1}$ and $\alpha_k < \beta_k$. Then $a \odot_\epsilon \underline{\alpha} < a \odot_\epsilon \underline{\beta}$ directly from the definition. This ends the proof of statement (a) in the case $a = b$. We note that in particular, from Proposition 3.3.1, we have proved that

$$\widehat{\underline{I}}^*_\delta(a) = a \odot_\epsilon (0^L)^\infty \leq a \odot_\epsilon \underline{\alpha} \leq a \odot_\epsilon (1^L)^\infty = \widehat{\underline{I}}_\epsilon(a)$$

for each $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$. Now we assume that $a \neq b$. Take $c \in (a, b)$ irrational. Then since $\widehat{\underline{I}}_\epsilon(c) = \widehat{\underline{I}}^*_\delta(c)$ (see Lemma 2.3.3), from Lemma 3.3.5(b) we get that $\widehat{\underline{I}}_\epsilon(a) < \widehat{\underline{I}}_\epsilon(c) = \widehat{\underline{I}}^*_\delta(c) < \widehat{\underline{I}}^*_\delta(b)$. So, from above we have

$$a \odot_\epsilon \underline{\alpha} \leq \widehat{\underline{I}}_\epsilon(a) < \widehat{\underline{I}}^*_\delta(b) \leq b \odot_\epsilon \underline{\beta}.$$

This concludes the proof of statement (a). Statement (b) follows in a similar way.

Now, we prove the first statement of (c). Without loss of generality we may assume that $a \in (0,1]$. If $a \notin \mathbf{Q}^*$ then the statement follows from the definition of $\odot_\epsilon$ and Lemma 3.3.6. Now, assume that $a \in \mathbf{Q}^*$. From Proposition 3.3.2 and Lemma 3.3.6 we also have that $a \odot_\epsilon (0^L)^\infty, a \odot_\epsilon (1^L)^\infty \in \mathcal{B}_\epsilon(0) \subset \mathcal{E}_\epsilon$. Therefore, we may assume that $\underline{\alpha} \notin \left\{(0^L)^\infty, (1^L)^\infty\right\}$. Since $\underline{\alpha}$ is minimal, we have $\underline{\alpha} = 0^L \ldots$. Indeed, otherwise we have $S^n(\underline{\alpha}) \geq \underline{\alpha} = 1^L \ldots$ for each $n \geq 0$. Hence $\underline{\alpha} = (1^L)^\infty$; a contradiction. Consequently, $a \odot_\epsilon \underline{\alpha} = 0^L \underline{r}(a) 0^L 1^L \ldots$. To end the proof of the first statement of (c) we have to prove that $S^j(a \odot_\epsilon \underline{\alpha}) \geq a \odot_\epsilon \underline{\alpha}$ for each $j \geq 1$. Let $\underline{\alpha} = \alpha_1 \alpha_2 \ldots$ and $a = p/q$ with $(p,q) = 1$ and $m \geq 1$. Then

$$S^{qm}(a \odot_\epsilon \underline{\alpha}) = \widehat{\alpha}_m \underline{r}(a) \alpha_{m+1} \widehat{\alpha}_{m+1} \ldots .$$

If $\alpha_m = 1^L$, then $\widehat{\alpha}_m = 0^L$ and, since $\underline{\alpha}$ is minimal, we have $S^{qm}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. If $\alpha_m = 0^L$ and $\widehat{\alpha}_m = 1^L$ then clearly, we are done. Now we look at

$$S^{mq-1}(a \odot_\epsilon \underline{\alpha}) = \alpha_m \widehat{\alpha}_m \underline{r}(a) \alpha_{m+1} \widehat{\alpha}_{m+1} \ldots .$$

If $\alpha_m = 1^L$, obviously $S^{mq-1}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. Assume that $\alpha_m = 0^L$. Then $\alpha_m \widehat{\alpha}_m = 0^L 1^L$ and the desired inequality follows from Lemma 3.3.13(a) (recall that we are assuming that $a \in (0,1]$ and $a \in \mathbf{Q}^*$; that is $\epsilon_1(a) = 0$). Now, assume that $1 < j \leq q-1$. Then

$$S^{(m-1)q+j-1}(a \odot_\epsilon \underline{\alpha}) = \epsilon_j(a)^L \ldots \epsilon_{q-1}(a)^L \alpha_m \widehat{\alpha}_m \ldots$$

and, from Lemma 3.3.13(b), we get $S^{(m-1)q+j-1}(\underline{\alpha} \odot_\epsilon a) \geq a \odot_\epsilon \underline{\alpha}$. This ends the proof of the first statement of (c). The fact that $\rho(\underline{\alpha} \odot_\epsilon a) = a$ follows straighfordwardly from the definition of $\odot_\epsilon$ and the fact that $\rho(\widehat{\underline{I}}_\epsilon(a)) = \rho(\widehat{\underline{I}}^*_\delta(a)) = a$. This ends the proof of (c). Statement (d) follows in a similar way.

Now, we prove (e). Assume that $a = p/q$ with $(p,q) = 1$ and set $\underline{\alpha} = \alpha_1\alpha_2\dots$ and $\underline{\beta} = \beta_1\beta_2\dots$. Since $a \in \mathbf{Q}^*$ we have that $E(a) = \widetilde{E}(a) = \epsilon_1(a)$. Hence,

$$a \odot_\epsilon \underline{\alpha} = \epsilon_1(a)^L \underline{r}(a)\alpha_1\widehat{\alpha}_1\underline{r}(a)\alpha_2\widehat{\alpha}_2\dots$$

and

$$a \odot_\delta \underline{\beta} = (\epsilon_1(a)+1)^L \underline{r}(a)\beta_1\widehat{\beta}_1\underline{r}(a)\beta_2\widehat{\beta}_2\dots .$$

Since $\underline{\alpha} \neq (1^L)^\infty$ and is minimal and $\underline{\beta} \neq (0^L)^\infty$ and is maximal, in a similar way as before we obtain that $\underline{\alpha} = 0^L\dots$ and $\underline{\beta} = 1^L\dots$. Therefore $\underline{\alpha} < \underline{\beta}$ and $(a \odot_\epsilon \underline{\alpha})' < a \odot_\delta \underline{\beta}$. Moreover, since $S^n(\underline{\alpha}) \leq \underline{\beta}$, we obtain $S^n(a \odot_\epsilon \underline{\alpha}) \leq a \odot_\delta \underline{\beta}$ in a similar way as above by using Lemma 3.3.13(c) instead of Lemma 3.3.13(a) and Lemma 3.3.13(d) instead of Lemma 3.3.13(b). On the other hand, from $S^n(\underline{\beta}) \geq \underline{\alpha}$ and Lemma 3.3.13(a)–(b) we obtain $S^n(a \odot_\delta \underline{\beta}) \geq a \odot_\epsilon \underline{\alpha}$. Then statement (d) follows from the definition of $\mathcal{E}^* \subset \mathcal{E}$ ∎

Now we will introduce the notation that in the next section will allow us to speak about iterated $\odot-$products.

Let $\overrightarrow{v} = (v_1, \dots, v_n) \in \prod_{i=1}^n (0,1]$ and $\underline{\alpha} \in \mathcal{B}_\epsilon(0)$. We note that if $\underline{\beta} \in \mathcal{B}_\epsilon(0)$ then, by Theorem 3.3.3(c) and the definition of $\odot_\epsilon$, $v_i \odot_\epsilon \underline{\beta}$ also lies in $\mathcal{B}_\epsilon(0)$. Therefore, the sequence

$$v_1 \odot_\epsilon v_2 \odot_\epsilon (\dots(v_{n-1} \odot_\epsilon (v_n \odot_\epsilon \underline{\alpha}))\dots)$$

is well defined. We will denote it by $\overrightarrow{v} \odot_\epsilon \underline{\alpha}$. Now we take $\overrightarrow{v} = (v_1, \dots, v_n) \in \prod_{i=1}^n (k, k+1]$ with $k \in \mathbf{Z}$ and we extend the notation $\overrightarrow{v} \odot_\epsilon \underline{\alpha}$ to this case as follows. Let $\widetilde{D}(\overrightarrow{v}) = (\widetilde{D}(v_1), \dots, \widetilde{D}(v_n))$. Then we set

$$\overrightarrow{v} \odot_\epsilon \underline{\alpha} = \pi_k^{-1}(\widetilde{D}(\overrightarrow{v}) \odot_\epsilon \underline{\alpha}).$$

In a similar way let $\overrightarrow{v} = (v_1, \dots, v_n) \in \prod_{i=1}^n [0,1)$ and $\underline{\alpha} \in \mathcal{B}_\delta(0)$. Then, by using Theorem 3.3.3(d), we define

$$\overrightarrow{v} \odot_\delta \underline{\alpha} = v_1 \odot_\delta (v_2 \odot_\delta (\dots(v_{n-1} \odot_\delta (v_n \odot_\delta \underline{\alpha}))\dots).$$

If $\overrightarrow{v} = (v_1, \ldots, v_n) \in \prod_{i=1}^{n} [k, k+1)$ with $k \in \mathbf{Z}$ and let $D(\overrightarrow{v}) = (D(v_1), \ldots, D(v_n))$. Then we set

$$\overrightarrow{v} \odot_\epsilon \underline{\alpha} = \pi_k^{-1}(D(\overrightarrow{v}) \odot_\delta \underline{\alpha}).$$

We note that if $v \in \mathbf{R}$ then from Theorem 3.3.3(a)–(b) and Corollary 3.3.9 we have that $v \odot_\epsilon \widehat{\underline{I}}_\delta^*(a) \leq v \odot_\epsilon \widehat{\underline{I}}_\epsilon(a)$ and $v \odot_\delta \widehat{\underline{I}}_\delta(a) \leq v \odot_\delta \widehat{\underline{I}}_\epsilon^*(a)$ for all $a \in (0,1)$. Therefore we can consider the following two closed intervals $[v \odot_\epsilon \widehat{\underline{I}}_\delta^*(a), v \odot_\epsilon \widehat{\underline{I}}_\epsilon(a)] \subset \mathcal{E}_\epsilon$ and $[v \odot_\delta \widehat{\underline{I}}_\delta(a), v \odot_\delta \widehat{\underline{I}}_\epsilon^*(a)] \subset \mathcal{E}_\delta$. Additionally we can define recursively the following set of intervals.

Let $\overrightarrow{v} = (v_1, \ldots, v_n) \in \prod_{i=1}^{n} (0,1] \cap \mathbf{Q}^*$ and $a \in (0,1]$. Then the interval

$$[\overrightarrow{v} \odot_\epsilon \widehat{\underline{I}}_\delta^*(a), \overrightarrow{v} \odot_\epsilon \widehat{\underline{I}}_\epsilon(a)]$$

in $\mathcal{E}_\epsilon$ will be denoted by $\mathcal{Q}_\epsilon(a, \overrightarrow{v})$. If $\overrightarrow{v} = (v_1, \ldots, v_n) \in \prod_{i=1}^{n}(k, k+1] \cap \mathbf{Q}^*$ and $a \in (k, k+1]$ with $k \in \mathbf{Z}$, then we denote the interval

$$[\pi_k^{-1}(\widetilde{D}(\overrightarrow{v}) \odot_\epsilon \widehat{\underline{I}}_\delta^*(\widetilde{D}(a))), \pi_k^{-1}(\widetilde{D}(\overrightarrow{v}) \odot_\epsilon \widehat{\underline{I}}_\epsilon(\widetilde{D}(a)))]$$

in $\mathcal{E}_\epsilon$ by $\mathcal{Q}_\epsilon(a, \overrightarrow{v})$.

