Essays on Asset Allocation

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Introduction

The aim of this work is to study the portfolio decisions of an investor who face a risky market, a topic that is a current subject of research in finance. In this regard, Modern Portfolio Theory provides us with the solution to the short term portfolio problem. The resulting strategy is suitable for one period horizon investments, reason by which it is named *myopic*. Similar results can be obtained if we extend the model to the more realistic situation of an investor in a multi period environment who is able to rebalancing the portfolio composition. This is true, for instance, in an economy where the investment opportunities does not change over time, and whose individual's risk preferences are represented by a power utility function. In particular it holds when some statistical characteristics of the rates of return such as the mean and the variance are identical and independently distributed. Despite the mathematical tractability of this approach, the real financial market does not behaves in such a way. A judicious study of financial time series suggest the presence of common properties, the so called *stylized facts*, which are irrespective of markets or instruments. As examples of these we find fluctuation over time of rates of return, heavy tails, volatility clustering, and leverage effects, among others. These changing market conditions are reflected in the portfolio decisions and accounted for by a new term in the portfolio allocation solution, the *inter temporal hedging demand*. In that context, we intend to study the financial market in order to determine the consequences of those factors on portfolio decisions. We do this by varying the underlying mathematical specification of the model in such a way that the aforementioned characteristics are modeled explicitly.

The first chapter is motivated by the recent turmoil in the financial markets, where we have observed large losses across markets. This phenomena, regarded as *systemic risk*, is the risk of occurrence of rare, large, and highly correlated jumps whose mechanism relies on the interactions between markets. To deal with this, we define a continuos multivariate model that includes a common jump component to all assets. The jump accounts for instantaneous and simultaneous occurrence of unexpected events in the market. The asset specific jump amplitude are defined as an independent random variable, reflecting the individual response to the severity of the shock. To gauge the impact of systemic risk to the portfolio, we match the first two moments of the jump-diffusion model to that of the pure diffusion model and compare the solutions.

As market data we use the monthly series of a low book-to-market (growth stocks) portfolio, the high book-to-market (value stocks) portfolio, and an intermediate portfolio of Fama and French. We find that the effects of systemic jumps may be potentially substantial as long as market equity returns experiment very large average (negative) sizes. However, it does not seem to be relevant that stock markets experience very frequent jumps if they are not large enough to impact the most levered portfolios. All potentially relevant effects are concentrated in portfolios financed with a considerable amount of leverage. In fact, for conservative investors with low leverage positions the potential effects of systemic jumps on the optimal allocation of resources is not substantial even under large average size jumps. Finally, the value premium is particularly high when the average size of the jumps of value stocks is positive, large and relatively infrequent, while the average size of growth stocks is also very large but negative. It seems therefore plausible to conclude that the magnitude of the value premium is closely related to the characteristics of the jumps experienced by value and growth stocks.

In chapter two, we follow the same line of including additional features of financial series to the model, and consider a time varying volatility. We also expect to enhance the the jump model's ability to capture skewness. Thus, besides to the jump term, we include a stochastic volatility term to the asset price dynamics. As in chapter one, this specification results in an incomplete market model. And, although dynamic programing method is still available to address the problem, we sought to solve the model through the use of duality methods which leads to deeper insights to the portfolio solution.

From the theoretical point of view, we obtain the expression for the portfolio allocation rule along with the market price of risk, the market price of volatility risk and the market price of jump risk. We also find the explicit expressions for the the mean, standard deviation, skewness and excess kurtosis in terms of the market model parameters. The portfolio rule reduces to the standard myopic rule when the correlation between the asset prices and volatility is zero, and the frequency of extreme event vanishes. The market price of risk is found to be composed by the Brownian market price of risk plus a jump contribution. The market price of volatility risk is approximately proportional to the market price of risk for a very low risk adverse investor and vanishes if the innovations in the returns are perfectly correlated with the instantaneous volatility.

For the empirical analysis, besides the growth and value series, we also include the Standard and Poor's Composite index for comparison purposes. The numerical results show that series' mean is dominated by the diffusion mean compensated by the long run variance. Volatility depends on model's volatility parameters, i.e., jump volatility, long term volatility, volatility of stochastic volatility, and mean reversion. Finally, skewness and kurtosis strongly depends on jump mean, and on jump volatility respectively, and also on Poisson intensity. A closer look at Growth and Value series reveals that the largest variance of Growth series is mainly due to the low frequency of unexpected large events in Value series. Additionally, the long term volatility of the Growth series is larger. It is worth to mention that shocks with small jump mean's absolute value happen more frequently than larger shocks, in agreement with the results in the first chapter.

The portfolio weights are found to be low compared with those of a standard diffusion model. This is because the investor perceives more risk coming from jumps and stochastic volatility. We calculate a myopic demand as the ratio of the risk premium over the instantaneous volatility times the risk aversion. It proves to be greater than the total portfolio weight. The discrepancy could be explained if we define an intertemporal hedging demand as the difference between the total portfolio weight and the myopic contribution. Given that the expectations of asset performance worsen, the intertemporal hedging demand is negative thus reducing the participation of the myopic component. This reduction diminishes as γ increases for Growth and Value series, and is constant for the S&P500.

In the third chapter we deal with time varying excess returns, we aim to set the basis of study for the time varying relative risk aversion as a useful tool for portfolio allocation. To that end, we estimate the consumption based, external habit model of Campbell and Cochrane. This model accounts for time varying and countercyclical expected returns, as well as the high equity premium with a low and steady riskfree rate. This model has the feature of deliver a counter cyclical varying risk aversion, and allows predictability of asset returns. Afterwards, we test two specifications of pricing models that includes surplus and risk aversion, under contemporaneous and ultimate consumption risk.

As market data we examine the 25 portfolios formed on size and book to market by Fama and French. Numerical results shows that curvature exhibits low values, in spite of the high relative risk aversion. Its magnitude lessens as the lag of the ultimate consumptions series increases. We render the surplus ratio, the stochastic discount factor, and the time varying relative risk aversion. The linear relationship between the price over dividend and the surplus ratio in the Campbell and Cochrane model is most closely followed under ultimate consumption series for twelve months lags. The stochastic discount factor exhibits a business cycle, attaining maxima at the start of recessions and dropping at the end of them. Finally, the risk aversion proved to be time varying and countercyclical.

We estimate two factor pricing models derived from the habit model specification. The data analysis suggest that ultimate consumption risk specification with time varying aversion seems to explain relatively well the cross section of average returns. Additionally, we conclude that the excess returns seems to be more sensitive to risk aversion than to consumption growth. However, the estimated intercept for the models is statistically significant indicating an overall rejection of them.

THE EFFECTS OF SYSTEMIC RISK ON THE ALLOCATION BETWEEN VALUE AND GROWTH PORTFOLIOS

Abstract. Given the striking effects of the recent financial turmoil, and the importance of value and growth portfolios for both local and international portfolio allocation, we investigate the effects of systemic jumps on the optimal portfolio investment strategies across value and growth equity portfolios. We find that the cost of ignoring systemic jumps is not substantial, unless the portfolio is highly levered and the average size amplitude of the jump is large enough. From the optimal asset allocation point of view, it seems more important the effects of few but relatively large jumps than highly frequent but small jumps. Indeed, the period in which the value premium is higher coincides with a period of few, but large, and a positive average size jump for value stocks, and a negative and very large average size jump for growth stocks.

1. Introduction

The value premium is one of the most relevant anomalies discussed in the asset pricing literature. Value stocks, which are characterized by high book-to-market ratios, earn higher average returns than growth stocks. In principle, growth options strongly depend upon future economic conditions which suggest that growth stocks should have higher betas than value stocks. Using monthly data from January 1963 to December 2010, the market beta of the Fama-French growth portfolio is 1.067 while the market beta of the value portfolio is 1.068. Although contrary to the theoretical prediction these betas are basically the same, it turns out that the annualized average return of the growth portfolio is 9.6% while the annualized average return of the value portfolio is 16.5%. This represents a value premium of 6.9% which is even higher that the well known market equity premium magnitude of 5.5% for the same sample period.[1]A key issue of the research agenda of the asset pricing literature is to

[1]These numbers correspond to the 10 Fama-French monthly portfolios sorted by book-to-market, and using a value-weighted scheme to obtain the portfolio returns. The value premium using an equally weighted approach is an even much higher 14.0% on annual basis.

understand why value stocks earn higher average returns than growth stocks. This paper does not pretend to answer this question.[2]On the contrary, our paper takes this anomaly as given to investigate optimal asset allocation decisions between value and growth portfolios. In particular, the paper investigates the effects of systemic jumps on the asset allocation these two key characteristics of equity returns.

