THE BIFURCATIONS OF A PIECEWISE MONOTONE FAMILY OF CIRCLE MAPS RELATED TO THE VAN DER POL EQUATION

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Abstract

By using symbolic dynamics we describe the bifurcations of a family of continuous circle maps. This provides an approximation to the description of the qualitative behavior for a system of the Van der Pol type.

1 Introduction.

We study the bifurcations in a three parameter family $f = f(.,b,\delta)$ of C^0 maps of the circle into itself of degree one, with the parameters ranging in $b_1 \leq b \leq b_2, 0 < \delta \leq \overline{\delta}$, and satisfying the following properties:

There exist $\gamma > 1,\ k > 1/\gamma,\ c > 0$ and an interval $\Delta \subset \mathbf{S^1}$ whose endpoints depend on b and δ such that $|\Delta| < \delta$ and

$$f'(x) > k\gamma \text{ for all } x \in \Delta \tag{1.1}$$

$$-1 + c < f'(x) < -1/\gamma \text{ for all } x \in \mathbf{S^1} \setminus \Delta \tag{1.2}$$

$$-d/db[f(x_i(b), b, \delta) - x_i(b)] > \omega > 0,\ i = 1, 2 \tag{1.3}$$

where $x_1(b)$ and $x_2(b)$, are the endpoints of Δ, all differentiable in b, and $\omega = \omega(\delta)$ is independent of b (see Figure 1.1).

This family is a piecewise-differentiable version of Levi's circle maps (see [L] p.30-31 or [GH] p.74-82) which is used to study the following system of the Van der Pol type with periodic forcing term (see [L]):

$$\epsilon\ddot{x} + \Phi(x)\dot{x} + \epsilon x = bp(t) \tag{1.4}$$

where $\epsilon > 0$ is a small parameter, Φ (damping) is negative for $|x| < 1$ and positive elsewhere, $p(t)$ is periodic of period T and b varies in some finite interval $[b_1, b_2]$. In particular Φ and p can be chosen close (in some sense) to the functions $\Phi_0 = sgn(x^2 - 1),\ p_0 = sgn \sin(2\pi t/T)$.

Figure 1.1:

In [ALS] the following result is given. It characterizes the dynamics of f for certain values of b (compare with [L]).

Theorem 1.1 *If the map f satisfies (1.1)-(1.3) then for $\overline{\delta}$ small enough the interval $[b_1, b_2]$ consist of two alternating types of intervals A_k, B_k separated by (short) gaps g_k :*

$$[b_1, b_2] = A_1 \cup g_1 \cup B_1 \cup g_2 \cup A_2 \cup g_3 \cup, \dots, \cup A_n \cup g_{2n-1} \cup B_n,$$

such that:

(A) *For $b \in A_k$ the map f has exactly two fixed points, one stable and another unstable. Moreover, the basin of attraction of the stable fixed point isthe whole circle except the unstable fixed point.*

(B) *For $b \in B_k$ the map f has four fixed points, two stable and two unstable. Moreover, these two unstable fixed points belong to a Cantor set C such that $f|_C$ is topologically conjugated to a certain subshift of finite type.*

In fact in [ALS] it is given a first approach to the characterization of the bifurcations of f as b crosses the gaps g_k. The goal of this paper is to give a complete characterization of these bifurcations. Our main result is the following:

Theorem 1.2 *Let $g_k = (g_{k,1}, g_{k,2})$. For every gap g_k there exist α_k, β_k such that $g_{k,1} < \alpha_k \leq \beta_k < g_{k,2}$ and*

(a) If $b \in g_k$, thenf has exactly two fixed points, one stable and another unstable.

(b) If $b \in (g_{k,1}, \alpha_k]$ then the basin of attraction of the stable fixed point is either the whole circle except the unstable fixed point or the whole circle except the unstable fixed point union or $x_i(b)$ with $i = 1$ or 2.

(c) If $b \in (\beta_k, g_{k,2})$ then there exist a Cantor set C (which depend on b), containing the unstable fixed point and such that $f|_C$ is topologically conjugated to a subshift of finite type. Moreover, the basin of attraction of the stable fixed point is either the complementary of the Cantor set C or the complementary of the Cantor set C union $\cup_{n=0}^{\infty} f^{-n}(x_i(b))$ with $i = 1$ or 2.

(d) If $\alpha_k \neq \beta_k$ then the interval $(\alpha_k,\ \beta_k]$ consists of two sets $H_k,\ L_k$ such that $(\alpha_k, \beta_k] = H_k \cup L_k$ and L_k (resp. H_k) is closed (resp. open) in $(\alpha_k, \beta_k]$ and if b belongs to L_k (resp. H_k) then the dynamics of f is analogous to the case $b \in (g_{k,1}, \alpha_k]$ (resp. $b \in (\beta_k, g_{k,2})$).