In a similar way let $\overrightarrow{v} = (v_1, \ldots, v_n) \in \prod_{i=1}^{n}[0,1) \cap \mathbf{Q}^*$ and $a \in [0,1)$. Then we denote the interval

$$[\overrightarrow{v} \odot_\delta \widehat{\underline{I}}_\delta(a), \overrightarrow{v} \odot_\delta \widehat{\underline{I}}_\epsilon^*(a)] \subset \mathcal{E}_\delta$$

by $\mathcal{Q}_\delta(a, \overrightarrow{v})$. If $\overrightarrow{v} = (v_1, \ldots, v_n) \in \prod_{i=1}^{n}[k, k+1) \cap \mathbf{Q}^*$ and $a \in [k, k+1)$ with $k \in \mathbf{Z}$, then we denote the interval

$$[\pi_k^{-1}(\widetilde{D}(\overrightarrow{v}) \odot_\delta \widehat{\underline{I}}_\delta(\widetilde{D}(a))), \pi_k^{-1}(\widetilde{D}(\overrightarrow{v}) \odot_\delta \widehat{\underline{I}}_\delta(\widetilde{D}(a)))] \subset \mathcal{E}_\delta$$

by $\mathcal{Q}_\delta(a, \overrightarrow{v})$.

3.4 Bifurcation structure in the Arnol'd tongues

In this section we will use the products defined in the previous two sections to describe the internal structure of the "boxes" $\mathcal{Q}_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$. In particular this gives the structure of the symbolic Arnol'd tongues. To do it we will use the unimodal boxes of sequences of the form $a \odot_\epsilon \underline{\alpha}$ and $a \odot_\delta \underline{\alpha}$. We recall that in Subsection 3.2.3 we have defined the unimodal box of a periodic sequence $\underline{\gamma}$ from $\mathcal{P}_\epsilon$ (respectively $\mathcal{P}_\delta$) as $\underline{\gamma} \star_\epsilon \mathcal{K} = \underline{\gamma} \star_\epsilon [L^\infty, RL^\infty]$ (respectively $\underline{\gamma} \star_\delta \mathcal{K} = \underline{\gamma} \star_\delta [L^\infty, RL^\infty]$).Thus, in order that the unimodal boxes of $a \odot_\epsilon \underline{\alpha}$ and $a \odot_\delta \underline{\alpha}$ are defined it is necessary that these sequences are periodic. The next result characterizes the periodic sequences of the form $a \odot_\epsilon \underline{\alpha}$ and $a \odot_\delta \underline{\alpha}$. It will be proved in Subsection 3.4.1.

Proposition 3.4.1 *Let $a \in \mathbf{R}$. The following statements hold.*

(a) Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \setminus \{(0^L)^\infty\}$ be periodic. If $a \notin \mathbf{Q}$ then $a \odot_\epsilon \underline{\alpha}$ is not periodic. If $a \in \mathbf{Z}$ then $a \odot_\epsilon \underline{\alpha}$ is periodic if and only if $\underline{\alpha} = (1^L)^\infty$. Moreover, $a^M \in P_\epsilon$ and $a \odot_\epsilon (1^L)^\infty = a^M \star_\epsilon L^\infty$. If $a \in \mathbf{Q}^*$ then $a \odot_\epsilon \underline{\alpha}$ is periodic. Moreover, there exists $\underline{\beta} d^M \in P_\epsilon$ such that $a \odot_\epsilon \underline{\alpha} = \underline{\beta} d^M \star_\epsilon L^\infty$.

(b) Let $\underline{\alpha} \in \mathcal{B}_\delta(0) \setminus \{(1^L)^\infty\}$ be periodic. If $a \notin \mathbf{Q}$ then $a \odot_\delta \underline{\alpha}$ is not periodic. If $a \in \mathbf{Z}$ then $a \odot_\delta \underline{\alpha}$ is periodic if and only if $\underline{\alpha} = (0^L)^\infty$. Moreover, $a^C \in P_\delta$ and $a \odot_\delta (0^L)^\infty = a^C \star_\delta L^\infty$. If $a \in \mathbf{Q}^*$ then $a \odot_\delta \underline{\alpha}$ is periodic. Moreover, there exists $\underline{\beta} d^C \in P_\delta$ such that $a \odot_\delta \underline{\alpha} = \underline{\beta} d^C \star_\delta L^\infty$.

Now we can define the *unimodal box* of a sequence of the form $a \odot_\epsilon \underline{\alpha}$ as follows. Let $a \in \mathbf{Q}$ and $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \setminus \{(0^L)^\infty\}$ be periodic. Then, with the notation of Proposition 3.4.1(a), we set

$$\mathcal{U}_\epsilon(a \odot_\epsilon \underline{\alpha}) = \begin{cases} \underline{\beta} d^M \star_\epsilon \mathcal{K} & \text{if } a \in \mathbf{Q}^*, \\ a^M \star_\epsilon \mathcal{K} & \text{if } a \in \mathbf{Z} \text{ and } \underline{\alpha} = (1^L)^\infty. \end{cases}$$

Let now $\underline{\alpha} \in \mathcal{B}_\delta(0) \setminus \{(1^L)^\infty\}$ be periodic. With the notation of Proposition 3.4.1(b), we set

$$\mathcal{U}_\delta(a \odot_\delta \underline{\alpha}) = \begin{cases} \underline{\beta} d^C \star_\delta \mathcal{K} & \text{if } a \in \mathbf{Q}^*, \\ a^C \star_\delta \mathcal{K} & \text{if } a \in \mathbf{Z} \text{ and } \underline{\alpha} = (0^L)^\infty. \end{cases}$$

The next theorem already gives a first approximation to the symbolic structure of the "boxes" $\mathcal{Q}_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$ (and hence to $\mathcal{E}_\epsilon$ and $\mathcal{E}_\delta$). It will be proved in Subsection 3.4.2

Theorem 3.4.2 *Let $a \in \mathbf{R}$. Then the following statements hold.*

(a) If $a \notin \mathbf{Q}$ then $\mathcal{Q}_\epsilon(a) = \{\widehat{\underline{I}}_\delta(a)\}$.

(b) If $a \in \mathbf{Z}$ then $\mathcal{Q}_\epsilon(a) \supset \mathcal{U}_\epsilon(\widehat{\underline{I}}_\epsilon(a))$.

(c) *If* $a \in \mathbf{Q}^*$ *then* $\mathcal{Q}_\epsilon(a) = \{\widehat{\underline{I}}^*_\delta(a)\} \cup (\cup_{c\in(\widetilde{E}(a),\widetilde{E}(a)+1]}\mathcal{Q}_\epsilon(c,a))$*. Moreover if* $c < c'$ *then for each* $\underline{\alpha} \in \mathcal{Q}_\epsilon(c,a)$ *and* $\underline{\beta} \in \mathcal{Q}_\epsilon(c',a)$ *we have that* $\underline{\alpha} < \underline{\beta}$*.*

(d) *If* $a \notin \mathbf{Q}$ *then* $\mathcal{Q}_\delta(a) = \{\widehat{\underline{I}}_\delta(a)\}$*.*

(e) *If* $a \in \mathbf{Z}$ *then* $\mathcal{Q}_\delta(a) \supset \mathcal{U}_\delta(\widehat{\underline{I}}_\delta(a))$*.*

(f) *If* $a \in \mathbf{Q}^*$ *then* $\mathcal{Q}_\delta(a) = \{\widehat{\underline{I}}^*_\epsilon(a)\} \cup (\cup_{c\in[E(a),E(a)+1)}\mathcal{Q}_\delta(c,a))$*. Moreover if* $c < c'$ *then for each* $\underline{\alpha} \in \mathcal{Q}_\delta(c,a)$ *and* $\underline{\beta} \in \mathcal{Q}_\delta(c',a)$ *we have that* $\underline{\alpha} < \underline{\beta}$*.*

The iterative use of Theorem 3.4.2 already gives the full structure of $Q_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$ for $a \in \mathbf{Q}^*$. Indeed, the structure of the "inside boxes" of the form $a \odot_\epsilon \mathcal{Q}_\epsilon(c)$ and $a \odot_\delta \mathcal{Q}_\delta(c)$ can be deduced from Theorem 3.4.2 and Theorem 3.3.3. Therefore, we obtain the following result which is the main result of this chapter. It already describes the bifurcation pattern when the sequence $\widehat{\underline{I}}_F(0^+)$ (respectively $\widehat{\underline{I}}_F(c_F^-)$) crosses the boxes $\mathcal{Q}_\epsilon(a)$ (respectively $\mathcal{Q}_\delta(a)$) with $a \in \mathbf{Q}^*$ (that is, when the left (respectively right) endpoint of R_F goes through $a \in \mathbf{Q}^*$).

Theorem 3.4.3 *Let* $F \in \mathcal{A}$ *be such that* $R_F = [x_1, y_1]$ *for some* $x_1, y_1 \in \mathbf{R}$*. Then the following statements hold.*

(a) *If* $x_1 \in \mathbf{Q}^*$ *and* $\widetilde{E}(x_1) = k$ *then for* $\widehat{\underline{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_1)$ *one and only one of the following statements hold:*

(a.1) *There exists* $\{x_n\}_{n>1} \in (k, k+1] \cap \mathbf{Q}^*$ *such that* $\widehat{\underline{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_n, \overrightarrow{a}_{n-1})$ *for each* $n \geq 2$ *where* $\overrightarrow{a}_{n-1}$ *denotes the vector* $(x_1, \ldots, x_{n-1})$*. Moreover for all* $n \geq 2$ *we have that* $\mathcal{Q}_\epsilon(x_1) \supset \mathcal{Q}_\epsilon(x_n, \overrightarrow{a}_{n-1}) \supset \mathcal{Q}_\epsilon(x_{n+1}, \overrightarrow{a}_n)$*.*

(a.2) *There exist* $n \geq 2$ *and a vector* $\overrightarrow{a}_{n-1} = (x_1, \ldots, x_{n-1})$ *with* $x_2, \ldots, x_n \in (k, k+1]$ *and* $x_i \in \mathbf{Q}^*$ *for* $i = 1, \ldots, n-1$ *such that* $\widehat{\underline{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_k, (x_1, \ldots, x_{k-1}))$ *for* $k = 2, 3, \ldots, n-1$ *and one and only one of the following statements hold:*

(a.2.1) $\widehat{\underline{I}}_F(0^+)$ *is equal to* $\overrightarrow{a}_{n-2} \odot_\epsilon \widehat{\underline{I}}^*_\delta(\widetilde{D}(x_{n-1}))$ *if* $n \geq 3$ *and* $\widehat{\underline{I}}^*_\delta(x_1)$ *if* $n = 2$*.*

(a.2.2) $x_n \in \mathbf{Z}$ *and* $\widehat{\underline{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_n, \overrightarrow{a}_{n-1}) \supset \mathcal{U}_\epsilon(\overrightarrow{a}_{n-1} \odot_\epsilon \widehat{\underline{I}}_\epsilon(\widetilde{D}(x_n))$*.*

(a.2.3) $x_n \notin \mathbf{Q}$ *and* $\widehat{\underline{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_n, \overrightarrow{a}_{n-1}) = \{\overrightarrow{a}_{n-1} \odot_\epsilon \widehat{\underline{I}}_\epsilon(\widetilde{D}(x_n))\}$*.*

(b) *If* $y_1 \in \mathbf{Q}^*$ *and* $\widetilde{E}(y_1) = k'$ *then for* $\widehat{\underline{I}}_F(c_F^-) \in \mathcal{Q}_\delta(y_1)$ *one and only one of the following statements hold.*

(b.1) *There exists* $\{y_n\}_{n>1} \in [k', k'+1) \cap \mathbf{Q}^*$ *such that* $\widehat{\underline{I}}_F(c_F^-) \in \mathcal{Q}_\delta(y_n, \overrightarrow{b}_{n-1})$ *for each* $n \geq 2$ *where* $\overrightarrow{b}_{n-1}$ *denotes the vector* $(y_1, \ldots, y_{n-1})$*. Moreover for all* $n \geq 2$ *we have* $\mathcal{Q}_\delta(y_1) \supset \mathcal{Q}_\delta(y_n, \overrightarrow{b}_{n-1}) \supset \mathcal{Q}_\delta(y_{n+1}, \overrightarrow{b}_n)$*.*

(b.2) *There exist* $n \geq 2$ *and a vector* $\overrightarrow{b}_{n-1} = (y_1, \ldots, y_{n-1})$ *with* $y_2, \ldots, y_n \in [k, k+1)$ *and* $y_i \in \mathbf{Q}^*$ *for* $i = 1, \ldots, n-1$ *such that* $\widehat{\underline{I}}_F(c_F^-) \in \mathcal{Q}_\delta(y_k, (y_1, \ldots, y_{k-1}))$ *for* $k = 2, 3, \ldots, n-1$ *and one and only of the following statements hold:*

(b.2.1) $\widehat{\underline{I}}_F(c_F^-)$ *is equal to* $\overrightarrow{b}_{n-2} \odot_\delta \widehat{\underline{I}}^*_\epsilon(D(y_{n-1}))$ *if* $n \geq 3$ *and* $\widehat{\underline{I}}^*_\epsilon(y_1)$ *if* $n = 2$*.*

(b.2.2) $y_n \in \mathbf{Z}$ *and* $\widehat{\underline{I}}_F(c_F^-) \in \mathcal{Q}_\delta(y_n, \overrightarrow{b}_{n-1}) \supset \mathcal{U}_\epsilon(\overrightarrow{b}_{n-1} \odot_\delta \widehat{\underline{I}}_\delta(D(y_n)))$*.*

(b.2.3) $y_n \notin \mathbf{Q}$ *and* $\widehat{\underline{I}}_F(c_F^-) \in \mathcal{Q}_\delta(y_n, \overrightarrow{b}_{n-1}) = \{\overrightarrow{b}_{n-1} \odot_\delta \widehat{\underline{I}}_\delta(D(y_n))\}$*.*

Theorem 3.4.3 will be proved in Subsection 3.4.2.