Recent financial crisis has shown that the failures of large institutions can generate large costs on the overall financial system. Systemic risk is one of the main issues to be resolved by the new regulation of financial markets over the world. It seems widely accepted that previous regulation focuses excessively on individual institutions ignoring critical interactions between institutions.[3] These interactions are the leading source of systemic risk around the world. From this point of view, it is important to analyze the impact of systemic risk on portfolio asset allocation among potential institutional investors. Given that two of the most popular investment strategies of large institutional investment employ value and growth assets, this paper analyzes how relevant is to ignore systemic risk on the ex-post performance of these strategies. We understand systemic risk as the risk arising from infrequent but arbitrarily large jumps that are highly correlated across the world and across a large number of assets.

We borrow from the mathematical jump-diffusion model developed by Das and Uppal (2004) to recognize that jumps occurs at the same time all over the world and across all portfolios, but allowing that the size of the jump may be different across them. These authors derive the optimal portfolio weights when equity returns follow a systemic jump-diffusion process. We calibrate the model to three monthly

[2]Zhang (2005) argues that costly reversibility and countercyclical market price of risk cause assets in place to be harder to reduce so they, in fact, become riskier than growth options. A related argument is given by Yogo (2006) who employs non-separability between durable and non-durable consumption to show that value stocks are more pro-cyclical than growth stocks. This implies that they perform especially badly during economic downturns. However, recent evidence provided by Belo et al. (2010) shows that the q-theory dynamic investment framework fails to explain the value spread. This is the case despite the fact that Chen and Zhang (2009) argue that a simple three-factor model inspired on the q-theory of investment is able to explain anomalies associated with short-term prior returns, financial distress, net stock issues, asset growth, earnings surprises, and some valuation ratios.

[3]See the Squam Lake Report prepared by French and others (2010) for a clarifying discussion of these issues.

Fama-French book-to-market portfolios where portfolio one is composed of securities with low bookto-market (growth stocks), portfolio 5 contains intermediate book-to-market assets, and portfolio 10 includes securities with high book-to-market (value stocks). For robustness, we also employ U.S. dollar monthly value and growth portfolios constructed by MSCI for three important geographic regions around the world, Europe, North America, and the Pacific.

It is well known that to properly describe equity-index returns one must allow for discrete jumps. As shown, among others, by Andersen, Benzoni, and Lund (2002) jumps play an important role for understanding of U.S. market returns over and above stochastic volatility and the negative relationship between return and volatility shocks. González, Novales, and Rubio (2011) also show that the relevance of jumps characterizes the French (CAC), German (DAX), and Spanish (IBEX-35) equity-index European returns. One may therefore expect jumps to impact optimal asset allocation between value and growth portfolios. Our evidence shows that this is not necessarily the case. We find that the effects of systemic jumps are indeed not negligible from 1982 to 1997 for low levels of risk aversion (highly levered portfolios) when the frequency of jumps is small but its average size is large. In fact, this period is characterized by an especially large value premium. These effects may therefore seem to be particularly relevant for the allocation of funds between growth and value portfolios. When systemic jumps are recognized, the average investor should optimally go long in value and short in growth in higher proportions than when assuming a pure diffusion process. However, and rather surprisingly, an average investor would have not been penalized ignoring systemic jumps from 1997 to 2010 when the frequency of jumps is much higher but the average magnitude is also smaller. The magnitude of the average jumps size seems to be very important to assess the impact of systemic jumps on asset allocation between value and growth portfolios. In the overall period and in both sub-periods, independently of using a pure diffusion or a jump-diffusion process, value stocks dominate growth stocks especially for highly levered portfolios.

This paper is structured as follows. In Section 2, we discuss a model of equity returns that allows for systemic risk. Moreover, we also describe optimal portfolio weights when equity returns have a systemic risk component. Section 3 discusses the estimation procedure, and Section 4 presents the key results of the paper using the Fama-French book-to-market portfolios. Section 5 provides a robustness analysis using value and growth portfolios across three geographic regions in the world. Concluding remarks are in Section 6.

2. Asset Returns, Systemic Risk, and the Optimal Allocation of Equity-Returns

This section first present a model of asset equity returns which is based on the asset pricing model proposed by Das and Uppal (2004). This model introduces systemic risk by imposing jumps that occur simultaneously across all assets but also allowing for a varying distribution of the jump size across all portfolios. Secondly, we discuss optimal portfolio allocation given that the underlying assets follow a jump-diffusion process with systemic jumps. We compare these results relative to the case in which asset returns follow a pure diffusion process.

2.1. **Asset Equity Returns and Systemic Risk.**

There is an instantaneous riskless asset which follows the return process given by,

$$\frac{dP}{P} = rdt, \tag{2.1}$$

where r is the constant continuous riskless rate of return. Moreover, there are N risky assets in the economy. Each of them follows a pure-diffusion process given by the well known expression,

$$\frac{dS_j}{S_j} = \mu_j^d dt + \sigma_j^d dZ_j; \qquad j = 1, 2, \dots N \tag{2.2}$$

where S_j is the price of asset j, Z_j is a Brownian motion, μ_j^d is the drift, and σ_j^d is the volatility, and superscript d highlights the pure diffusion character of the process. We denote by Σ^d the $N \times N$ covariance matrix of the diffusion components where the typical component of this matrix is $\sigma^d_{ij} = \rho^d_{ij}\sigma^d_i\sigma^d_j$ with ρ^d_{ij} being the correlation between the Brownian shocks dZ_j and Z_i. In matrix notation, μ^d is the $N-$vector of expected returns, $\Sigma^d = \sigma^d\rho^d(\sigma^d)'$, where σ^d is the diagonal matrix of volatilities, and ρ^d is the symmetric matrix of correlations.

We next allow for unexpected rare systemic events by introducing a jump to the process given by (2.2). Following Das and Uppal (2004), we assume that the jump arrives at the same time across all equity portfolios, and that all portfolios jump in the same direction. Then,

$$\frac{dS_j}{S_j} = \mu_j dt + \sigma_j dZ_j + (\widetilde{J}_j - 1)dQ(\lambda); \qquad j = 1, 2, \ldots N \tag{2.3}$$

where Q is a Poisson process with common constant intensity λ and $(\widetilde{J}_j - 1)$ is the random jump magnitude that generates the percentage change in the price of asset j if the Poisson event is observed. It is important to note that the arrival of jumps occur at the same time for all equity portfolios. We assume that the Brownian shock, the Poisson jump, and the jump amplitude $\widetilde{J}_j$ are independent, and that $J_j \equiv \ln(\widetilde{J}_j)$ has a Normal distribution Ψ with constant mean η_j and variance ν^2_j. Therefore, the distribution of the jump size is allowed to be different for each portfolio, although all jumps arrive at the same time.

We define μ and $\Sigma = \sigma\rho\sigma'$ as the drift $N-$vector and the $N \times N$ covariance matrix of the diffusion components of (2.3) respectively. It must be noted that they are now the drift and the covariance matrix of the diffusion components when there are jumps in the return process. In this case, we also have an additional drift, μ^J, and an additional covariance matrix, Σ^J, from the jump components of the process. Given that we select the parameters of the jump-diffusion process in (2.3) such that the first two moments for this process match exactly the first two moments of the pure diffusion process in (2.2), it must be the case that,

$$\begin{aligned} \mu^d &= \mu + \mu^J \\ \Sigma^d &= \Sigma + \Sigma^J \end{aligned} \tag{2.4}$$