The interest in studying the map f as $b \in g_k$ is due to the fact that by applying Levi's techniques to it we can obtain, to a certain level of approximation, the behavior of the flow (1.4) when b belongs to the bifurcation gaps g_k.

The paper is organized as follows. In Section 2 we give some definitions and preliminary results. In Section 3 we give the proof of Theorem 1.2.

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2 Definitions and preliminary results.

First we give some definitions about circle maps.

Definition 2.1 If $f \in C^0(\mathbf{S^1}, \mathbf{S^1})$, then a lifting of f is any continuous map $F : \mathbf{R} \longrightarrow \mathbf{R}$ such that for all $X \in \mathbf{R}$, $e(F(X)) = f(e(X))$ where $e : \mathbf{R} \longrightarrow \mathbf{S^1}$ is defined by $e(X) = \exp(2\pi i X)$.

From now one we will use lower case letters to denote points in $\mathbf{S^1}$ and for the corresponding point in the covering space $\mathbf{R}$ we will use the corresponding upper case letter. Also, F will denote the lifting of f. Since f has degree one for all b, we have that $F(X+1) = F(X) + 1$ for all $X \in \mathbf{R}$.

To define the intervals $A_k,\ B_k,\ g_k$ we need the following result.

Lemma 2.2 *([L]) If f satisfies (1.1)-(1.3), and if $\overline{\delta}$ is small enough, then*

$$1 + C_1 < F(X_2) - F(X_1) < 2 - C_2 \tag{2.5}$$

and the constant $0 < C_1 < 1$ is independent on b and δ.

From now one we assume that $\overline{\delta}$ is such that Lemma 2.2 holds. Then we have that for f only one of the following three cases can occur: (see Figure 2.1)

Case A. The set $I = f^{-1}(\Delta) \cap \Delta$ is an interval, such that $f(I) = \Delta$ and the endpoints of I map onto the endpoints of Δ.

Case B. The set I is a union of two disjoint intervals I_1 and I_2 so that the endpoints of each I_i map onto the endpoints of Δ.

Case g. $f(x_i) \in Int\Delta$, for $i = 1$ or 2 (i.e. the set I is a union of two disjoint intervals I_1 and I_2 so that the endpoints I_1 map onto the endpoints of Δ and $f(I_2)$ is strictly contained in Δ).

Call $A_k,\ B_k,\ g_k$ the maximal intervals of b for which the corresponding alternative holds.

Since the endpoints of $f(\Delta)$ move monotonically (clockwise) with respect to the endpoints of Δ (see (1.3)) the intervals alternate as stated in Theorem 1.1.

We shall study the bifurcations when b crosses a gap g_{2k-1} from A_k to B_k (e.g. $f(x_1(b)) \in Int\Delta$). In a similar way, we may study them when b crosses a gap g_{2k} from B_{k-1} to A_k. We describe these bifurcations in terms of symbolic dynamics. So we use the following definitions.

Figure 2.1:

Definition 2.3 Let $S = 1, 2, \ldots, m$ and $T = (t_{i,j})$ an $m \times m$ matrix such that $t_{i,j} \in \{0, 1\}$. We denote by $\sum_T = \sum(S, T)$ the set of infinite sequences $\mathbf{a} = (a_i)_{i=0}^{\infty}$ such that $a_i \in S$ and $t_{a_i a_{i+1}} = 1$ for all $i \in \mathbf{Z},\ i \geq 0$. We define the shift map $\sigma : \sum_T \longrightarrow \sum_T$ by $\sigma(\mathbf{a}) = (a_i)_{i=1}^{\infty}$. Then the set $\sum_T$ with the shift map σ is called a *subshift of finite type with transition matrix* T. If $t_{i,j} = 1$ for all i, j, then we call it *full shift on m symbols.* The set $\sum_T$ has a metric defined by $d(\mathbf{a}, \mathbf{b}) = \sum_{i=0}^{\infty} \gamma_i 2^{-i}$ where $\gamma_i = 0$ if $a_i = b_i$ and $\gamma_i = 1$ if $a_i \neq b_i$. Then $\sum_T$ is a Hausdorff compact space and σ is a homeomorphism.

Definition 2.4 Let $f \in C(\mathbf{S^1}, \mathbf{S^1})$ and let $\sum \subset \mathbf{S^1}$ be an invariant set (i.e. $f(\sum) \subset \sum$) we say that $f|_{\sum}$ is topologically conjugated to a subshift of finite type $\sigma|_{\sum_T}$ if there is a homeomorphism $h : \sum_T \longrightarrow \sum$ such that $f \circ h = h \circ \sigma$.