3.4.1 Proof of Proposition 3.4.1

To prove Proposition 3.4.1 we need three preliminary results. The next lemma follows easily.

Lemma 3.4.4 *Let* $\underline{\alpha} = \alpha_1\alpha_2\ldots, \underline{\beta} = \alpha_1\beta_2\ldots \in \mathcal{AD}$ *be such that* $\underline{\alpha} < \underline{\beta}$*. Then the following statements hold.*

(a) *If* $\alpha_1 = d^L$ *then* $S(\underline{\alpha}) < S(\underline{\beta})$*.*

(b) *If* $\alpha_1 = d^R$ *then* $S(\underline{\alpha}) > S(\underline{\beta})$*.*

The following proposition characterizes the sequences in $\mathcal{P}_\epsilon$ and $\mathcal{P}_\delta$.

Proposition 3.4.5 *The following statements hold.*

(a) *Let* $\underline{\beta} \in \Xi$ *be such that* $\underline{\gamma} = \underline{\beta}d^M$ *is minimal satisfying that if* $S^{j-1}(\underline{\gamma}) = d^R\ldots$ *for some* $j = 0, 1, \ldots, |\underline{\gamma}| - 1$*, then* $S^j(\underline{\gamma}) \geq \underline{\gamma}'$*. Then there exists* $F \in \mathcal{A}$ *such that* $\widehat{\underline{I}}_F(0) = \underline{\gamma}$*. Moreover* $\underline{\gamma} \in \mathcal{P}_\epsilon$*.*

(b) *Let* $\underline{\beta} \in \Xi$ *be such that* $\underline{\gamma} = \underline{\beta}d^C$ *is maximal. Then there exists* $F \in \mathcal{A}$ *such that* $\widehat{\underline{I}}_F(c_F) = \underline{\gamma}$*. Moreover* $\underline{\gamma} \in \mathcal{P}_\delta$*.*

Proof. We will prove statement (a). Statement (b) follows similarly. The strategy of the proof will be to construct effectively a map $F \in \mathcal{A}$ such that $\widehat{\underline{I}}_F(0) = \underline{\gamma}$. We proceed as follows. Set $\underline{\gamma} = d_1^{s_1}d_2^{s_2}\ldots d_{n-1}^{s_{n-1}}d_n^{s_n}$ with $s_n = M$. Let $k \in \mathbf{Z}$ be such that $\max\{|d_i| : i = 1, \ldots, n\} < k$ and let $c \in (0, 1)$. Now, for $j = 0, 1, \ldots n-1$, we choose points $x(S^j(\underline{\gamma})) \in [0, 1)$ such that

1. $x(\underline{\gamma}) = 0$,

2. if for $j = 1, \ldots, n-1$ we have $S^{j-1}(\underline{\gamma}) = d_j^L \ldots$ (respectively $S^{j-1}(\underline{\gamma}) = d_j^R \ldots$) then $x(S^j(\underline{\gamma})) \in (0, c)$ (respectively $x(S^j(\underline{\gamma})) \in (c, 1)$),

3. if for $i \neq j$, $i, j \in \{1, 2, \ldots, n-1\}$ we have $x(S^i(\underline{\gamma})), x(S^j(\underline{\gamma})) \in [0, c)$ (respectively $x(S^i(\underline{\gamma})), x(S^j(\underline{\gamma})) \in (c, 1)$), then $x(S^i(\underline{\gamma})) < x(S^j(\underline{\gamma}))$ if and only if $S^i(\underline{\gamma}) < S^j(\underline{\gamma})$ (respectively $S^i(\underline{\gamma}) > S^j(\underline{\gamma})$).

We note that, by the minimality of $\underline{\gamma}$, we have $x(\underline{\gamma}) < x(S^j(\underline{\gamma}))$ for $j = 1, 2, , \ldots n-1$. Therefore we can write

$$x(\underline{\gamma}) < x(S^{j_1}(\underline{\gamma})) < \ldots < x(S^{j_k}(\underline{\gamma})) < c < x(S^{j_{k+1}}(\underline{\gamma})) < \ldots < x(S^{j_{n-1}}(\underline{\gamma})).$$

Then we set $j_0 = 0$ and we take $F \in \mathcal{L}$ such that $F(c) = k$, $F(x(S^{j_t}(\underline{\gamma})) = x(S^{j_t+1}(\underline{\gamma})) + d_{j_t+1}$ if $j_t \neq n-1$, $F(x(S^{n-1}(\underline{\gamma})) = d_n$ and F is affine in $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]$ for $t \in \{0, 1, \ldots, n-1\} \backslash \{k\}$ and in $[x(S^{j_k}(\underline{\gamma})), c]$ and $[c, x(S^{j_{k+1}}(\underline{\gamma}))]$. Now, we claim that $F \in \mathcal{A}$. To prove it note that $F(c) = k > F(x(S^j(\underline{\gamma}))$ for $j = 0, \ldots, n-1$. Then $F|_{[x(S^{j_k}(\underline{\gamma})), c]}$ is strictly increasing and $F|_{[c, x(S^{j_{k+1}}(\underline{\gamma}))]}$ is strictly decreasing. Let t be such that $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))] \subset [0, c)$. We have $S^{j_t}(\underline{\gamma}) = d_{j_t+1}^{s_{j_t+1}} \ldots < d_{j_{t+1}+1}^{s_{j_{t+1}+1}} \ldots = S^{j_{t+1}}(\underline{\gamma})$. If either $d_{j_t+1} < d_{j_{t+1}+1}$ or $d_{j_t+1} = d_{j_{t+1}+1}$ and $s_{j_t+1} < s_{j_{t+1}+1}$, then clearly $F(x(S^{j_t}(\underline{\gamma}))) < F(x(S^{j_{t+1}}(\underline{\gamma})))$. Now, assume $d_{j_t+1}^{s_{j_t+1}} = d_{j_{t+1}+1}^{s_{j_{t+1}+1}}$. From Lemma 3.4.4 we have that either $S^{j_t+1}(\underline{\gamma}) < S^{j_{t+1}+1}(\underline{\gamma})$ if $s_{j_t+1} = L$ or $S^{j_t+1}(\underline{\gamma}) > S^{j_{t+1}+1}(\underline{\gamma})$ if $s_{j_t+1} = R$. In both cases $x(S^{j_t+1}(\underline{\gamma})) < x(S^{j_{t+1}+1}(\underline{\gamma}))$ and, in consequence, $F(x(S^{j_t}(\underline{\gamma}))) < F(x(S^{j_{t+1}}(\underline{\gamma})))$. Thus $F|_{[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]}$ is strictly increasing. In a similar way we can prove that if $[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))] \subset (c, 1)$ then $F|_{[x(S^{j_t}(\underline{\gamma})), x(S^{j_{t+1}}(\underline{\gamma}))]}$ is strictly decreasing. To end the proof of the claim we have to prove that $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$. Since $x(S^{j_{n-1}}(\underline{\gamma})) \in (c, 1)$ we have that $S^{j_{n-1}-1}(\underline{\gamma}) = d_{j_{n-1}}^R \ldots$. Then $S^{j_{n-1}}(\underline{\gamma}) > \underline{\gamma}'$. If either $d_{j_{n-1}+1} > (d_1 + 1)$ or $d_{j_{n-1}+1} = (d_1 + 1)$ and $s_{j_{n-1}+1} = R > L = s_1$ then, since $F(1) = F(0) + 1 = x(S(\underline{\gamma})) + d_1 + 1$ and $F(x(S^{j_{n-1}}(\underline{\gamma}))) = x(S^{j_{n-1}+1}(\underline{\gamma})) + d_{j_{n-1}+1}$, we have that $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$. On the other hand, assume that $d_{j_{n-1}+1}^{s_{j_{n-1}+1}} = (d_1 + 1)^{s_1}$. We obtain that $F(x(S^{j_{n-1}}(\underline{\gamma}))) > F(1)$ as above by using Lemma 3.4.4. This ends the proof of the claim. Lastly, we have $\widehat{\underline{I}}_F(0) = \underline{\gamma}$ by construction. Also, from Lemma 3.2.2(a) we have that $\underline{\gamma} \in \mathcal{P}_\epsilon$. This ends the proof of the proposition. ∎

The next lemma characterizes the periodic sequences in $\mathcal{B}_\epsilon(0)$ and $\mathcal{B}_\delta(0)$.

Lemma 3.4.6 *The following statements hold.*

(a) Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(0^L)^\infty, (1^L)^\infty\}$ *be periodic. Then* $\underline{\alpha} = (0^L \underline{\beta} 1^L)^\infty$ *for some* $\underline{\beta} \in \Xi$.

(b) Let $\underline{\alpha} \in \mathcal{B}_\delta(0) \backslash \{(0^L)^\infty, (1^L)^\infty\}$ *be periodic. Then* $\underline{\alpha} = (1^L \underline{\beta} 0^L)^\infty$ *for some* $\underline{\beta} \in \Xi$.

Proof. Clearly $\underline{\alpha}$ is of the form $(d_1^L \underline{\beta} d_n^L)^\infty$ with $\underline{\beta} \in \Xi$. Assume that $d_1 = 1$. Since $\underline{\alpha}$ is minimal we have that $\underline{\alpha} = 1^L \ldots \leq S^j(\underline{\alpha})$ for all j. Then $S^j(\underline{\alpha}) = 1^L \ldots$ for all j and, in consequence, $\underline{\alpha} = (1^L)^\infty$; a contradiction. Hence $d_1 = 0$. Now, assume that $d_n = 0$. Then $\underline{\alpha} = (0^L \underline{\beta} 0^L)^\infty$. If $\underline{\beta}$ is the empty sequence then $\underline{\alpha} = (0^L)^\infty$; a contradiction. Now assume that $\underline{\beta}$ is not the empty sequence and set $\underline{\beta} = \beta_2 \ldots \beta_{n-1}$. Since $\underline{\alpha}$ is minimal $\underline{\alpha} = 0^L \beta_2 \ldots \leq 0^L 0^L \beta_2 \ldots = S^{n-1}(\underline{\alpha})$. Thus $\beta_2 = 0^L$. Proceeding inductively we obtain that $\beta_i = 0^L$ for $i = 2, \ldots, n-1$. Thus $\underline{\alpha} = (0^L)^\infty$; a contradiction. This ends the proof of (a). Statement (b) follows in a similar way. ∎