2.2. **Portfolio Allocation.** Our representative investor maximizes the expected power utility defined on terminal wealth, W_T, given by the well known expression $U(W_T) = W_T^{1-\gamma}/(1-\gamma)$, where $\gamma > 0$ is the constant relative risk aversion coefficient. We first briefly describe the optimal portfolio weights using the pure-diffusion process given by (2.2) Denoting the vector of the proportions of wealth invested in risky equity portfolios by ω, the optimal portfolio problem at t is

$$V(W_t, t) = \max_{\{\omega\}} E\left(\frac{W_T^{1-\gamma}}{1-\gamma}\right), \tag{2.5}$$

subject to the dynamic budget constraint

$$\frac{dW_t}{W_t} = (\omega' R^d + r)dt + \omega' \sigma^d dZ_t, \tag{2.6}$$

Where $R^d = \mu^d - 1_N r$ is the $N-$vector of excess returns, σ^d is the diagonal matrix of volatilities, dZ_t is the vector of diffusion shocks, and 1_N is an $N-$vector of ones. The solution is the vector of portfolio weights obtained by Merton (1969) corresponding to the standard diffusion process (2.2),

$$\omega_{op}^d = \frac{1}{\gamma}(\Sigma^d)^{-1} R^d \tag{2.7}$$

When the process includes a jump component as in expression (2.3), the dynamics of wealth for the initial wealth $W_0 = 1$ can be written as

$$\frac{dW_t}{W_t} = (\omega' R + r)dt + \omega' \sigma dZ_t + \omega' \widetilde{J}_t dQ(\lambda), \tag{2.8}$$

where $R = \mu - 1_N r$ is the $N-$vector of excess returns , σ is the diagonal matrix of volatilities, dZ_t is the vector of diffusion shocks under a jump-diffusion process, and $\widetilde{J}_t = (\widetilde{J}_1 - 1, \widetilde{J}_2 - 1, \dots, \widetilde{J}_N - 1)'$ is the vector of random jump amplitudes for the N equity portfolios.

One can employ stochastic dynamic programming to solve for the optimal weights,[4]

$$0 = \max_{\{\omega\}} \left\{ \frac{1}{dt} E[dV(W_t, t)] \right\},$$

where in this equation, we can use the generalized jump-diffusion Itos lemma to calculate the differential of the value function, dV.[5] Then, the Hamilton-Jacobi-Bellman equation is

$$(2.9) \quad 0 = \max_{\{\omega\}} \left\{ \frac{\partial V(W_t, t)}{\partial t} + (\omega' R + r) W_t \frac{\partial V(W_t, t)}{\partial W_t} \right.$$
$$\left. \frac{1}{2} \omega' \Sigma \omega \, W_t^2 \frac{\partial^2 V(W_t, t)}{\partial W_t^2} + \lambda E[V(W_t + W_t \omega' \widetilde{J}_t, t) - V(W_t, t)] \right\}$$

The impact of the jumps in the return process is given by the last term of equation (2.9). This term employs the fact that $EdQ_t = \lambda dt$, and the assumption of independence of the Poisson jump and the jump amplitude except for the fact that the jump size is conditional on the Poisson event happening. As usual in this type of problems, one can guess the solution to the value function as having the following form:

$$(2.10) \qquad V(W_t, t) = F(t) \frac{W_t^{1-\gamma}}{1-\gamma}$$

Replacing this solution into the Hamilton-Jacobi-Bellman equation we get:

$$(2.11) \quad 0 = \max_{\{\omega\}} \left\{ \frac{1}{F(t)} \frac{dF(t)}{dt} + (1-\gamma)(\omega' R + r) \right.$$
$$\left. -\gamma(1-\gamma) \frac{1}{2} \omega' \Sigma \omega + \lambda E[(1 + \omega' \widetilde{J}_t)^{1-\gamma} - 1] \right\}$$

By differentiating with respect to ω we obtain the optimal weights with systemic jumps as the solution for each time t to the system of N nonlinear equations which

[4] see the Appendix for an alternative procedure to solve the portfolio problem by calculating directly the conditional expectation.

[5] Alternatively, we may employ the infinitesimal generator of the jump-diffusion process of Duffie et al. (2000).

must be solved numerically:

$$0 = R - \gamma\Sigma\omega_{op} + \lambda E[\widetilde{J}_t(1 + \omega'_{op}\widetilde{J}_t)^{-\gamma}] \tag{2.12}$$

It should be pointed out that if we replace equation (2.10) into the Hamilton-Jacobi-Bellman equation and evaluate at ω_{op} we obtain:

$$\frac{1}{F(t)}\frac{dF(t}{dt} = -k,$$

where,

$$k = (1-\gamma)(\omega'_{op}R + r) - \gamma(1-\gamma)\frac{1}{2}\omega'_{op}\Sigma\omega_{op} + \lambda E[(1 + \omega'_{op}\widetilde{J}_t)^{1-\gamma} - 1]\}$$

We next employ the boundary condition $F(T) = 1$ to obtain,

$$F(\omega, t) = e^{k(T-t)} = e^{k\tau}$$

Therefore, the value function is given by the expression

$$V(W_t, t) = e^{k\tau}\frac{W_t^{1-\gamma}}{1-\gamma} \tag{2.13}$$

2.3. **Certainty Equivalent Cost.** As an additional analysis of the effects of ignoring systemic risk on asset allocation, we can also calculate the certainty equivalent cost (CEQ hereafter) of following an allocation strategy that ignores the simultaneous jumps occurred in the data. The CEQ gives the additional amount in U.S dollars that must be added to match the expected utility of terminal wealth under the pure-diffusion suboptimal allocation to that under the optimal strategy with the jump-diffusion process of equity returns. In other words, as in Das and Uppal (2004), we calculate CEQ as the marginal amount of money that equalizes pure-diffusion expected utility with the jump-diffusion expected utility. The compensating CEQ wealth is therefore computed by equating the following expressions:

$$V((1 + CEQ)W_t, t; \omega^d_{op}) = V(W_t, t; \omega_{op}) \tag{2.14}$$

Then, using expression (2.10) we have that

$$CEQ = \left(\frac{F(t, \omega_{op})}{F(t, \omega^d_{op})} \right)^{1/(1-\gamma)} - 1 \tag{2.15}$$

3. The Estimation Procedure to Analyze the Effects of Systemic Risk

Let us start again with the pure-diffusion case given by equation (2.2). The parameters to be estimated are $\{\mu^d, \Sigma^d\}$ where μ^d and Σ^d are the $N-$vector of expected returns and the $N \times N$ covariance matrix of the diffusion components. The moment conditions for individual equity portfolios are given by

$$\begin{aligned} E_t\left(\frac{dS_j}{S_j}\right) &= \mu^d_j dt \\ E_t\left(\frac{dS_i}{S_i} \times \frac{dS_j}{S_j}\right) &= \sigma^d_{ij} dt = \rho^d_{ij}\sigma^d_i \sigma^d_j dt \end{aligned} \tag{3.1}$$

This implies that moment conditions $\{\mu^d, \Sigma^d\}$ can be estimated directly from the means and the covariance of the actual sample series available. On the other hand, to derive the four unconditional moments of the jump-diffusion process given by (2.3), we follow Das and Uppal (2004) to identify the characteristic function which can be differentiated to obtain the moments of the equity portfolio returns process. We next present a detailed exposition of the procedure employed to obtain the moments under the systemic-jump-diffusion process.

3.1. **The Characteristic Function.** We first write the previous process in log returns with $X_j = \ln(S_j)$. It is well known that the pure-diffusion process in (2.2) becomes

$$dX_j = \alpha^d_j dt + \sigma^d_j dZ_j, \tag{3.2}$$

where $\alpha^d_j = \mu^d_j - \frac{1}{2}(\sigma^d_j)^2$. As before, the model in matrix notation is $dX = \alpha^d dt + \sigma^d dZ$, where α^d is the $N-$vector of expected returns, and σ^d is the diagonal matrix of volatilities. On the other hand, the jump-diffusion return process can be written as

$$dX_j = \alpha_j dt + \sigma_j dZ_j + J_j dQ, \tag{3.3}$$

where $\alpha_j = \mu_j - \frac{1}{2}\sigma_j^2$. Hence, in matrix notation, the continuously compounded asset return vector for the jump-diffusion model satisfies the following stochastic differential equation $dX = \alpha dt + \sigma dZ + JdQ$ where α is the $N-$vector of expected returns, σ is the diagonal matrix of volatilities for the jump-diffusion case, and J is the $N-$vector of jump amplitudes. The $N-$vector of the average size of the jumps amplitude is denoted by η, while the diagonal variance matrix of the size of the jumps is denoted by ν.

As mentioned before, the theoretical moments for the jump-diffusion process are calculated using the characteristic function which in turn can be derived from the Kolmogorov theorem. We formally work in a probability space $(\Omega, \mathfrak{F}, P)$ where, in the jump-diffusion case, both the process Q and the Brownian motion Z generate the filtration $\mathfrak{F}_t$. The conditional characteristic function of the process X conditioned on $\mathfrak{F}_t$ is defined as the expected value of $e^{i\zeta' X_T}$, where $\zeta = (\zeta_1, \ldots, \zeta_N)'$ is the argument of the characteristic function, Φ_t, given by

$$(\zeta, X_T, T, t) \to \Phi(\tau, \zeta, X_t) = E(e^{i\zeta' X_T} \big| \mathfrak{F}_t), \tag{3.4}$$

where $\tau = T - t$.