3 Proof of Main Theorem.

Let f be a continuous map of the circle into itself which satisfies (1.1)-(1.3). Assume that $b \in g_k$. We note that for $b \in A_k \cup g_k$, then f has exactly two fixed points one stable and the other unstable (see Case A, Case g and Figure 2.1). From now one we denote by $u(b)$ the unstable fixed point and by $s(b)$ the stable fixed point. By the definition of the intervals A_k and g_k we have that $s(b) \in \mathbf{S^1} \setminus \Delta$ and $u(b) \in Int\Delta$. Let $W = \{x \in \mathbf{S^1} : \lim_{n\to\infty} f^n(x) = s(b)\}$ (i.e. W is the basin of attraction of the stable fixed point).

Now, we will use the lifting F of the map f, and so we have to fix our notation. Without loss of generality we may assume that $0 \in e^{-1}(x_2(b))$. Then $\overline{\Delta}$ denotes the interval $e^{-1}(\Delta) \cap [0, 1]$. Also, $U(b)$ (resp. $X_1(b)$) denotes the only element of $e^{-1}(u(b)) \cap \overline{\Delta}$ (resp. $e^{-1}(x_1(b)) \cap \overline{\Delta}$). Lastly, we choose the lifting F such that $F(U(b)) = U(b) + 1$ (see Figure 3.1).

Also, we recall that if $b \in g_k$ then $f(x_1(b)) \in Int\Delta$. The following lemma is not difficult to prove (see Figure 3.2)

Lemma 3.1 *The following statements hold.*

(a) If $F(X_1(b)) > U(b)$ then, $W = \mathbf{S^1} \setminus \{u(b)\}$.

(b) If $F(X_1(b)) = U(b)$ then, $W = \mathbf{S^1} \setminus \{u(b), x_1(b)\}$.

Remark 3.2 We note that the situation described in the Lemma 3.1.(a) is similar to the case when $b \in A_k$ and persists in a small open interval contained in g_k.

Lemma 3.3 *If $F(X_1(b)) < U(b)$ then there are two points $V(b)$, $Q(b)$ such that $0 < Q(b) < X_1(b) < V(b) < U(b)$ and $F(Q(b)) = F(V(b)) = U(b)$.*

Proof. Observe that $F(X_1(b)) = \inf_{x \in [0,1]} F(X)$. Since $F|_{\overline{\Delta}}$ is strictly increasing, $u(b)$ is the only fixed point in Δ and $F(U(b)) = U(b) + 1$ we have $F(1) > 2$ (see Figure 3.1). Hence $F(0) > 1$. By using the intermediate value theorem we find two points $V(b) > X_1(b)$ and $Q(b) < X_1(b)$ such that $F(V(b) = F(Q(b)) = U(b)$. Also, $V(b) < U(b)$ because $F|_{\overline{\Delta}}$ is strictly increasing. ∎

Let $x, y \in \mathbf{S^1}$. We denote by $[x, y]$ (resp. (x, y)) the closed (resp. open) arc from x to y counterclockwise. We call it a closed (resp. open) interval of $\mathbf{S^1}$. Let $q(b) = e(Q(b))$, $v(b) = e(V(b))$ and $I = [q(b), u(b)]$ (see Figure 3.3). Clearly $F([V(b), U(b)] = [U(b), U(b) + 1]$. Then there is a unique point $R(b) \in (V(b), U(b))$ such that $F(R(b)) = Q(b) + 1$. Let $r(b) = e(R(b))$. So $f([r(b), u(b)]) = I$.

Observe that $\mathbf{S^1} \setminus I$ is contained in W and let A_0 denote the interval $(v(b), r(b))$. Then the following lemma follows from the fact that $f(A_0) = \mathbf{S^1} \setminus I$ and $f(I \setminus A_0) = I$ (see Figure 3.3).

Lemma 3.4 *Let $F(X_1(b)) < U(b)$. Then, $(\mathbf{S^1} \setminus I) \cup A_0$ is contained in W. Moreover, $W \cap I = \cup_{i=0}^{\infty} f^{-i}(A_0)$.*

We denote by W_I the open set $W \cap I$ (in I).

Proposition 3.5 *W is a open dense set in $\mathbf{S^1}$.*

Proof. From Lemma 3.4 we have that W_I is open. Then the proposition will follow by showing that W_I is dense in $I \setminus x_1(b)$. Suppose not. Then $D = (I \setminus x_1(b)) \setminus Cl(W)$ is a countable union of open intervals (in I). Number these intervals and let d_i be the length of the $i - th$ one. Then $\sum_{i=1}^{\infty} d_i \leq 1$ and each d_i is positive. So $\lim_{i \to \infty} d_i = 0$. Hence there is a i_0 with the property that $d_i \leq d_{i_0}$ for all i. By using that $f(x_1(b)) \in \Delta$ we have that if $x \in (q(b), v(b))$, then $f(x) \in (x_1(b), u(b))$. From (1.1) and (1.2) we obtain that $(f^2)'|_D > k > 1$. Now observe that $f^2(D) \subset D$ and that f^2 restricted to the $i_0 - th$ interval of D maps this interval to a larger interval because $(f^2)' > 1$. But such an interval can not be in D. This is the required contradiction. ∎

Let $\sum = \mathbf{S^1} \setminus W$. Clearly, $\sum = I \setminus W_I$.