Proof of Proposition 3.4.1. We will only prove statement (a). Statement (b) follows in a similar way. The fact that $a \odot_\delta \underline{\alpha}$ is not periodic when $a \notin \mathbf{Q}$ and when $a \in \mathbf{Z}$ is periodic if and only if $\underline{\alpha} = (1^L)^\infty$ follows from the definitions of $\odot_\epsilon$ and of the sequences $\underline{\widehat{I}}_\delta^*(a)$ and $\underline{\widehat{I}}_\epsilon(a)$. The third statement follows directly from the definitions. Now we prove the last two statements. Assume that $a \in \mathbf{Q}^*$. If $\underline{\alpha} = (1^L)^\infty$ then $a \odot_\epsilon \underline{\alpha}$ is periodic by Proposition 3.3.1 and Lemma 3.3.5(a). Moreover if $a = p/q$ with $(p,q) = 1$ then $a \odot_\delta \underline{\alpha} = (\epsilon_1(a)^L \epsilon_2(a)^L \ldots \epsilon_q(a)^L)^\infty$. Let $\underline{\alpha} \in \mathcal{B}_\epsilon(0) \backslash \{(1^L)^\infty\}$. By Lemma 3.4.6(a) we get $\underline{\alpha} = (0^L \alpha_2 \ldots \alpha_{n-1} 1^L)^\infty$. Without loss of generality assume that $\widetilde{E}(a) = 0$. Then

$$a \odot_\epsilon \underline{\alpha} = (0^L \underline{r}(a) 0^L 1^L \underline{r}(a) \alpha_2 \ldots \underline{r}(a) \alpha_{n-1} \widehat{\alpha}_{n-1} \underline{r}(a) 1^L)^\infty$$

is periodic. Now, let $\underline{\gamma} = 0^L \underline{r}(a) 0^L 1^L \underline{r}(a) \alpha_2 \ldots \underline{r}(a) \alpha_{n-1} \widehat{\alpha}_{n-1} \underline{r}(a) 1^M$. Clearly, $a \odot_\epsilon \underline{\alpha} = \underline{\gamma} \star_\epsilon L^\infty$. Since, from Proposition 2.3.8(b), $0^L \underline{r}(a) 1^M$ is a minimal sequence, by using Lemma 3.3.13(a)–(b), we have that $\underline{\gamma}$ is a minimal sequence (note that $\epsilon_1(a) = 0$). Then by Proposition 3.4.5(a) we have that $\underline{\gamma} \in \mathcal{P}_\epsilon$. ∎

3.4.2 Proof of Theorems 3.4.1 and 3.4.3

To prove Theorem 3.4.2 we will use the following technical lemma.

Lemma 3.4.7 *Let $a \in \mathbf{R}$. Then*

$$(\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\epsilon(a)] = \cup_{b \in (\widetilde{E}(a), \widetilde{E}(a)+1]} \mathcal{Q}_\epsilon(b, a)$$

and

$$[\widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)) = \cup_{b \in [E(a), E(a)+1)} \mathcal{Q}_\delta(b, a).$$

Proof. From the definition of $\odot_\epsilon$ and Corollary 3.3.10 we may assume $\widetilde{E}(a) = 0$. Since $[(0^L)^\infty, (1^L)^\infty] = [\widehat{\underline{I}}_\epsilon(0), \widehat{\underline{I}}_\epsilon(1)] = [\widehat{\underline{I}}_\delta(0), \widehat{\underline{I}}_\delta(1)]$ from Theorem 2.3.4 and Lemma 3.3.5(b) we have that $((0^L)^\infty, (1^L)^\infty] = \cup_{a \in (0,1]} \mathcal{Q}_\epsilon(a)$ and $[(0^L)^\infty, (1^L)^\infty) = \cup_{a \in [0,1)} \mathcal{Q}_\delta(a)$. Then, by using Proposition 3.3.1 and Theorem 3.3.3(a)–(b) we have that $(\widehat{\underline{I}}^*_\delta(b), \widehat{\underline{I}}_\epsilon(b)] = (b \odot_\epsilon (0^L)^\infty, b \odot_\epsilon (1^L)^\infty] = \cup_{a \in (0,1]} \mathcal{Q}_\epsilon(a, b)$ and $[\widehat{\underline{I}}_\delta(b), \widehat{\underline{I}}^*_\epsilon(b)) = [b \odot_\epsilon (0^L)^\infty, b \odot_\epsilon (1^L)^\infty) = \cup_{a \in (0,1]} \mathcal{Q}_\delta(a, b)$. ∎

Proof of Theorem 3.4.2. We prove (a)–(c). Statements (d)–(f) follow in a similar way. Clearly if $a \notin \mathbf{Q}$ then $\mathcal{Q}_\epsilon(a) = \{\widehat{\underline{I}}_\epsilon(a)\}$ because $\widehat{\underline{I}}^*_\delta(a) = \widehat{\underline{I}}_\epsilon(a)$. This proves (a). Now, let $a \in \mathbf{Z}$. Since $a \odot_\epsilon (1^L)^\infty = \widehat{\underline{I}}_\epsilon(a)$, then $\mathcal{U}_\epsilon(\widehat{\underline{I}}_\epsilon(a)) = \mathcal{U}_\epsilon(a \odot_\epsilon (1^L)^\infty) = [a^M \star_\epsilon RL^\infty, a^M \star_\epsilon L^\infty]$. As $a^M \star_\epsilon RL^\infty = (a-1)^R \ldots > (a-1)^L (a^L)^\infty = \widehat{\underline{I}}^*_\delta(a)$, statement (b) follows. Let now $a \in \mathbf{Q}^*$. From Lemma **??** the first part of (c) follows. The second part follows from Theorem 3.3.3(a) and Theorem 2.3.4. This ends the proof of theorem. ∎

The rest of this subsection is devoted to prove Theorem 1.5.3.

Proposition 3.4.8 *Let $k \in \mathbf{Z}$ and let $\{x_n\}_{n \in \mathbf{N}} \in (k, k+1) \cap \mathbf{Q}^*$ be a sequence. Let $\overrightarrow{a}_n = (x_1, \ldots, x_n)$ for $n \in \mathbf{N}$. Then for all $i \geq 1$,*

$$\mathcal{Q}_\epsilon(x_1) \supset \mathcal{Q}_\epsilon(x_{i+1}, \overrightarrow{a}_i) \supset \mathcal{Q}_\epsilon(x_{i+2}, \overrightarrow{a}_{i+1})$$

and

$$\mathcal{Q}_\delta(x_1) \supset \mathcal{Q}_\delta(x_{i+1}, \overrightarrow{a}_i) \supset \mathcal{Q}_\delta(x_{i+2}, \overrightarrow{a}_{i+1}).$$

Proof. As before, without loss of generality we assume that $x_n \in (0,1)$ for all $n \in \mathbf{N}$. ByLemma 3.3.5(b), by using standard arguments, we see that

$$(0^L)^\infty = \widehat{\underline{I}}_\epsilon(0) < \widehat{\underline{I}}^*_\delta(x_n) < \widehat{\underline{I}}_\epsilon(x_n) < \widehat{\underline{I}}_\epsilon(1) = (1^L)^\infty$$

for all $n \in \mathbf{N}$. Thereforefrom Theorem 3.3.3(a) and Proposition 3.3.1, we have that

$$\widehat{\underline{I}}^*_\delta(x_n) < x_n \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+1}) < x_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+1}) < \widehat{\underline{I}}_\epsilon(x_n). \tag{3.4.6}$$

If we use Theorem 3.3.3(a) in (3.4.6) replacing n by $n+1$ we obtain that

$$x_n \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+1}) < x_n \odot_\epsilon (x_{n+1} \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+2})) <$$
$$x_n \odot_\epsilon (x_{n+1} \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+2})) < x_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+1}).$$

From (3.4.6) the first statement follows in the case $n = 1$. Assume now that the first statement follows for $n \geq 1$. Then we have that

$$\overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+1}) < \overrightarrow{a}_{n+1} \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+2})) <$$
$$\overrightarrow{a}_{n+1} \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+2})) < \overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+1}).$$

By using (3.4.6) with $n+1$ instead of n, from Theorem 3.3.3(a), we obtain that

$$\overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+1}) < \overrightarrow{a}_n \odot_\epsilon (x_{n+1} \odot_\epsilon \widehat{\underline{I}}^*_\delta(x_{n+2}))) <$$
$$\overrightarrow{a}_n \odot_\epsilon (x_{n+1} \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+2}))) < \overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(x_{n+1}).$$

This concludes the proof of the first statement. The second one follows in a similar way. ∎

Proposition 3.4.9 *Let $k \in \mathbf{Z}$, $n \in \mathbf{N}$ and let $x_1, \ldots x_n \in (k, k+1) \cap \mathbf{Q}^*$. Set $\overrightarrow{a}_n = (x_1, \ldots, x_n)$. Then, for each $c \in (k, k+1)$ such that $c \notin \mathbf{Q}$, we have that*

$$\mathcal{Q}_\epsilon(c, \overrightarrow{a}_n) = \{\overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(\widetilde{D}(c))\}$$

and

$$\mathcal{Q}_\delta(c, \overrightarrow{a}_n) = \{\overrightarrow{a}_n \odot_\delta \widehat{\underline{I}}_\delta(D(c))\}.$$

Also, $U_\epsilon(\overrightarrow{a}_n \odot_\epsilon \widehat{\underline{I}}_\epsilon(1)) \subset \mathcal{Q}_\epsilon(k+1, \overrightarrow{a}_n)$ and $U_\delta(\overrightarrow{a}_n \odot_\delta \widehat{\underline{I}}_\delta(0)) \subset \mathcal{Q}_\delta(k, \overrightarrow{a}_n)$.

Proof. As in the previous results, whitout loss of generality assume that $k = 0$. Then by the definition of $\mathcal{Q}_\epsilon(c, \overrightarrow{a}_n)$ and $\mathcal{Q}_\delta(c, \overrightarrow{a}_n)$, and Lemma 2.3.3 the statement follows for c irrational.

To end the proof of the proposition we will show that

$$\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}^*_\delta(1) < \underline{\beta} 1^M \star_\epsilon RL^\infty < \underline{\beta} 1^M \star_\epsilon L^\infty = \overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1).$$

Recall that, from Proposition 3.4.1(a), $\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1) = \underline{\beta} 1^M \star_\epsilon L^\infty$. Since $\underline{\widehat{I}}_\epsilon(1) = (1^L)^\infty$ from Proposition 3.4.1(a) we have that $x_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1) = \underline{\widehat{I}}_\epsilon(x_n) = (0^L \underline{r}(x_n) 1^L)^\infty$ is periodic. Assume that $\underline{r}(x_i)$ has length k_i for $i = 1, 2, \ldots, n$. Let $(0^L \underline{r}(x_n) 1^L)^\infty = (\beta_{1,n} \ldots \beta_{k_n+1,n} 1^L)^\infty = (\underline{\beta}_n 1^L)^\infty$ and let

$$\underline{\beta}_{n-1} = \overbrace{0^L \underline{r}(x_{n-1}) \beta_{1,n} \widehat{\beta_{1,n}} \underline{r}(x_{n-1}) \ldots \underline{r}(x_{n-1}) \beta_{k_n+1,n} \widehat{\beta_{k_n+1,n}} \underline{r}(x_{n-1})}^{1+(k_n+1)k_{n-1}+2(k_n+1)}.$$

Then, from the definition of $\odot_\epsilon$, we have that

$$x_{n-1} \odot_\epsilon (x_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1)) = (\underline{\beta}_{n-1} 1^L)^\infty$$

and

$$x_{n-1} \odot_\epsilon (x_n \odot_\epsilon \underline{\widehat{I}}^*_\delta(1)) = \underline{\beta}_{n-1} 0^L 1^L \ldots .$$

Proceeding inductively we obtain that

$$\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1) = (\underline{\beta}_1 1^L)^\infty$$

and

$$\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}^*_\delta(1) = \underline{\beta}_1 0^L 1^L \ldots .$$

By Proposition 3.4.1(a) we can write $\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1) = \underline{\beta}_1 1^M \star_\epsilon L^\infty$. Then we have that $\underline{\beta}_1 1^M \star_\epsilon RL^\infty = \underline{\beta}_1 0^R \ldots$. In consequence $\underline{\beta}_1 1^M \star_\epsilon RL^\infty > \overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}^*_\delta(1)$ and $U_\epsilon(\overrightarrow{a}_n \odot_\epsilon \underline{\widehat{I}}_\epsilon(1)) \subset \mathcal{Q}_\epsilon(c, \overrightarrow{a}_n)$. The second inclusion follows in a similar way. This ends the proof of the proposition. ∎

Proof of Theorem 3.4.3. We prove statement (a). Statement (b) follows in a similar way. Without loss of generality assume that $\widetilde{E}(x_1) = 0$. From Theorem 3.4.2(c) we have that

$$\mathcal{Q}_\epsilon(x_1) = \{\underline{\widehat{I}}^*_\delta(x_1)\} \cup (\cup_{x_2 \in (0,1]} \mathcal{Q}_\epsilon(x_2, x_1)).$$

Then we have one and only one of the following four possibilities:

(1.1) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_2, x_1) = \mathcal{Q}_\epsilon(x_2, \overrightarrow{a}_1)$ with $x_2 \in \mathbf{Q}^*$.