From the process X, we know that α, σ, and λ are constant on X. Therefore, under some technical regularity conditions discussed in Duffie, Pan, and Singleton (2000), the conditional characteristic function has an exponential affine form given by

$$\Phi_t(\tau, \zeta, X_t) = e^{A_t(\tau,\zeta) + B_t'(\tau,\zeta) X_t} \tag{3.5}$$

According to the Feynman-Kac theorem, the conditional characteristic function is the solution to[6]

$$\ell\Phi_t = -\frac{\partial \Phi_t}{\partial t}, \tag{3.6}$$

where ℓ is the infinitesimal generator for the jump-diffusion process X given by

$$\ell\Phi_t = \left(\frac{\partial \Phi_t}{\partial X}\right)' \alpha + \frac{1}{2}\text{Trace}\left(\sigma\rho\sigma'\frac{\partial^2 \Phi_t}{\partial X^2}\right) + \lambda \int_{-\infty}^{\infty} \left[\Phi_t(X+Y) - \Phi_t(X)\right] d\Psi(Y), \tag{3.7}$$

where, as already pointed out, Ψ is the jump amplitude Normal distribution. The boundary condition for the differential equation is the value of the conditional characteristic function at the equity portfolio horizon $\Phi_T(0,\zeta,X_T) = e^{i\zeta' X_T}$. Thus, from (3.5), it must be the case that $A_T(0,\zeta) = 0$ and $B_T(0,\zeta) = i\zeta$. In order to find out the expressions for the functions $A_t(\tau,\zeta)$ and $B_t(\tau,\zeta)$, equation (3.5) is replaced into equation (3.7) to obtain

$$\left[\frac{\partial A_t}{\partial \tau} + \left(\frac{\partial B_t}{\partial \tau}\right)' X_t\right] = B_t'\alpha + \frac{1}{2}B_t'\Sigma B_t + \lambda \int_{-\infty}^{\infty} \left[e^{B_t'(\tau,\zeta)Y} - 1\right] d\Psi(Y), \tag{3.8}$$

where $\Sigma = \sigma\rho\sigma'$.

Thus, the two ordinary differential equations are

$$\begin{aligned} \frac{\partial A_t}{\partial \tau} &= B_t'\alpha + \frac{1}{2}B_t'\Sigma B_t + \lambda \int_{-\infty}^{\infty} \left[e^{B_t'(\tau,\zeta)Y} - 1\right] d\Psi(Y) \\ \frac{\partial B_t}{\partial \tau} &= 0 \end{aligned} \tag{3.9}$$

The second equation in (3.9) implies that B does not depend on τ. Thus, using the boundary condition, we obtain $B_t(\tau,\zeta) = i\zeta$. We now replace B in the first equation, we integrate, and we use the remaining boundary condition to get

$$A_t(\tau,\zeta) = \left[-\frac{1}{2}\zeta'\Sigma B_t + i\zeta'\alpha + \lambda \int_{\mathbb{R}^N} \left[e^{i\zeta' Y} - 1\right] d\Psi(Y)\right]\tau \tag{3.10}$$

The integral in the right hand side of (3.10) can be recognized as the jump amplitude's characteristic function, given that the random variable J follows a Normal

[6]see the Appendix for an alternative procedure to calculate the conditional characteristic function directly from the conditional expectation.

distribution:

$$\int_{\mathbb{R}^N} e^{i\zeta' Y} d\Psi(Y) = \exp\left(i\zeta'\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right)$$

Therefore,

$$A_t(\tau,\zeta) = \left[-\frac{1}{2}\zeta'\Sigma B_t + i\zeta'\alpha + \lambda \exp\left(i\zeta'\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) - \lambda\right]\tau, \tag{3.11}$$

and, finally, the characteristic function is given by

$$\Phi_t(\tau,\zeta,X_t) = \exp\left(\left[-\frac{1}{2}\zeta'\Sigma B_t + i\zeta'\alpha + \lambda \exp\left(i\zeta'\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) - \lambda\right]\tau + i\zeta' X_t\right) \tag{3.12}$$

3.2. **Unconditional Moments.** Using the characteristic function one can derive the K co-moments throughout the following expression:

$$E[X_1^{k_1}, X_2^{k_2}, \ldots, X_N^{k_N} \big| \mathfrak{F}_t] = i^{-K} \left.\frac{\partial^K \Phi_t(\tau,\zeta,X_t)}{\partial\zeta_1^{k_1}, \partial\zeta_2^{k_2}, \cdots, \partial\zeta_N^{k_N}}\right|_{\zeta=0}, \tag{3.13}$$

where $K = \sum_{j=1}^{N} k_j$. The gradient and the Hessian, $\partial\Phi/\partial\zeta$ and $\partial^2\Phi/\partial\zeta\zeta'$ are used to find the first and second moments respectively. For the one-period investment horizon, $\tau = 1$, and using the conversion from the non-central to central moments we obtain the mean, covariance, co-skewness, and excess kurtosis:[7]

$$\mu_1 = \alpha + \lambda\eta = \left(\mu - \frac{1}{2}\sigma^2 I_N\right) + \lambda\eta \tag{3.14}$$

$$\mu_2 = \Sigma + \lambda(\eta\eta' + \nu I \nu') \tag{3.15}$$

$$\mu_3 = \lambda\sigma^{-1} I \sigma^{-2} \circ \left[2(\nu\eta)(\nu I_N)' + (\eta^{\circ 2} + \nu^2 I_N)\eta'\right] \tag{3.16}$$

$$\mu_4 = \lambda\sigma^{-4}\left(3\nu^4 I_N + 6\nu^2\eta^{\circ 2} + \eta^{\circ 4}\right) \tag{3.17}$$

[7]See the Appendix at the end of the paper for alternative procedures to obtain the unconditional moments. These derivations extend and complete the mathematical technicalities suggested by Das and Uppal (2004).

In the expressions above, I is the $N \times N$ matrix of ones, and $\circ$ denotes the $N-$times element-by-element multiplication. It should be noted that μ_1 and μ_4 are $N-$vectors, and μ_2 and μ_3 are $N \times N$ matrices.

If we now compare the mean and covariance for the jump-diffusion process above with those for the pure-diffusion processes for $\lambda = 0$, we observe that

$$\mu^J = \lambda\eta \tag{3.18}$$

$$\Sigma^J = \lambda(\eta\eta' + \nu I \nu') \tag{3.19}$$

Then, the diffusion moments of the jump-diffusion process are retrieved using the expressions in (2.4) as:

$$\mu = \mu^d - \lambda\eta \tag{3.20}$$

$$\Sigma = \Sigma^d - \lambda(\eta\eta' + \nu I \nu') \tag{3.21}$$

From the moment conditions in equations (3.14) to (3.17) the parameters to be estimated are $\{\alpha, \Sigma, \eta, \nu, \lambda\}$. For the universe of N assets there are N jump amplitude means and N jump amplitude volatilities. This represents $2N + 1$ parameters to be estimated including the Poisson intensity λ. On the other hand, there are N^2 coskewness moments and N excess kurtosis moments for a total of $N(N+1)$ moment conditions to be employed in the generalized method of moment (GMM) estimation procedure.

3.3. **Sampling Moments.** The sampling mean and co-moments of a variable X with respect to variable Y are:

$$m^X = \frac{1}{T}\sum_{t=1}^{T} X_t$$

$$m_2^{XY} = \frac{1}{T-1}\sum_{t=1}^{T}(X_t - m^X)(Y_t - m^Y)$$

$$m_r^{XY} = \theta_r \frac{1}{T-1} \sum_{t=1}^{T} \varepsilon_t^X (\varepsilon_t^Y)^{r-1},$$

for $r \geq 3$ and $\varepsilon_t^X = \frac{X_t - m^X}{\sqrt{m_2^{XY}}}$, where θ_r is the adjustment term for the unbiasedness correction.