Corollary 3.6 *The set $\sum$ is a closed totally disconnected invariant set.*

Next we use symbolic dynamics to describe the behavior of f in $\sum$. To do this we define $K_1(b) = \cup_{n=0}^{\infty} f^{-n}(x_1(b))$.

Theorem 3.7 *Let $b \in g_k$ such that $F(X_1(b)) < U(b)$. Then there is a sequence $R_1, \ldots, R_m$ with $m = m(x_1(b))$ of closed pairwise disjoint intervals in I such that*

(a) $\sum \subset (\cup_{i=1}^{m} R_i)) \cup \{x_1(b)\}$
(b) $f|_{\sum \setminus K_1(b)}$ is topologically conjugate to $\sigma|_{\sum_T}$, a subshift of finite type.

Figure 3.1:

Figure 3.2:

Figure 3.3:

Corollary 3.8 *Let $x_1(b) \in W_I$. Then there is a sequence $R_1, \ldots, R_m$ with $m = m(x_1(b))$ of closed pairwise disjoint intervals in I such that*

(a) $\sum \subset (\cup_{i=1}^m R_i)$

(b) $f|_{\sum}$ *is topologically conjugate to* $\sigma|_{\sum_T}$, *a subshift of finite type.*

Proof. It follows from the fact that $K_1(b) \subset W_I$ and, hence, $\sum \cap K_1(b) = \emptyset$. ■

Proof of Theorem 3.7. First of all we construct the sequence $R_1, \ldots, R_m$. Let V be the connected component of W_I such that $f(x_1(b)) \in Cl(V)$ and $f((q(b), v(b)) \cap V \neq \emptyset$ and let $V' = (y, z)$ be such that $f(y) = f(z) = \sup V$ (see Figure 3.4). Note that $x_1(b) \in V'$ and, if $x_1(b) \in W_I$, then V' is a connected component of W_I. Otherwise V' is $x_1(b)$ union two connected components of W_I. Clearly, there exists a non-negative integer l such that $f^l(V) = A_0$ (recall that $W_I = \cup_{i=0}^{\infty} f^{-i}(A_0)$. Observe that for all n such that $0 \leq n \leq l$, $f^n(V)$ is an open interval and the endpoints of $f^n(V)$ map onto the endpoints of $f^{n+1}(V)$. Moreover, $V' \cup V \cup f(V) \cup, \ldots, \cup f^l(V) \subset W_I \cup \{x_1(b)\}$.

The complement of $V' \cup V \cup f(V) \cup, \ldots, \cup f^l(V)$ in I is union of a finite sequence of closed pairwise disjoint intervals. Call them $R_1, \ldots, R_m$. Let $R = \cup_{i=1}^m R_i$. Clearly, $\sum \subset R \cup \{x_1(b)\}$ and statement (a) is proved.

The map f is monotonic on each closed interval R_i and we have that $f^{-1}(R) \subset R$. Moreover, for all i, j the set $R_i \cap f^{-1}(R_j)$ has at most one connected component. Define the $m \times m$ matrix $T = (t_{i,j})$ by $t_{i,j} = 1$ if $R_i \cap f^{-1}(R_j) \neq \emptyset$ and $t_{i,j} = 0$ if $R_i \cap f^{-1}(R_j) = \emptyset$. Let $(\sum_T, \sigma)$ be the subshift of finite type with transition matrix T (see Definition 2.3). Now, statement (b) follows in the standard way. ■

Figure 3.4:

Proof of Theorem 1.2. From Case g it follows immediately statement (a) (see also Figure 1.1). Let $\alpha_k = \inf\{b \in A_k \cup g_k : F(X_1(b) = U(b)\}$. Since $F(X_1(b)) > U(b)$ for $b \in A_k$ and $u(b) \in Int\Delta$ we have $\alpha_k \in g_k$. From Lemma 3.1 it follows statement (b). Let $\beta_k = \sup\{b \in A_k \cup g_k : F(X_1(b)) = U(b)\}$. Clearly, $\alpha_k \leq \beta_k$. From Theorem 3.7 and Corollary 3.8 it follows statement (c). If $\alpha_k \neq \beta_k$, set $L_k = \{b \in (\alpha_k, \beta_k] : F(X_1(b)) \geq U(b)\}$. Since $F(X_1(b)) - U(b)$ depends continuously on b we have that L_k is closed in $(\alpha_k, \beta_k]$. From statements (b) and (c) it follows (d). ■

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