(1.2) $\underline{\widehat{I}}_F(0^+) = \underline{\widehat{I}}^*_\delta(x_1)$.

(1.3) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(1, x_1) = \mathcal{Q}_\epsilon(1, \overrightarrow{a}_1)$.

(1.4) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_2, x_1) = Q_\epsilon(x_2, \overrightarrow{a}_1)$ with $x_2 \notin \mathbf{Q}$.

Now, assume that (1.1) holds. Since

$$\begin{aligned} \mathcal{Q}_\epsilon(x_2, x_1) &= [x_1 \odot_\epsilon \underline{\widehat{I}}^*_\delta(x_2), x_1 \odot_\epsilon \underline{\widehat{I}}_\epsilon(x_2)] \\ &= [x_1 \odot_\epsilon (x_2 \odot_\epsilon (0^L)^\infty), x_1 \odot_\epsilon (x_2 \odot_\epsilon (1^L)^\infty)] \end{aligned}$$

we have that

$$\mathcal{Q}_\epsilon(x_2, x_1) = \{\overrightarrow{a}_2 \odot_\epsilon (0^L)^\infty\} \cup (\overrightarrow{a}_2 \odot_\epsilon \underline{\widehat{I}}_\epsilon(0), \overrightarrow{a}_2 \odot_\epsilon \underline{\widehat{I}}_\epsilon(1)]$$

is equal to $\{\overrightarrow{a}_2 \odot_\epsilon (0^L)^\infty\} \cup (\cup_{x_3 \in (0,1]} [\overrightarrow{a}_2 \odot_\epsilon \underline{\widehat{I}}^*_\delta(x_3), \overrightarrow{a}_2 \odot_\epsilon \underline{\widehat{I}}_\epsilon(x_3)])$. Thus

$$\mathcal{Q}_\epsilon(x_2, x_1) = \{\overrightarrow{a}_2 \odot_\epsilon (0^L)^\infty\} \cup (\cup_{x_3 \in (0,1]} \mathcal{Q}_\epsilon(x_3, \overrightarrow{a}_2)).$$

Then one and only one of the following four possibilities hold.

(2.1) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_3, \overrightarrow{a}_2)$ with $x_3 \in \mathbf{Q}^*$.

(2.2) $\underline{\widehat{I}}_F(0^+) = \overrightarrow{a}_2 \odot_\epsilon (0^L)^\infty = \overrightarrow{a}_1 \odot_\epsilon \underline{\widehat{I}}^*_\delta(x_2)$.

(2.3) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(1, \overrightarrow{a}_2)$.

(2.4) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_3, \overrightarrow{a}_2)$ with $x_3 \notin \mathbf{Q}$.

Proceeding inductively we have that if (n-1.1) holds then one and only one of the following four possibilities hold.

(n.1) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_{n+1}, \overrightarrow{a}_n)$ with $x_{n+1} \in \mathbf{Q}^*$.

(n.2) $\underline{\widehat{I}}_F(0^+) = \overrightarrow{a}_{n-1} \odot_\epsilon \underline{\widehat{I}}^*_\delta(x_n)$.

(n.3) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(1, \overrightarrow{a}_n)$.

(n.4) $\underline{\widehat{I}}_F(0^+) \in \mathcal{Q}_\epsilon(x_{n+1}, \overrightarrow{a}_n)$ with $x_{n+1} \notin \mathbf{Q}$.

We note that statement (a.1) is equivalent to say that statement (n.1) holds for all $n \geq 1$. From above we have that either (n.1) holds for all $n \geq 1$, or there exists $n \geq 2$ such that (n-1.1) does not hold and one and only one of the three conditions (n-1.2)–(n-1.4) hold. Then the theorem follows from Propositions 3.4.8 and 3.4.9. ∎

3.4.3 Concluding remarks

In this chapter we have described the structure of the boxes $\mathcal{Q}_\epsilon(a)$ and $\mathcal{Q}_\delta(a)$. This gives a good information on the bifurcations occurring when the left (respectively right) endpoints of the rotation interval goes trough a rational. However to describe the self–similar structures of the Arnol'd tongues, a deeper knowledge of the topology of the sets $T_\epsilon(a)$ and $T_\delta(a)$ (and hence of $\mathcal{E}$) is needed. In this context an open problem in to characterize the symbolic structure of the part of the integer boxes which is the complement of the unimodal ones. That is, $\mathcal{Q}_\epsilon(k)\backslash\mathcal{U}_\epsilon(\widehat{\underline{I}}_\epsilon(k))$ and $\mathcal{Q}_\delta(k)\backslash\mathcal{U}_\delta(\widehat{\underline{I}}_\delta(k))$ (see Theorem 3.4.2(b) and (e)) for $k \in \mathbf{Z}$.

On the other hand, Theorem 3.4.3 can be viewed as a refinement of Theorem 2.3.4 (see [5]) thus giving a better approximation on topological entropy and the set of periodic points of the map under consideration.

3.5 Appendix

Proof Of Theorem 3.1.1. Clearly if $\underline{\alpha} \in \mathcal{E}_\epsilon$ then from the definition of $\mathcal{E}$ we only have to prove the "only if " part. To do it let $\underline{\alpha} = d_1^{s_1} d_2^{s_2} \dots$ be a minimal sequence satisfying that if some $n \geq 0$, $S^n(\underline{\alpha}) = d^R \dots$, then $S^{n+1}(\underline{\alpha}) \geq \underline{\alpha}'$. Since $\underline{\alpha}$ is an admissible sequence there exists $k \in \mathbf{N}$ such that for all $i \geq 1$, $\mid d_i \mid \leq k$. Clearly $S^n(\underline{\alpha}) \leq ((k+1)^L)^\infty$ for all $n \geq 0$. Thus $(\underline{\alpha}, ((k+1)^L)^\infty) \in \mathcal{E}$. This ends the proof of (a). Statement (b) follows in a similar way. ∎

Chapter 4

Topological entropy

4.1 Introduction

In [17] Hockett and Holmes describe certain bifurcations of a continuous one-parameter family of degree one circle maps in terms of the relation between the parameter and the rotation interval of these maps. To carry on their study they use the natural extension of the "Kneading Theory" of Milnor and Thurston [20] to the family of maps they consider. This extension is based in the use of an "ad hoc" coding. In order to maintain small the number of symbols of this coding (and, therefore, to maintain the difficulty of the computations at a reasonable level) the authors have to impose a restriction on the "height" of the maps under consideration (see Section 2.2 for a precise definition of "height").

The purpose of this chapter is to obtain a simple formula for the topological entropy of the maps from the family considered by Hockett and Holmes in [17]. To do this, instead of working in their framework, we shall use the coding introduced by Alsedà and Mañosas in [5] together with the appropriate extension of the Kneading Theory given in Chapter 1 to this coding. The advantage of this approach is that it allows us to work with circle maps of degree one of arbitrary "height" without increasing too much the difficulty of the computations. Therefore, we shall be able to find a simple entropy formula for a much wider class of maps. This formula depends in a simple way on the kneading pair of the map under consideration (see again Section 2.2 for a precise definition of a kneading pair).

Now we are going to define the class $\mathcal{M}$ of maps we shall consider. We will say that $F \in \mathcal{M}$ if:

(A) $F \in \mathcal{A}$.

(B) There exists a closed interval A_F of length at most 1 such that $c_F \in \mathrm{Int}(A_F)$ and $F(A_F) \subset A_F + m$ for some $m \in \mathbf{Z}$.

We recall that each map from $\mathcal{L}$ is conjugated by a translation to a map from $\mathcal{L}$ having the minimum at 0. Therefore, the fact that in (A) we fix that F has a minimum in 0 is not restrictive.

The chapter is organized as follows. In Section 4.2 we recall the definition of topological entropy for continuous maps of a compact space into itself. In Section 4.3 we define the appropriate Kneading Theory for the class $\mathcal{A}$ in order to compute the topological entropy. Finally, in Section 4.4 we state the formula to compute the topological entropy for maps in $\mathcal{M}$ and in Section 4.5 we prove it.

4.2 The topological entropy

There are several definitions of topological entropy. We shall use the classical definition, due to Adler, Konheim and McAndrew [1].

Let X be a compact (usually metric) topological space, and let $f : X \longrightarrow X$ be a continuous map. A set $\mathcal{Y}$ of subsets of X is called a *cover* if their union is X. For open covers $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ of X we denote:

$$\begin{aligned} \bigvee_{i=1}^{n} \mathcal{Y}_i &= \mathcal{Y}_1 \vee \mathcal{Y}_2 \vee \ldots \vee \mathcal{Y}_n \\ &= \{A_1 \cap A_2 \cap \ldots \cap A_n : A_i \in \mathcal{Y}_i, 1 \le i \le n,\ A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset\}. \end{aligned}$$

Note that $\vee_{i=1}^{n} \mathcal{Y}_i$ is also an open cover.

For an open cover $\mathcal{Y}$ we denote $f^{-n}(\mathcal{Y}) = \{f^{-n}(A) : A \in \mathcal{Y}\}$ and $\mathcal{Y}^n = \vee_{i=0}^{n} f^{-i}(\mathcal{Y})$. For each i, $f^{-i}(\mathcal{Y})$ is an open cover, so $\mathcal{Y}^n$ is also an open cover. If we want to indicate that we use the map f, we write $\mathcal{Y}_f^n$ for $\mathcal{Y}^n$. Next, we denote by $\mathcal{N}(A)$ the minimal possible cardinality of a *subcover* chosen from $\mathcal{Y}$ (i.e. a subset of $\mathcal{Y}$ which is also a cover of X). If $\mathcal{Y}$ is a cover of X and $Y \subset X$ then we denote by $\mathcal{Y}|_Y$ the cover $\{A \cap Y : A \in \mathcal{Y}\}$ of Y.

The following simple inequalities hold:

$$\mathcal{N}(\mathcal{Y} \vee \mathcal{J}) \leq \mathcal{N}(\mathcal{Y})\mathcal{N}(\mathcal{J}), \tag{4.2.1}$$

$$\mathcal{N}(f^{-n}(\mathcal{Y})) \leq \mathcal{N}(\mathcal{Y}). \tag{4.2.2}$$

We have $\mathcal{Y}^{k+n} = \mathcal{Y}^k \vee f^{-k}(\mathcal{Y}^n)$ and hence the next useful inequality

$$\mathcal{N}(\mathcal{Y}^{k+n}) \leq \mathcal{N}(\mathcal{Y}^k)\mathcal{N}(\mathcal{Y}^n). \tag{4.2.3}$$

We have to use a simple analytic lemma. A sequence $(\alpha_n)_{n=1}^{\infty}$ of non-negative real numbers is called *subadditive* if for each n and k we have $\alpha_{k+n} \leq \alpha_k + \alpha_n$.

Lemma 4.2.1 *If $(\alpha_n)_{n=1}^{\infty}$ is a subadditive sequence then the limit*

$$\lim_{n\to\infty} \frac{\alpha_n}{n}$$

exists and is equal to $\inf_n \alpha_n/n$.

By (4.2.3) and Lemma 4.2.1, the limit

$$h(f, \mathcal{Y}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\mathcal{Y}) \tag{4.2.4}$$

exists and is equal to the infimum of $(1/n)\log \mathcal{N}(\mathcal{Y}^n)$. Clearly, $h(f, \mathcal{Y}) \geq 0$. The number $h(f, \mathcal{Y})$ is called *the (topological) entropy of f on the cover $\mathcal{Y}$*. Now we can take

$$h(f) = \sup h(f, \mathcal{Y}) \tag{4.2.5}$$

where the supremum is taken over all open covers $\mathcal{Y}$ of X. The number $h(f)$ is called *the topological entropy of f*. It is also non-negative.