4. Optimal Allocation for Value and Growth Portfolios with Systemic Risk: Empirical Results

We first estimate the model using 3 portfolios from the 10 monthly book-to-market sorted portfolios taken from Kenneth French's web page. Portfolio 1 contains the companies with low book-to-market, while portfolio 10 includes assets with high book- to-market. We refer to portfolio 1 as the growth portfolio, and portfolio 10 as the value portfolio. We also employ portfolio 5 denoted as the intermediate portfolio. In order to pay special attention to these characteristics we use the equally-weighted scheme of the individual stocks rather than the more popular value-weighted portfolios. This weighting approach also amplifies the value effect anomaly.

4.1. **Parameter Estimates for the Jump-Diffusion Return Process.** Table 1 reports the descriptive statistics of the three monthly Fama-French book-to- market portfolios for the full sample period from January 1982 to October 2010, and two sub-periods from January 1982 to February 1997, and from March 1997 to October 2010. Both sub-periods contain episodes with large negative shocks. The first sub-period includes the market crash of October 1987, the Gulf War I in August 1990, and the Mexican crisis in December 1994. On other hand, the second sub-period contains the Asian crisis of July 1997, the Russian crisis of August 1998, the bursting of the dot.com bubble, the terrorist attack of September 2001, the outbreak of the Gulf War II in March 2003, the beginning of the sub-prime crisis, and the Lehmann Brothers default in September 2008.

Panel A of Table 1 shows that the use of the equally-weighted growth and value portfolios leads to an impressive value premium of 15.8% on annual basis for the full sample period. Moreover, the value premium is 13.0% and 18.4% for the second and first sub-periods respectively. As expected, the annualized volatility of the growth portfolio is higher than the corresponding volatility of the value portfolio is all three sample periods. On annual basis, the growth volatility premium is 4.7% for the full period, and 5.6% and 3.9% for the second and first sub-periods respectively. Indeed, the growth portfolio seems to be riskier than the value portfolio. The problem is, of course, the enormous average return of the value portfolio.

The third and fourth moments of both portfolios also present an interesting behaviour. For the full period, the growth portfolio has positive skewness, while the value portfolio has a slightly negative skewness. Moreover, growth stocks have lower kurtosis than the value portfolio. The overall behaviour is the consequence of two very different sub-periods. From 1997 to 2010, the value portfolio has negative skewness, and, on the contrary, growth stocks have positive skewness. The excess kurtosis is similar for both portfolios. However, from 1982 to 1997, we report a very high negative skewness for the growth portfolio, and a positive skewness for value stocks. Similarly, the behaviour of excess kurtosis is also surprising. Value stocks present a much higher kurtosis than the growth portfolio. The changes observed in both skewness and kurtosis from one sub-period to the other for growth and value portfolios are striking and deserve further attention. For completeness, Figure 1 shows the density functions for both sub-periods of the growth, intermediate and value portfolios, and the QQ plots to assess the deviations of their returns from the Normal distribution.

Panel B of Table 1 reports the correlation coefficients among the three book-to-market portfolios. The results show that from 1997 to 2010 the value and growth portfolios are less correlated than in the previous sub-period.

Panel A of Table 2 contains the parameter estimates obtained by the generalized method of moments with the identity matrix using expressions (3.14) to (3.17). The estimated value for λ of 0.152 for the full sample period implies that on average the chance of a simultaneous jump in any month across book-to-market portfolios is about 15%, or one jump is expected every 6.6 months or, equivalently, 0.55 years.

The average size of the jump across three portfolios is -1.307, and it seems much higher (in absolute value) for the value portfolio than for the growth stocks. Although, the intermediate portfolio has the highest average size of the jump, the volatilities of the size of the jumps are higher for the extreme growth and value portfolios.

As before, the more interesting results come from the changing behavior of the growth and value stocks across sub-periods. The estimated value for λ of 0.938 from 1997 to 2010 implies that on average the chance of a simultaneous jump in any month across book-to-market portfolios is almost 94%, or one jump is expected every 1.1 months or 0.09 years. Hence, the frequency of simultaneous jumps in the last sub-periods is extremely high. However, during this sub-period, the average size of the jump is -0.243 which is a relatively low relative to the estimate for the full sample period. This relatively small average size is partly due to the compensation between the positive average size for growth stocks, and the negative average size for value stocks. Hence, form 1997 to 2010, growth stocks have on average positive jumps, while value stocks present on average negative jumps. At the same time, the volatility of the size of the jumps seems to be lower than the volatility for the full sample period. The crisis episodes of the last decade or so, seems to affect much more negatively value than growth stocks. This is clearly consistent with the negative (positive) skewness for the value (growth) stocks from 1997 to 2010. The economic implication is that value stocks seem to be more pro-cyclical than growth stocks.

Once again, the estimation results are very different for the first sub-period from 1982 to 1997. The estimated value for λ of 0.031 from 1982 to 1997 implies that, on average, the chance of a simultaneous jump in any month across book-to-market portfolios is about 3%, or one jump is expected every 32.3 months or 2.7 years. Hence, the frequency of simultaneous jumps in the first sub-period is much lower than in the most recent sub-period. This is, by itself, a relevant result. During the last fourteen years, there seems to be many more systemic jump episodes than during the previous fifteen years. Somehow surprisingly, however, the average size of the jump from 1982 to 1997 is strongly negative and much larger in absolute value than during the last sub-period. Also, the volatility of the size of the jumps is much larger than the volatility reported from 1997 to 2010. Interestingly, the average size of the jump is positive (negative) for value (growth) stocks which are precisely the opposite reported from 1997 to 2010. Again, this is consistent with the negative (positive) skewness for the growth (value) portfolio shown in Table 1. It is also important to point out that it is precisely in this sub-period when the value premium is as high as 18.4% on annual basis. The jumps of the first sub-period are therefore much less frequent but of larger magnitude than the jumps observed from 1997 to 2010. The relatively few but larger jumps affect more negatively growth stocks, while very frequent although smaller jumps impact more negatively value stocks. Again, it should be recalled that the value premium is much larger from 1982 to 1997 than from 1997 to 2010.

Panel B of Table 2 compares the reconstructed moments that are obtained by substituting the parameter estimates in the theoretical jump-diffusion model and the sample moments. As before, the comparison exercise is performed for each of the three sample periods. Overall, the jump-diffusion model captures very well the sample excess kurtosis for all time periods and portfolios. However, the model seems to have more problems fitting the magnitudes of the asymmetry of the distribution of returns. This is especially true for the value portfolio, where the theoretical process overstates the magnitude of skewness. This result comes basically from the bad performance of the model capturing the skewness of value stocks from 1997 to 2010. The jump-diffusion process generates much more negative skewness than the one observed in the data.

4.2. **Portfolio Weights.** We want to solve numerically equation (2.12) to obtain the optimal weights when the return process follows the jump-diffusion model of equation (2.3). The parameters we employ for the return process are those reported in Panel A of Table 2. We also assume that the annualized risk-free rate is equal to

6%, and we solve for optimal weights assuming 10 alternative values of the relative risk aversion coefficient. As the benchmark case, we use the pure-diffusion model where the optimal weights are given by the well known expression (2.7).

Panel A of Table 3 contains the equity portfolio allocation results for the benchmark case, while Panel B reports the results for the jump-diffusion model. The results show that, independently of the sample period employed and the level of risk aversion, value stocks receive a higher investment proportion of funds than growth stocks. The optimal allocation implies in all cases to go long in the value portfolio and short on the growth portfolio. This is also true whether we recognize simultaneous jumps across assets or not. This is, of course, what a zero-cost investment on the HML portfolio of Fama and French (1993) precisely does. Surprisingly, for the full sample period, the intermediate book-to-market portfolio dominates the value portfolio due to the large weights this portfolio gets from 1982 to 1997. For this first sub-period, it must be pointed out that, when we incorporate jumps, the value portfolio indeed gets more proportion of wealth than the intermediate portfolio but only for the most levered position. It is interesting to observe the important impact that jumps have in this case for the non-conservative investor. The proportion invested in the intermediate assets decreases from 28.7% without jumps to 17.4% with jumps, while the same proportions go from 14.0% to 18.5% for the value portfolio. Finally, the value portfolio dominates the investment in the risky component of the optimal asset allocation for the 1997 to 2010 sub-period independently of recognizing jumps or not.

From Panel C of Table 3, where we report the differences in weights between the benchmark case and the jump-diffusion model, we observe that, for the overall sample period, the recognition of jumps diminishes the differences in the sense of increasing the long position on the value portfolio while, at the same, it suggests not short-selling as much on the growth portfolio.[8] The recognition of jumps implies to

[8]A negative sign for the value portfolio in Panel C indicates that the recognition of simultaneous jumps makes the investor to allocate more funds in the value portfolio. A negative sign on the growth portfolio implies that the recommendation would be to short-sale growth stocks in less proportion than the case of the pure-diffusion benchmark.

put relatively fewer funds in the risky portfolio for all levels of risk aversion, although this seems to be especially the case for the non-conservative investors and, therefore, for the most levered positions. However, the main point is that the value portfolio gets more weights for all γ. At the end, at least from 1982 to 2010, the use of systemic jumps in the optimal allocation of funds, increases the proportion of value stocks, reduces the amount of short-selling in the growth stocks, and diminishes the proportion of funds invested in the intermediate portfolio.

As in other cases, the results of the full sample period seem to be the consequences of two different sub-periods. The effects of jumps from 1997 to 2010 are, to all effects, negligible. The differences of optimal weights between the benchmark case and the jump-diffusion model are therefore basically zero except, if anything, for the highly levered position. The main effects of jumps come from the first sub-period. From 1982 to 1997, there were very few jumps but with a relatively very large average size. In fact, the average jump size for the value portfolio is positive, which makes the proportion of funds invested in value to increase importantly and, especially, for the most levered portfolios. These effects can be seen directly from Panel C of Table 3 and they are also reflected in Figure 2. By recognizing systemic jumps, optimal asset allocation increases significantly for value stocks and for all levels of risk aversion, although more for the less conservative more levered case. At the same time, jumps reduce considerably the proportion invested in the intermediate portfolio. As a consequence of recognizing jumps, there is also a reduction in the short-selling proportions of growth stocks. The same conclusion is obtained by the results shown in Table 4. We report the percentages of the growth, intermediate, and value portfolios in the total proportion invested in risky assets. We observe that the spread between the proportions invested in value and growth assets is higher in the first than in the second sub-period. It is also the case, that the spread in the first sub-period is particularly important for the most levered portfolios, although the relevance of risk aversion is negligible in the last sub-period.

We may conclude that systemic jumps across portfolios seem to be important for asset allocation as long as the average magnitude of the jumps is large enough. From the point of view of asset allocation, it is therefore more important the amplitude than the frequency of the jumps. In our specific sample, jumps turn out to be relevant for value stocks given the combination of a large and positive average jump size. Jumps also seems to favor value stocks even with a negative average size jump as in the full sample period, but this is relatively less important and, in any case, the possible effects are just concentrated in highly levered portfolios.

4.3. **Certainty Equivalent Costs.** We next analyze the effect on utility of the optimal portfolio strategy that recognizes the occurrence of systemic jumps across book-to-market assets relative to the strategy that ignores these simultaneous jumps. We employ the CEQ given in expression (2.15) that calculates the additional wealth per $\$1,000$ of investment needed to raise the expected utility of terminal wealth under the non-optimal portfolio strategy to that under the optimal investment strategy. We consider the effects for investment horizons of 1 to 5 years and for levels of risk aversion of 2 to 10.

Table 5 reports the results. As before, the effects of frequent but relatively small jumps observed from 1997 to 2010 is negligible. Even the U.S. dollar consequences of the non-optimal strategy over the full sample period are very small. For highly risk-taken investors and over a 5 years horizon, the cost of ignoring jumps is only about $2.00. All the relevant effects come, once again, from the 1982 to 1997 sub-period in which the average size jump seems to be high enough to impact the portfolio allocation of equity portfolios. As we observe, the CEQ decreases as risk aversion increases. This suggests that, as he becomes more risk averse, the investor holds a smaller proportion of his wealth in risky portfolios and, therefore, both the exposure to simultaneous jumps and the effects on CEQ are smaller. However, when the investor is willing to accept higher risks, his optimal portfolio becomes much more levered to buy additional risky assets. Then, the consequences of ignoring systemic jumps should be higher. This is exactly what we observe from Table 5 and Figure

3. For highly levered portfolios, the CEQ goes from \$48.40 for a 1-year horizon to \$266.50 for the longest horizon per \$1,000 of investment. It should not be surprising that the higher the leverage position of an investment is, the higher the impact of jumps on portfolio strategies. It should be recognized however, that the dollar effects diminish very rapidly as the investor becomes more risk averse. For $\gamma = 4$, even for the longest horizon of 5 years, the CEQ is \$30.42 per \$1,000 which does not seem to be substantial.

4.4. **Sample Average Riskless Rate.** In the previous discussion, we impose a 0.5% monthly riskless rate for the full sample period and for both sub-periods. Although this is a reasonable riskless rate for the first sub-period, it may be too high for the second sub-period. For this reason, we estimate again the model from 1997 to 2010 imposing the actual average riskless rate of 0.5% per month or 3% per year. It must be noted that the interest rate affects the parameter estimates and, therefore, it may have consequences for the general conclusions about the optimal portfolio allocation during the second sub-period.

Table 6 reports the results affected by the riskless rate. The empirical evidence is almost identical to the evidence contained in Tables 2, 3, and 5. The average size of jumps is even slightly lower, and the frequency of the jumps is now 1.231 relative to the previous estimate of 0.938. Thus, the frequency of the simultaneous jumps is higher than the one reported in Table 2. Once again, this sub-period is characterized by many jumps of small average amplitude. The effects of jumps on the portfolio weights and on the cost of ignoring jumps are very similar to the previously reported results. As expected, given that now the risk-free investment offers a lower rate, the optimal amount of the risky portfolios is higher with respect to the allocation shown in previous tables. However, the effects about the distribution of resources among book-to-market portfolios with or without jumps are negligible. All our previous conclusions remain the same even with a much lower riskless rate.

4.5. **Specification Tests with a Pre-specified Weighting Matrix.** Up to now, the statistical performance of the model has been very informal. Intentionally, our discussion has been based mainly on economic intuition rather than statistical formality. We finally want to test the overall fit of the model. The test-statistic is the GMM test of overidentification restrictions.

We denote by $f_t(\theta)$ the $K-$vector of moment conditions containing the pricing errors generated by the jump-diffusion model at time t, and by θ the set of parameters to be estimated. The corresponding sample averages are denoted by $g_T(\theta)$. Then, the GMM estimator procedure minimizes the quadratic form $g_T(\theta)' W_T g_T(\theta)$ where W_T is a weighting squared matrix. The evaluation of the model performance is carried out by testing the null hypothesis $T[Dist(\theta)] = 0$, with $Dist = g'_T(\theta) W_T g_T(\theta)$ where the weighting matrix, W, is in our case the identity matrix. If the weighting matrix is optimal, $T[Dist(\widehat{\theta})]$ is asymptotically distributed as a Chi-square with $K - P - 1$ degrees of freedom, where P is the number of parameters. However, for any other weighting matrix (including the identity matrix), the distribution of the test statistic is unknown. Jagannathan and Wang (1996) show that, in this case, $T[Dist(\widehat{\theta})]^2$ is asymptotically distributed as a weighted sum of $K - P - 1$ independent Chi-squares random variables with one degree of freedom. That is

$$T\left[Dist(\widehat{\theta})\right]^2 \xrightarrow{d} \sum_{i=1}^{K-P-1} \lambda_i \chi_i^2(1) \tag{4.1}$$

where λ_i , for $i = 1, 2, \ldots, K - P - 1$, are the positive eigenvalues of the following matrix:

$$A = S_T^{1/2} W_T^{1/2} \left[I_K - (W_T^{1/2})^{-1} D_T (D'_T W_T D_T)^{-1} D'_T W_T^{1/2} \right] (W_T^{1/2})' (S_T^{1/2})',$$

in which $X^{1/2}$ means the upper-triangular matrix from the Choleski decomposition of X, and I_K is a $K-$dimensional identity matrix. Moreover, S_T and D_T are given by,

$$S_T = \frac{\sum_{t=1}^{T} f_t(\theta) f_t(\theta)'}{T}$$

$$D_T = \frac{\sum_{t=1}^{T} \partial f_t(\theta) \partial f_t(\theta)'}{T}$$

Therefore, in order to test the different models we estimate, we proceed in the following way. First, we estimate the matrix A and compute its nonzero $K - P - 1$ eigenvalues. Second, we generate $\{\upsilon_{hi}\}$, $h = 1, 2, \ldots, 100$, $i = 1, 2, \ldots, K - P - 1$, independent random draws from a $\chi^2(1)$ distribution. For each h, $u_h = \sum_{i=1}^{K-P-1} \lambda_i \upsilon_{hi}$ is computed. Then we compute the number of cases for which $u_h > T[Dist(\widehat{\theta})]^2$. Let p denote the percentage of this number. We repeat this procedure 1000 times. Finally, the $p-$value for the specification test of the model is the average of the p values for the 1000 replications.