Let $F \in \mathcal{L}$ and assume that F is a lifting of f. We define the *topological entropy* of F, $h(F)$, as the topological entropy of f (see [1] or [35]).

The topological entropy measures the complexity of the map in the sense that it measures the exponential growth rate of the number of "$\varepsilon-$different" pieces of orbits of length n when n tends to infinity. For a piecewise monotone map of the interval it measures the exponential growth rate of the number of pieces of monotonicity of the iterates of the map (see [36]). Roughly speaking, it also measures the exponential growth rate of the number of periodic orbits, as we increase their periods.

4.3 Kneading Theory and Topological entropy for maps in $\mathcal{A}$

Now, we are going to outline the extension of the kneading theory of Milnor and Thurston [20] to the class $\mathcal{A}$. These techniques have been used already by Alsedá and Mañosas in [5] to obtain lower bounds of the topological entropy depending on the rotation interval for the class of maps $\mathcal{A}$.

We say that $F \in \mathcal{L}$ is *piecewise monotone* if there are $0 = c_0 < c_1 < \cdots < c_n = 1$, such that, $F|_{[c_{i-1},c_i]}$ is strictly monotone for $i = 1, 2, \dots n$. Assume that each interval $[c_{i-1}, c_i]$ is maximal having the above property and satisfying that $(E \circ F)|_{(c_{i-1},c_i)}$ is constant. Note that the points of the set $\{c_i + k, i = 0, 1, \dots n; k \in \mathbf{Z}\}$ are either local minima, local maxima, points of $\mathbf{Z}$, or points of $F^{-1}(\mathbf{Z})$. We call those points *turning points of* F. From now on, the set of turning points of F will be denoted by $\Delta(F)$. We note that if F is piecewise monotone then F^n is piecewise monotone for all $n \geq 0$. Also, any map from $\mathcal{A}$ is piecewise monotone.

Let $F \in \mathcal{A}$ be with height p_F (see Section 2.2). Then $\Delta(F) = \mathbf{Z} \cup F^{-1}(\mathbf{Z}) \cup c_F + \mathbf{Z}$. We note that if $x \in \Delta(F)$ then $x + \mathbf{Z} \subset \Delta(F)$. Moreover, $\Delta(F) \cap [0, 1]$ can be written as $\{c_0, c_1, c_2, \dots, c_{2p+1}\}$ with $0 = c_0 < c_1 < \dots < c_{p+1} = c_F < \dots < c_{2p+1} = 1$, $F(c_1) = E(F(0)) + 1 = E(F(c_F)) - p + 1$ and $F(c_i) = F(c_{2p+1-i}) = E(F(c_F)) - p + i$ for $i = 2, 3, \dots, p$ (see Figure 4.3.1).

Now we define the notion of address we are going to use. It is essentially the same that has been introduced in Chapter 2 except for the fact that, in order to compute topological entropy easily, we code each turnig point in $(0, 1)$ with a special symbol. For $x \in \mathbf{R}$ we set $A_F(x) = (s(x), d(x))$, where $d(x) = E(F(x)) - E(x)$ and

$$s(x) = \begin{cases} L & \text{if } x - E(x) < c_F \text{ and } x \notin \Delta(F), \\ R & \text{if } x - E(x) > c_F \text{ and } x \notin \Delta(F), \\ c_i & \text{if } D(x) = c_i. \end{cases}$$

Since $F|_{[c_{i-1},c_i]}$ is monotone and $(E \circ F)|_{[c_{i-1},c_i]}$ is constant for all $i = 1, 2, \dots, 2p + 1$, each point from an interval of the form $(c_{i-1}, c_i) + m$ with $m \in \mathbf{Z}$ has the same address.

Figure 4.3.1: The turning points of a map from $\mathcal{A}$.

Next we are going to define the *invariant coordinate of a point.* Let

$$A = (s, d) \in \{L, R, c_0, c_1, c_2, \ldots, c_{2p+1}\} \times \mathbf{Z}.$$

We set

$$\epsilon(A) = \begin{cases} 1 & \text{if } s = L, \\ -1 & \text{if } s = R, \\ 0 & \text{otherwise,} \end{cases}$$

$\kappa_0(x) = A_F(x)$, and

$$\kappa_n(x) = \left[\prod_{i=0}^{n-1} \epsilon(A_F(F^i(x)))\right] A_F(F^n(x))$$

for each $n \in \mathbf{N}$. Then the formal power series $\sum_{n=0}^{\infty} \kappa_n(x) t^n$ will be called the *invariant coordinate of* x and will be denoted by $\kappa_F(x)$ (or simply $\kappa(x)$ when no confusion will be possible). Note that $\kappa(x) = \kappa(x+m)$ for all $m \in \mathbf{Z}$.

Let $\mathcal{V}$ be the set of all pairs of the form (s, d) with $d \in \mathbf{Z}$ and $s \in \{L, R\}$. We note that for $F \in \mathcal{A}$ and for $x \notin \Delta(F)$, $A_F(x) \in \mathcal{V}$.

It is not difficult to show that for each $n \geq 0$ there exists $\delta(n) > 0$ such that $\kappa_n(y)$ takes a constant value, denoted by $\kappa(x^+)$, for all $y \in (x, x+\delta(n))$. Then, for $x \in \mathbf{R}$ we set

$$\kappa(x^+) = \kappa_F(x^+) = \sum_{n=0}^{\infty} \kappa_n(x^+)t^n.$$

In a similar way we define $\kappa(x^-)$.

We note that if $F^n(x) \notin \Delta(F)$ for all $n \geq 0$ (that is, $s(F^n(x)) \in \{L, R\}$ for all $n \geq 0$) then $\kappa(x^+) = \kappa(x^-) = \kappa(x)$. As for the invariant coordinate we have that $\kappa(x^+) = \kappa((x+m)^+)$ and $\kappa(x^-) = \kappa((x+m)^-)$ for all $m \in \mathbf{Z}$. The sequences $\kappa(0^+)$ and $\kappa(c_F^-)$ will play a special role in our study.

Remark 4.3.1 For each $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in (0, \delta)$ there exists $y \in (-\epsilon, 0)$ with the property that $F(x) = F(y)$. Therefore, $\kappa_0(0^+) = (L, E(F(0)))$, $\kappa_0(0^-) = (R, E(F(0))+1)$ and $\kappa_n(0^+) = -\kappa_n(0^-)$ for all $n > 0$. In a similar way we obtain that $\kappa_0(c_F^+) = (R, E(F(c_F)))$, $\kappa_0(c_F^-) = (L, E(F(c_F)))$ and $\kappa_n(c_F^+) = -\kappa_n(c_F^-)$ for all $n > 0$. Furthermore, assume that $F \in \mathcal{A}$ has kneading pair $(\underline{\widehat{I}}_F(0^+), \underline{\widehat{I}}_F(c_F^-))$ with $\underline{\widehat{I}}_F(0^+) = d_{1,1}^{s_{1,1}} d_{1,2}^{s_{1,2}} \ldots$ and $\underline{\widehat{I}}_F(c_F^-) = d_{2,1}^{s_{2,1}} d_{2,2}^{s_{2,2}} \ldots$. We note that then $\kappa_0(0^+) = (L, d_{1,1})$ and $\kappa_i(0^+) = \pm(s_{1,i}, d_{1,i+1})$ for each $i \geq 1$. Also, $\kappa_0(c_F^-) = (L, d_{2,1})$ and $\kappa_i(c_F^-) = \pm(s_{2,i}, d_{2,i+1})$ for each $i \geq 1$. Therefore, from the kneading pair of F we get easily the sequences $\kappa(0^-)$ and $\kappa(c_F^+)$. $\square$

By setting $L < R$ we can define an ordering in $\mathcal{V}$ as follows. Let (s, d) and (t, m) be elements of $\mathcal{V}$ such that $(s, d) \neq (t, m)$. We say that $(s, d) < (t, m)$ if either

$$\begin{array}{ll} s < t & \text{or} \\ s = t = L & \text{and } d < m, \text{ or} \\ s = t = R & \text{and } d > m. \end{array}$$

If none of these holds we say that $(s, d) > (t, m)$. We note that this ordering has the property that if $x, y \notin \Delta(F)$ and $x < y$, then $A_F(x) \leq A_F(y)$.

For a map $F \in \mathcal{A}$, we shall denote by $\mathcal{V}_F$ the set of all addresses of all points of $\mathbf{R} \setminus \Delta(F)$. Note that $\mathcal{V}_F \subset \mathcal{V}$ and $\mathrm{Card}\mathcal{V}_F = 2p+1$. We also shall write the elements of $\mathcal{V}_F$ as $I_1 < I_2 < \ldots < I_{2p+1}$.

Now, for each $i \in \{1, 2, \ldots, 2p\}$, we define the *i-th kneading invariant of* F to be $\nu(c_i) = \kappa_F(c_i^+) - \kappa_F(c_i^-)$. Note that $\nu(c_i)$ is a power series with coefficients in $\mathbf{Z}[[\mathcal{V}_F]]$. Thus we can write

$\nu(c_i) = \sum_{j=1}^{n} \nu_i^j(t) I_j$ with $\nu_i^j(t) \in \mathbf{Z}[[t]]$ for all i, j. The $(2p+1) \times (2p)$ matrix $\mathcal{K}_F(t) = (\nu_i^j(t))$ will be called the *kneading matrix of* F. Let $D_F^i(t)$ be the determinant which is obtained by deleting the i-th row of $\mathcal{K}_F(t)$.

The expression

$$\mathcal{D}_F(t) = \frac{(-1)^{i+1}}{(1-\epsilon(I_i)t)} D_F^i(t)$$

will be called the *kneading determinant of* F. It is well known [20] (see also [5]) that the above expression does not depend on i. Thus, the kneading determinant of F is well defined. From [20] and [36] (see also [5]) we obtain the following result.

Theorem 4.3.2 *Let* $F \in \mathcal{A}$. *If* $\mathcal{D}_F(t)$ *does not vanish in* $(0,1)$ *then* $h(F) = 0$. *Otherwise,* $h(F) = \log \frac{1}{\alpha}$ *where* α *is the smallest zero of* $\mathcal{D}_F(t)$ *in* $(0,1)$.

Theorem 4.3.2 is the key point to obtain our formula to compute the topological entropy. This is the goal of the next section. However, we will end this section with a simple result on the topological entropy for maps in $\mathcal{A}$.

Proposition 4.3.3 *Let* $F \in \mathcal{A}$ *be such that* $((\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. *Then* $h(F) = 0$.

Proof. Since $((\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$ we have that $p_F = 1$. So, we can write $\kappa(0^+) = I_1 + k(t)$, $\kappa(0^-) = I_3 - k(t)$, $\kappa(c_F^+) = I_3 - k(t)$ and $\kappa(c_F^-) = I_2 + k(t)$ where $k(t) = \sum_{j=1}^{3} p_j(t) I_j$ with $p_j(t) \in \mathbf{Z}[[t]]$ for $j = 1, 2, 3$. Since $F(c_1) = 1$ we have that $\kappa(c_1^+) = I_2 + \kappa(0^+)$ and $\kappa(c_1^-) = I_1 + \kappa(0^-)$. Therefore, $\nu(c_1) = (I_2 + I_1 + k(t)) - (I_1 + I_3 - k(t)) = I_2 - I_3 + 2k(t)$. On the other hand, since $c_2 = c_F$, $\nu(c_2) = (I_3 - k(t)) - (I_2 + k(t)) = I_3 - I_2 - 2k(t)$. Thus, $\nu(c_1) = -\nu(c_2)$ and, hence, $\mathcal{D}_F(t) = 0$. Therefore, $h(F) = 0$ from Theorem 4.3.2. ∎

4.4 The Topological entropy formula for maps in $\mathcal{M}$

This section will be devoted to establish the formula for the topological entropy we are looking for. To do this we shall obtain a formula for the kneading determinant of F and we will use Theorem 4.3.2.

Since A_F has length at most 1 and $c_F \in \mathrm{Int}(A_F)$ we have $A_F \subset (c_F - 1, c_F + 1)$. On the other hand, since $F(A_F) \subset A_F + m$ we know that $F(A_F)$ also has length at most 1. Therefore, $0 \notin A_F$. Otherwise, $F(A_F) \supset F((0, c_F))$. In view of (A) and (B) we have that $F(c_F) > F(1) = F(0) + 1$.