It turns out that we are not able to reject the jump-diffusion model in any of the alternative sample periods employed in the estimation. The p-values for the full sample period, the first sub-period, and the second sub-period are 0.257, 0.103, and 0.676 respectively. These results suggest that the jump-diffusion model fits the actual data better from 1997 to 2010 than from 1982 to 1997. This is the case despite the poor reconstruction of the actual skewess for the value portfolio from 1997 to 2010.

5. The Effects of Systemic Jumps for Value and Growth Portfolios across Geographical Regions

We next check whether the previous results are sensitive to the choice of data. We extend the analysis to a sample of international monthly return data which include value and growth portfolios from Europe, North America, and the Pacific regions. Additionally, this may be a more direct way for testing the effects of systemic weights all over the world.

It is important to point out that this data comes from Morgan Stanley Capital International (MSCI). They provide monthly data on indices for value and growth portfolios which do not include only the extreme value and growth, as in the portfolios constructed by Fama and French. In fact, every single asset traded in a particular region is included either on the value or the growth portfolios. Of course, to be part of the value (growth) portfolio, the asset has a book-to-market ratio above (below) the overall median of the book-to-market distribution. But, once again, it is important to understand that they are not the portfolios generating the well know value premium for the U.S. market.

Table 7 contains the descriptive statistics of these geographical portfolios. Panel A shows that the usual value premium is only found for the Pacific region. The premium is a relatively small 3.78% per year, and it is accompanied by the annualized growth premium volatility of 3.0%. For Europe there is a very slight value premium, although is volatility is also slightly higher for value stocks. The North American region presents a surprising although very small growth premium. As expected, these results suggest that the value premium for value stocks relative to growth stocks is due to the extreme portfolios. The results are similar across sub-periods. Moreover, the value portfolios always have negative skewness, and their magnitudes are quite similar to the skewness of growth stocks for Europe and North America. The only exception is the Pacific region. Growth stocks in this area tend to have larger negative skewness than value stocks, except for the sub-period from 1982 to 1997 in which growth stocks in the Pacific have a small but positive skewness. Excess kurtosis tends to be rather low, especially in the Pacific stock exchanges. The only exception is the North America region during the first sub-period for both value and growth stocks. It is also the case that excess kurtosis is higher during the first sub-period for all cases except for value stocks in the Pacific.

The correlation coefficients contained in Panel B of Table 7 show that returns in Europe and North America tend to be more correlated than the returns with the Pacific area. The lowest correlation is between North America and the Pacific, and this is particularly the case for the first sub-period. There is an important increase in the correlation coefficients across regions during the most recent sub-period. It may be a consequence of the recent financial crisis in which the correlation coefficients have increased among stock exchanges all over the world.[9] It is also important to point out that these results are practically the same for both value and growth portfolios.

Panel A of Table 8 reports the parameter estimates of the jump-diffusion model with systemic risk. As with the Fama-French dataset, the result show many more frequent jumps in the last sub-period than in the first sub-period. The frequency of simultaneous jumps is only slightly higher for growth stocks than for value stocks. The estimated value for λ of 0.159 for the full sample period for value stocks implies that on average the chance of a simultaneous jump in any month across value stocks in Europe, North America and the Pacific is about 16%, or one jump is expected every 6.3 months or 0.52 years. On the other, hand the λ of 0.183 for growth stocks suggests one jump every 5.5 months.

The average size of the jump seems to be similar for the value and growth portfolios across geographical regions for the full sample period. However, the average size of the jump is larger (in absolute terms) for value stocks during the first sub-period, while the size (again in absolute value) of the jumps is larger for growth stocks in the second sub-period. The explanation of this difference lies on the surprising behavior of the stocks in the Pacific region. Indeed, while the average size of value stocks in this area is negative from 1982 to 1997, the size becomes positive for the growth portfolio. This decreases the average size of the negative jump of growth stocks across the regions from 1982 to 1997.

Panel B of Table 8 reports the comparison between the moments from the theoretical model and the actual sample moments. For Europe and North America, the jump-diffusion process captures extraordinarily well all moments independently of the sample period. However, the skewess of the Pacific area is not well explained by the model. This is especially the case for both portfolios from 1982 to 1997. The asymmetry generated by the jump-diffusion model tends to exaggerate both the negative and the positive skewness of the value and growth stocks respectively.

[9]It may also be a sign of higher international integration. However, this is a rather subtle point. Higher correlation does not necessarily imply higher integration among stock exchanges.

Tables 9 and 10 present the optimal portfolio weights and the CEQ across different geographical regions for value and growth portfolios. The results in both tables show that the effects of systemic jumps on portfolio allocation strategies are very weak. The performance of the value and growth portfolio reported in Table 7 is reflected now on the optimal allocation across different regions for both value and growth stocks. However, the effects of jumps for a given level of leverage hardly change the optimal weights across regions.

6. Conclusions

Given the tendency of globalization and increasing integration of financial markets, it is generally accepted that equity portfolios of different characteristics and also international equities are characterized by simultaneous jumps. We investigate the effects of these jumps on optimal portfolio allocation using value and growth stocks and two different sub-periods. It seems that the effects of systemic jumps may be potentially substantial as long as market equity returns experiment very large average (negative) sizes. However, it does not seem to be relevant that stock markets experience very frequent jumps if they are not large enough to impact the most levered portfolios. All potentially relevant effects are concentrated in portfolios financed with a considerable amount of leverage. In fact, for conservative investors with low leverage positions the potential effects of systemic jumps on the optimal allocation of resources is not substantial even under large average size jumps. Finally, the value premium is particularly high when the average size of the jumps of value stocks is positive, large and relatively infrequent, while the average size of growth stocks is also very large but negative. It seems therefore plausible to conclude that the magnitude of the value premium is closely related to the characteristics of the jumps experienced by value and growth stocks.

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Appendix A. The Central Moments under Alternative Derivation Procedures

The idea of the appendix is to show how to calculate the central moments directly from the characteristic function. This approach is more direct and efficient than the procedure employed by Das and Uppal (2004). At the outset we perform the derivatives of the value function, equation (2.10), and of the characteristic function, equation (3.5). Alternatively, we present a procedure to calculate directly the conditional expectations to find the characteristic function and to solve the portfolio problem.

We first obtain the partial differential equation using the derivatives of equation (2.10) as follows:

$$\partial_t V = \partial_t F(t)\frac{W_t^{1-\gamma}}{1-\gamma} = \frac{\partial_t F(t)}{F(t)}V$$

$$\partial_{Wt} V = (1-\gamma)\partial_t F(t)\frac{W_t^{1-\gamma}}{1-\gamma} = \frac{(1-\gamma)}{W_t}V$$

$$\partial_{WtWt} V = \partial_{Wt}(1-\gamma)F(t)\frac{W_t^{1-\gamma}}{1-\gamma} = -\gamma(1-\gamma)F(t)\frac{W_t^{1-\gamma}}{1-\gamma} = \frac{-\gamma(1-\gamma)}{W_t^2}V$$

$$\begin{aligned} V(W_t + W_t\omega'(\widetilde{J}_t - I_N), t) - V(W_t, t) &= \frac{F(t)}{1-\gamma}\left(\left(W_t + W_t\omega'(\widetilde{J}_t - I_N)\right)^{1-\gamma} - W_t^{1-\gamma}\right) \\ &= F(t)\frac{W_t^{1-\gamma}}{1-\gamma}\left(\left(1 + \omega'(\widetilde{J}_t - I_N)\right)^{1-\gamma} - 1\right) \\ &= V\left(\left(1 + \omega'(\widetilde{J}_t - I_N)\right)^{1-\gamma} - 1\right) \end{aligned}$$