Hence, $F(A_F)$ would have length larger than 1; a contradiction. Assume now that $1 \in A_F$. Let B_F denote the interval $[\max A_F, \min A_F + 1]$ (note that B_F degenerates to a point if A_F has length 1) and let G denote the map $F - m$. Since $G(A_F) \subset A_F$ we have that $G(\max A_F) \leq \max A_F$ and $G(\min A_F) \geq \min A_F$. Then, $G(\min A_F + 1) = G(\min A_F) + 1 \geq \min A_F + 1$ because $G \in \mathcal{L}$. Therefore, G has a fixed point in B_F. Let us call this fixed point u_F. Since $B_F \subset (1, c_F + 1)$, G is strictly increasing in B_F. Thus, $G([u_F - 1, u_F]) = [u_F - 1, u_F]$ and, hence, $G|_{[u_F-1,u_F]}$ is a bimodal map of the interval. We note that, in this case, F has degenerate rotation interval equals to $\{m\}$. For this case an entropy formula has been obtained by Mumbrú in [37]. Thus, in what follows we shall replace (C) by the following stronger condition:

(C) There exists a closed interval $A_F \subset (0,1)$ such that $c_F \in \operatorname{Int}(A_F)$ and $F(A_F) \subset A_F + m$ for some $m \in \mathbf{Z}$.

We note that then, if $F \in \mathcal{M}$, the interval map $(F - m)|_{A_F}$ is unimodal.

Prior to state the theorem giving the entropy formula for maps from $\mathcal{M}$, which is the main result of this chapter, we shall introduce some more notation.

Set $R_F(t) = t[\kappa(0^+) - \kappa(0^-)]$. Since $\kappa(0^+)$ and $\kappa(0^-)$ are formal power series with coefficients in $\mathbf{Z}[[\mathcal{V}_F]]$ so is $R_F(t)$. Hence, $R_F(t)$ can be written as $\sum_{i=1}^{2p_F+1} \phi_i(t) I_i$, where $\phi_i(t) \in \mathbf{Z}[[t]]$ for all $i = 1, 2, \ldots, 2p_F + 1$. Then we also set

$$P_F(t) = -1 + \sum_{i=1}^{p_F} (p_F - i + 1)\phi_i(t) - \sum_{i=p_F+3}^{2p_F+1} (i - p_F - 2)\phi_i(t).$$

Remark 4.4.1 The series $P_F(t)$ can be computed directly from $\kappa(0^+)$ and hence from $\underline{\widehat{I}}_F(0^+)$ (see Remark 4.3.1). To see this we note that, in a similar way as we did for $R_F(t)$, we can write $\kappa(0^+)$ as $\sum_{i=1}^{2p+1} \tilde{\phi}_i(t) I_i$ with $\tilde{\phi}_i(t) \in \mathbf{Z}[[t]]$ for all $i = 1, 2, \ldots, 2p_F + 1$. Then, by Remark 4.3.1, we have that $R_F(t) = tI_1 - tI_{2p+1} + 2t[\kappa(0^+) - I_1] = -tI_1 - tI_{2p+1} + 2t\kappa(0^+)$. Hence, $\phi_1(t) = -t + 2t\tilde{\phi}_1(t)$, $\phi_{2p+1}(t) = -t + 2t\tilde{\phi}_{2p+1}(t)$ and $\phi_i(t) = \tilde{\phi}_i(t)$ for $i = 2, 3, \ldots, 2p$. $\square$

From the definition of $\mathcal{M}$ (see (C)) we have that $\kappa(c_F^+)$ and $\kappa(c_F^-)$ are formal power series with coefficients in $\mathbf{Z}[[I_{p+1}, I_{p+2}]]$. Therefore, $\kappa(c_F^+) - \kappa(c_F^-)$ can be written as $K_F(t)I_{p+1} + \widetilde{K}_F(t)I_{p+2}$ with $K_F(t), \widetilde{K}_F(t) \in \mathbf{Z}[[t]]$.

Remark 4.4.2 The series $K_F(t)$ can be computed directly from $\kappa(c_F^-)$and hence from $\underline{\widehat{I}}_F(c_F^-)$ (see Remark 4.3.1). Indeed, if $\kappa(c_F^-) = \pi_1(t)I_{p+1} + \pi_2(t)I_{p+2}$ with $\pi_1(t), \pi_2(t) \in \mathbf{Z}[[t]]$ then, by

Remark 4.3.1, we have that $\kappa(c_F^+) = (1-\pi_1(t))I_{p+1} + (1-\pi_2(t))I_{p+2}$. Hence $K_F(t) = 1 - 2\pi_1(t)$. $\square$

If $K_F(t)$ vanishes in $(0,1)$ we shall denote by α_{K_F} the smallest zero of $K_F(t)$ in $(0,1)$. Otherwise we set $\alpha_{K_F} = 1$. In a similar way we define α_{P_F} by using $P_F(t)$ instead of $K_F(t)$.

The following theorem is the main result of this paper and gives the formula we are looking for.

Theorem 4.4.3 *For $F \in \mathcal{M}$ we have $h(F) = \log(\min\{\alpha_{K_F}, \alpha_{P_F}\})^{-1}$.*

We note that, in view of Remarks 4.4.1 and 4.4.2, the numbers α_{K_F} and α_{P_F} can be computed solely from the knowledge of $\underline{\widehat{I}}_F(0^+)$ and $\underline{\widehat{I}}_F(c_F^-)$. Therefore, Theorem 4.4.3 gives a formula for the topological entropy of a map from $\mathcal{M}$ *depending only on the kneading pair of the map under consideration.*

In view of Condition (C), for each $F \in \mathcal{M}$ we get that $F|_{A_F}$ is unimodal. Therefore, $\alpha_{K_F}^{-1} \leq 2$ (see for instance [36]). Hence, whenever $\alpha_{P_F}^{-1} \geq 2$ we shall have $h(F) = \log \alpha_{P_F}^{-1}$. Next we shall obtain sufficient conditions to assure the validity of this formula.

Corollary 4.4.4 *If the length of the rotation interval of $F \in \mathcal{M}$ is strictly larger that $1/2$ then $h(F) = \log \alpha_{P_F}^{-1}$.*

Proof. We note that the rotation interval of each map $F \in \mathcal{M}$ is of the form $[c, d_F]$ with $d_F = E(F(c_F))$. By Theorem B of [3] we get that $h(F) \geq \log \beta_{d_F - c}$ where β_{d_F-c} is the largest root of the equation

$$z + 1 - 2\sum_{n=0}^{\infty} z^{-E(n/(d_F-c))} = 0.$$

In view of Theorem C.(c) and Lemma 22 of [3] we obtain that β_{d_F-c} is larger than or equal to the largest root of the equation $x^3 - x^2 - 3x + 1 = 0$. This root is $2.1700864866\ldots$. This ends the proof of the corollary. ∎

We also note that if for $F \in \mathcal{M}$ we have $p_F \geq 2$ then the rotation interval of F has length larger than or equal to 1. Thus, from the above corollary, we obtain

Corollary 4.4.5 *Let $F \in \mathcal{M}$. If $p_F \geq 2$ then $h(F) = \log \alpha_{P_F}^{-1}$.*

Figure 4.4.2: A map F_μ which satisfies (a)–(c).

On the other hand, in the case of the family considered by Hockett and Holmes [17] it turns out that $\alpha_{K_F} = 1$ and, hence, the same formula for the topological entropy holds. To see this let us define precisely the family of maps they considered. Let $[\mu_0, \mu_1]$ be a closed proper interval of the real line and let $F_\mu = F(\mu, .) : [\mu_0, \mu_1] \times \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous one-parameter family satisfying the following conditions for each $\mu \in [\mu_0, \mu_1]$:

1. $F_\mu \in \mathcal{M} \cap \mathcal{C}^1(\mathbf{R}, \mathbf{R})$.
2. There exists $w_\mu \in A_{F_\mu}$ such that w_μ is an attractive fixed point of $(F_\mu - m)|_{A_F}$ and $\min A_F$ is a repulsive fixed point of $(F_\mu - m)|_{A_F}$.
3. There exist $a \in (0, c_F)$ and $b \in (c_F, 1)$ such that $F_\mu(b) = F_\mu(\min A_F) = F_\mu(a) + 1$ and $a + 1 > F_\mu(0) > b$ (see Figure 4.4.2).

We note that for such a family of maps one has $w_\mu < b$ and $E(F_\mu(c_F)) = E(F_\mu(w_\mu))$ for each $\mu \in [\mu_0, \mu_1]$. Therefore, $A_{F_\mu}(F_\mu^n(c_{F_\mu})) = I_{p+2}$ for all $n \geq 1$ and, hence, $\kappa(c_{F_\mu}^-) = I_{p+1} + \sum_{i=1}^{\infty}(-1)^{i-1}t^i I_{p+2}$. Thus, by Remark 4.4.2, $K_{F\mu}(t) = -1$. Therefore, in view of Theorem 4.4.3, we get the following

Corollary 4.4.6 *Let $F_\mu : [\mu_0, \mu_1] \times \mathbf{R} \longrightarrow \mathbf{R}$ be the continuous one-parameter family satisfying conditions (a)–(c). Then $h(F_\mu) = \log \alpha_{P_{F_\mu}}^{-1}$ for all $\mu \in [\mu_0, \mu_1]$.*

4.5 Proof of Theorem 4.4.3

In view of Theorem 4.3.2 we only have to show that the zeros of $K_F(t) \cdot P_F(t)$ and $\mathcal{D}_F(t)$ in $(0, 1)$ coincide. Before starting the proof of Theorem 4.4.3 we shall compute the kneading invariants of the map under consideration. Since $c_{p+1} = c_F$ we have that $\nu(c_{p+1}) = \nu(c_F) = \kappa(c_F^+) - \kappa(c_F^-) = K_F(t)I_{p+1} + \widetilde{K}_F(t)I_{p+2}$. The next lemma takes care of the computation of the rest of the kneading invariants.

Lemma 4.5.1 *For each $F \in \mathcal{M}$ we have $\nu(c_i) = I_{i+1} - I_i + R_F(t)$ for $i \neq p_F + 1$.*

Proof. First we compute $\nu(c_i)$ with $i \in \{1, 2, \ldots, p\}$. Since F is increasing in a neighborhood of c_i, $F(c_i) \in \mathbf{Z}$ and, $\kappa(x^+) = \kappa((x+m)^+)$ and $\kappa(x^-) = \kappa((x+m)^-)$ for all $x \in \mathbf{R}$ and $m \in \mathbf{Z}$ we have that $\kappa(c_i^+) = I_{i+1} + \epsilon(I_{i+1})t\kappa(0^+)$ and $\kappa(c_i^-) = I_i + \epsilon(I_i)t\kappa(0^-)$. Since $i \leq p$ we have $\epsilon(I_{i+1}) = \epsilon(I_i) = 1$ and, hence, $\nu(c_i) = I_{i+1} - I_i + t[\kappa(0^+) - \kappa(0^-)] = I_{i+1} - I_i + R_F(t)$. When $i \in \{p+2, \ldots, 2p\}$, since F is decreasing in a neighborhood of c_i, in a similar way we have $\kappa(c_i^+) = I_{i+1} + \epsilon(I_{i+1})t\kappa(0^-)$ and $\kappa(c_i^-) = I_i + \epsilon(I_i)t\kappa(0^+)$. Now we have $\epsilon(I_{i+1}) = \epsilon(I_i) = -1$ and, hence, $\nu(c_i) = I_{i+1} - I_i - t[\kappa(0^-) - \kappa(0^+)] = I_{i+1} - I_i + R_F(t)$. ∎

Proof of Theorem 4.4.3. We recall that $R_F(t) = \sum_{i=1}^{2p+1} \phi_i(t)I_i$ (in this proof p_F will be denoted by p for simplicity). Then, by Lemma 4.5.1, we have that $\mathcal{K}_F(t)$ is (in the following matrices, again for simplicity, $\phi_i(t)$ will be denoted by ϕ_i)

$$\begin{pmatrix}
\phi_1-1 & \phi_1 & \cdots & \phi_1 & 0 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2+1 & \phi_2-1 & \cdots & \phi_2 & 0 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3+1 & \cdots & \phi_3 & 0 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & 0 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p-1 & 0 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1}+1 & K_F(t) & \phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} & \phi_{p+1} \\
\phi_{p+2} & \phi_{p+2} & \cdots & \phi_{p+2} & \widetilde{K}_F(t) & \phi_{p+2}-1 & \phi_{p+2} & \cdots & \phi_{p+2} & \phi_{p+2} \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & 0 & \phi_{p+3}+1 & \phi_{p+3}-1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & 0 & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p}+1 & \phi_{2p}-1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & 0 & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1}+1
\end{pmatrix}.$$

Now, $D_F^{p+2}(t) =$

$$\begin{vmatrix}
\phi_1-1 & \phi_1 & \cdots & \phi_1 & 0 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2+1 & \phi_2-1 & \cdots & \phi_2 & 0 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3+1 & \cdots & \phi_3 & 0 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & 0 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p-1 & 0 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1}+1 & K_F(t) & \phi_{p+1} & \phi_{p+1} & \cdots & \phi_{p+1} & \phi_{p+1} \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & 0 & \phi_{p+3}+1 & \phi_{p+3}-1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & 0 & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p}+1 & \phi_{2p}-1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & 0 & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1}+1
\end{vmatrix}_{2p} =$$

$$(-1)^{2(p+1)} K_F(t) \begin{vmatrix}
\phi_1-1 & \phi_1 & \cdots & \phi_1 & \phi_1 & \phi_1 & \cdots & \phi_1 & \phi_1 \\
\phi_2+1 & \phi_2-1 & \cdots & \phi_2 & \phi_2 & \phi_2 & \cdots & \phi_2 & \phi_2 \\
\phi_3 & \phi_3+1 & \cdots & \phi_3 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 \\
\phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 & \phi_4 & \cdots & \phi_4 & \phi_4 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_p & \phi_p & \cdots & \phi_p-1 & \phi_p & \phi_p & \cdots & \phi_p & \phi_p \\
\phi_{p+3} & \phi_{p+3} & \cdots & \phi_{p+3} & \phi_{p+3}+1 & \phi_{p+3}-1 & \cdots & \phi_{p+3} & \phi_{p+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\phi_{2p} & \phi_{2p} & \cdots & \phi_{2p} & \phi_{2p} & \phi_{2p} & \cdots & \phi_{2p}+1 & \phi_{2p}-1 \\
\phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1} & \phi_{2p+1} & \cdots & \phi_{2p+1} & \phi_{2p+1}+1
\end{vmatrix}_{2p-1} =$$

$$K_F(t)\left|\begin{array}{ccccccc|cccccc}
\phi_1-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\phi_2+1 & -2 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\
\phi_3 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cline{2-12}
\phi_{p+3} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\phi_{p+4} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{2p-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\
\phi_{2p} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
\phi_{2p+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

If $p = 1$ then it follows that

$$D_F^{p+2}(t) = K_F(t)(\phi_1(t) - 1) = K_F(t) \cdot P_F(t).$$

Now, suppose that $p \geq 2$. Then by adding the first row of the determinant to the second one we get, $D_F^{p+2}(t) =$

$$K_F(t)\left|\begin{array}{ccccccc|cccccc}
\phi_1-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
\phi_2+\phi_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_3 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cline{2-12}
\phi_{p+3} & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\phi_{p+4} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{2p-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\
\phi_{2p} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\
\phi_{2p+1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

Let $u_k = \sum_{i=k}^{2p+1} \phi_i$ for $k = p+3, p+4, \ldots, 2p+1$. Then we have that $D_F^{p+2}(t) =$

$$K_F(t)\left|\begin{array}{cccccc|cccccc}
\phi_1-1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
\phi_2+\phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_3 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0\\
u_{2p+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

Now we add the p-th column of the above determinant to the $(p-1)$-th one. Then we add the $(p-1)$-th column of the new determinant to the $(p-2)$-th one and, by iterating this process $p-2$ times we get $D_F^{p+2}(t) =$

$$K_F(t)\left|\begin{array}{cccccc|cccccc}
\phi_1-1 & p-1 & p-2 & \cdots & 2 & 1 & 1 & 1 & \cdots & 1 & 1 & 1\\
\phi_2+\phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0\\
u_{2p+1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

Let $u = \phi_1 - 1 - \sum_{k=p+3}^{2p+1} u_k$. Then $D_F^{p+2}(t) =$

$$K_F(t)\left|\begin{array}{cccccc|cccccc}
u & p-1 & p-2 & \cdots & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_2+\phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0\\
\cline{2-11}
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0\\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
u_{2p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0\\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

We note that $\sum_{k=p+3}^{2p+1} u_k = \sum_{k=p+3}^{2p+1} \sum_{i=k}^{2p+1} \phi_i(t) = \sum_{i=p+3}^{2p+1}(i-p-2)\phi_i(t)$. Therefore, $P_F(t) = -1+\sum_{i=1}^{p}(p-i+1)\phi_i(t)-\sum_{i=p+3}^{2p+1}(i-p-2)\phi_i(t) = -1+\phi_1(t)+(p-1)(\phi_1(t)+\phi_2(t))+\sum_{i=3}^{p}(p-i+1)\phi_i(t)-\sum_{k=p+3}^{2p+1} u_k = u+(p-1)(\phi_1(t)+\phi_2(t))+\sum_{i=3}^{p}(p-i+1)\phi_i(t)$. Thus $D_F^{p+2}(t) =$

$$K_F(t) \left|\begin{array}{cccccc|cccccc}
P_F(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_2+\phi_1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_3 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\phi_{p-1} & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\phi_p & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
u_{p+3} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
u_{p+4} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{2p-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
u_{2p} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right|_{2p-1}.$$

Hence, $D_F^{p+2}(t)$ is equal to $(-1)^{p-1}K_F(t)\cdot P_F(t)$.

Since

$$\mathcal{D}_F(t) = \frac{(-1)^{p+3}}{(1-\epsilon(I_{p+2})t)} D_F^{p+2}(t) = \frac{(-1)^{p+3}}{(1-t)} D_F^{p+2}(t)$$

we have that the zeros of $\mathcal{D}_F(t)$ and $D_F^{p+2}(t)$ in $(0,1)$ coincide. Therefore, the zeros of $\mathcal{D}_F(t)$ and $K_F(t)\cdot P_F(t)$ in $(0,1)$ are the same. This ends the proof of Theorem 4.4.3. ∎

Bibliography

[1] Adler R.L., Konheim A.G., McAndrew M.H., *Topological entropy*, Trans. Amer. Soc. **114**, 1965

[2] Alsedà L., Llibre J., Misiurewicz M., *Combinatorial dynamics and entropy in dimension one*, Advances Series Nonlinear Dynamics, World Scientific, Singapore, 1993.

[3] Alsedà L., Llibre J., Misiurewicz M., Simó C., *Twist periodic orbits and topological entropy for continuous maps of the circle of degree one which have a fixed point*, Ergod. Th. & Dynam. Sys.,**5**, 1985.

[4] Alsedà L., Llibre J., Serra R., *Bifurcations for a circle map associated with the Van der Pol equation*, Sur la théorie de l'iteration et ses aplications, Colloque internationaux du CNRS No.**332**, Toulouse, 1982.

[5] Alsedà L., Mañosas F., *Kneading theory and rotation interval for a class of circle maps of degree one*, Nonlinearity **3**, 1990.

[6] Alsedà L., Mañosas F., Szlenk W., *A Characterization of the uniquely ergodic endomorphisms of the circle*, Proc. Amer. Math. Soc., 117, **3** 1993

[7] Boyland P.L., *Bifurcations of circle maps: Arnol'd Tongues, bistability and rotation intervals*, Commun. Math. Phys., 106, 353-381, 1986.

[8] Cartwright M.L., Littlewood J.E., *On nonlinear differential equations of second order II*, Ann. of Math. **48**, 1947.

[9] Collet P. Eckmann J.P., *Iterated maps on the interval as dynamical systems*, Progress in Physics, Birkhäuser, 1980

[10] Chenciner A., Gambaudo J.M., Tresser Ch., *Une remarque sur la structure des endomorphismes de degré* 1 *du cercle*, C. R. Acad. Sci., Paris series I **299**, 1984.

[11] Glass L., Guevara M.R., Belair J., Shrier A., *Global bifurcations of a periodic forced biological oscillator*, Physical Review A, Volume 29, **3**, 1348–1357, 1984.

[12] Glass L., Mackey M.C., *From clocks to chaos*, Princenton University Press, 1988.

[13] Glass L., Perez R., *Fine Structure of Phase Locking*, Physical Review Letters, Volume 48, **26**, 1772–1775, 1982.

[14] Guckenheimer J., *Symbolic dynamics and relaxation oscillations*, Physica 1D, 227-235, 1980.

[15] Guckenheimer J., *Sensitive dependence to initial conditions for one dimensional maps*, Commun. Math. Phys. , 70, 133–160, 1979

[16] Guckenheimer J., Holmes P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.

[17] Hockett K. Holmes P., *Bifurcations to badly ordered orbits in one parameter families of circle maps, or Angels fallen from the Devil's staircase*, Proc. Amer. Math. Soc. **102**, 1031–1051, 1988.

[18] Ito R., *Rotation sets are closed*, Math. Proc. Camb. Phil. Soc. 89, 1981.

[19] Jensen M.H., Bak P., Bohr T., *Transition to chaos by iteration of resonances in dissipative systems. I. Circle maps*, Physical Review A, Volume 30, **5**, 319–328, 1984.

[20] Milnor J. Thurston P., *On iterated maps on the interval*, I, II, "Dynamical Systems", Lect. Notes 1342, Springer, 1988.

[21] Katok A., *Some remarks on Birkhoff and Mather twist theorems*, Ergod. Theor. & Dynam. Sys. **2**. 1982.

[22] Kennedy M.P., Krieg K.R.,Chua L.O. *The Devil's Staircase: The Electrical Engineer's Fractal*, IEEE Trans. on Circuits and Systems., Vol **36**. 1133-1139, 1989.

[23] Malta I., *On Denjoy's theorem for endomorphisms*, Ergod. Th. & Dynam. Sys., **6**, 259–264, 1986.

[24] de Melo W., van Strien S., *One dimensional dynamics*, Springer-Verlag, 1993.

[25] Moser J., *Stable and random motions in dynamical systems*, Princenton University Press, 1973.

[26] Newhouse S.E., *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms*, IHES, 1977.

[27] Newhouse S.E.,Palis J., *Cycles and bifurcation theory*, Asterisque **31**, 1976.

[28] Newhouse S.E., Palis J., Takens F., *Stable families of dynamical systems I*, IHES 57, 1983.

[29] Misiurewicz M., *Rotations intervals for a class of maps of the real line into itself*, Ergod. Theor. Dynam. Sys. **6**. 1986.

[30] Levi M., *Qualitative analysis of the periodically forced relaxation oscillations*, Mem. Amer. Math. Soc. **244**, 1981.

[31] Singer D., *Stable orbits and bifurcation maps of the interval*, Siam J. Appl. Math., Vol 35, No. 2, 1978.

[32] Levinson N., *A second order differential equation with singular solutions*, Ann. of Math. **50**, 1949.

[33] Van der Pol B., *On relaxation oscillations*, Phil. Mag.**2**, 1926.

[34] Van der Pol B.,Van der Mark J., *Frecuency demultiplication*, Nature **120**, 1927.

[35] Denker M., Grillenberger C., Sigmund K., *Ergodic Theory in compact spaces*, Lecture Notes in Mathematics **527**, 1976.

[36] Misiurewicz M. Szlenk W.,*Entropy of piecewise monotone mappings*, Stud. Math., **67**, 45-63, 1980.

[37] Mumbrú P. *Periodic structure and topological entropy of bimodal maps (in Catalan)*, Ph. D. Thesis, Universitat Autònoma de Barcelona.

[38] Poincaré H., *Sur les curves définies par les équations differentielles*, Ouvres completes, Vol 1, 137-158, Gauthiers-Villars, Paris, 1952.

[39] Wall C.T.C., *A geometric introduction to topology*, Addison-Wesley, Reading, Mass., 1972.