We now obtain the ordinary differential equation for the characteristic function from the Feynman-Kac equation (3.6). We calculate the following derivatives:

$$\begin{aligned} \partial_\tau \Phi_t &= -\partial_t \Phi_t \\ &= \partial_\tau e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t} = \left[e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t}\right]\left[\partial_\tau A_t + \partial_\tau B_t' X_t\right] \\ &= \left[\partial_\tau A_t + \partial_\tau B_t' X_t\right]\Phi_t \end{aligned}$$

$$\partial_X \Phi_t = \partial_X e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t} = \left[e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t}\right]B_t$$

$$= B_t(\tau,\zeta)\Phi_t$$

$$\partial_{XX'}\Phi_t = \partial_X\partial_{X'}\Phi_t = \partial_{X'}\Phi_t B_t'(\tau,\zeta) = B_t(\tau,\zeta)B_t'(\tau,\zeta)\Phi_t$$

Additionally, the jump term becomes:

$$\begin{aligned}\Phi_t(X+Y) - \Phi_t(X) &= e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)(X_t+Y)} - e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t} \\ &= e^{A_t(\tau,\zeta)+B_t'(\tau,\zeta)X_t}\left[e^{B_t'(\tau,\zeta)Y} - 1\right] \\ &= \Phi_t\left[e^{B_t'(\tau,\zeta)Y} - 1\right]\end{aligned}$$

As an alternative to find the characteristic function we calculate the conditional expected value directly. For this, consider the process X_t, whose dynamics in scalar form are stated in equation (3.3), and whose vector form is given by $X_t = X_0 + \alpha t + \sigma Z_t + Y_t$ where $Y_t = \sum_{i=1}^{Q_t} J_i$. In particular, for $t = T$, we have

$$\begin{aligned}X_T &= X_0 + \alpha T + \sigma(Z_T - Z_t + Z_t) + Y_T - Y_t + Y_t \\ &\overset{d}{=} X_t + \alpha\tau + \sigma Z_\tau + Y_\tau\end{aligned}$$

where $\tau = T - t$, and "$\overset{d}{=}$" denotes equality in distribution. The process $\alpha\tau + \sigma Z_\tau + Y_\tau$ is of Lévy type represented by the Lévy triplet (α,σ,Ψ), where Ψ is a Lévy measure. Additionaly, $Z_\tau \sim \Psi(0,\Sigma\tau)$ and $Y_\tau \sim Po(\lambda\tau)$. Our purpose is to calculate the conditional expected value $E[e^{i\zeta' X_T}|\mathfrak{F}_t]$, i.e. the conditional characteristic function with $N \times 1$ vector parameter ζ. Recall that the jump component, the Brownian motion and the jump amplitude are independent. Given that the Brownian motion and the compound Poisson process have independent and stationary increments we have

$$E[e^{i\zeta' X_T}|\mathfrak{F}_t] = e^{i\zeta' X_t} E\left[e^{i\zeta'(\alpha\tau + \sigma Z_\tau + Y_\tau)}\right]$$

And from the Lévy-Kintchine theorem

$$E[e^{i\zeta' X_T}|\mathfrak{F}_t] = e^{i\zeta' X_t}\exp\left\{\left(i\zeta'\alpha - \frac{1}{2}\zeta'\Sigma\zeta + \lambda\int_{\mathbb{R}^N}(e^{i\zeta' z} - 1)d\Psi(z)\right)\tau\right\}$$

The integral in the right hand side is recognized as the jump amplitude's characteristic function. Given that the random variable J follows the Normal distribution Ψ, $J \sim \Psi(\eta, \Sigma^J)$

$$\int_{R^N} e^{i\zeta' z} d\Psi(z) = \exp\left\{ i\zeta'\eta - \frac{1}{2}\zeta'\Sigma^J\zeta \right\}$$

Hence, the conditional characteristic function $\Phi_t(\tau, \zeta, X_t) = E[e^{i\zeta' X_T}|\mathfrak{F}_t]$ is

$$\Phi_t(\tau, \zeta, X_t) = \exp\left\{ \left(i\zeta'\alpha - \frac{1}{2}\zeta'\Sigma\zeta + \lambda \exp\left\{ i\zeta'\eta - \frac{1}{2}\zeta'\Sigma^J\zeta \right\} - \lambda \right) \tau + i\zeta' X_t \right\}$$

Now we aim to solve the portfolio problem, for this we should calculate $E_t[u(W_T)]$ where $u(W_T) = W_T^{1-\gamma}/(1-\gamma)$, that is $E_t[u(W_T)] = E_t[W_T^{1-\gamma}]/(1-\gamma)$. And the dynamics of the wealth process is given by

$$\frac{dW_t}{W_t} = (\omega' R + r)dt + \omega'\sigma dZ_t + \omega'\widetilde{J} dQ_t$$

Applying Ito's lemma to $\ln W_t$

$$d\ln W_t = \left(\omega' R + r - \frac{1}{2}\omega'\Sigma\omega \right) dt + \omega'\sigma dZ_t + \ln(1 + \omega'\widetilde{J})dQ_t$$

For constant r, R, and σ the integral gives

$$W_t = \exp\left\{ W_0 + \left(\omega' R + r - \frac{1}{2}\omega'\Sigma\omega \right) t + \omega'\sigma Z_t + Y_t \right\}$$

Where W_0 is the initial wealth and $Y_t = \sum_{i=1}^{Q_t} \ln(1 + \omega'\widetilde{J}_i)$. The conditional expectation becomes

$$\begin{aligned}
E_t[u(W_T)] &= \frac{E_t[W_T^{1-\gamma}]}{1-\gamma} \\
&= \frac{1}{1-\gamma} E_t\left[\exp\left\{ (1-\gamma)\left(W_0 + \left(\omega' R + r - \frac{1}{2}\omega'\Sigma\omega \right) T + \omega'\sigma Z_T + Y_T \right) \right\} \right] \\
&= \exp\left\{ (1-\gamma)\left(\omega' R + r - \frac{1}{2}\omega'\Sigma\omega \right) \tau \right\} E_t\left[e^{(1-\gamma)\omega'\sigma Z_\tau + (1-\gamma)Y_\tau} \right] u(W_t) \\
&= \exp\left\{ (1-\gamma)(\omega' R + r)\tau - \frac{1}{2}\gamma(1-\gamma)\omega'\Sigma\omega\tau \right. \\
&\qquad \left. + \lambda\tau \int_{\mathbb{R}^N} (e^{(1-\gamma)\ln(1+\omega' z)} - 1) d\Psi(z) \right\} u(W_t)
\end{aligned}$$

The last step comes from the Lévy-Kintchine theorem. The integral on the right hand side is an expectation, i.e.

$$\int_{\mathbb{R}^N} ((1+\omega' z)^{(1-\gamma)} - 1)d\Psi(z) = E\left[(1+\omega'\widetilde{J})^{(1-\gamma)} - 1\right]$$

Therefore

$$E_t[u(W_T)] = \exp\left\{\left((1-\gamma)(\omega' R + r) - \frac{1}{2}\gamma(1-\gamma)\omega'\Sigma\omega \right.\right.$$
$$\left.\left. +\lambda E\left[(1+\omega'\widetilde{J})^{(1-\gamma)}\right] - \lambda\right)\tau\right\} u(W_t)$$

A.1. **Central moments using the derivatives of the cumulant function with respect to variables j and i.** The characteristic function is given by

$$\Phi_t(\tau,\zeta,X_t) = \exp\left(\left[i\zeta'\alpha - \frac{1}{2}\zeta'\Sigma\zeta + \lambda\left(\exp\left(i\zeta\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) - 1\right)\right]\tau + i\zeta X_t\right)$$

Setting $\tau = 1$, we can write

$$\Phi_t(\zeta, X) = \Phi_t(\tau = 1, \zeta, X_t) = e^{i\zeta'(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)},$$

where $C = \zeta'\beta - \frac{1}{2}\zeta'\Sigma^J\zeta$ and $\beta = i\eta$. We define the cumulant generating function as follows,

$$\Psi_t(\zeta, X_t) = ln(\Phi_t(\zeta, X_t)) = i\zeta'(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)$$

Its derivatives are given by,

$$\partial_{\zeta_j}\Psi_t(\zeta, X_t) = i(\alpha_j + X_{jt}) - \Sigma_j\zeta + \lambda e^C(\beta_j - \Sigma_j^J\zeta)$$

$$\partial_{\zeta_j\zeta_i}\Psi_t(\zeta, X_t) = -\sigma_{ji} + \lambda e^C\left\{(\beta_j - \Sigma_j^J\zeta)(\beta_i - \Sigma_i^J\zeta) - \nu_j\nu_i\right\}$$

$$\partial_{\zeta_j\zeta_i^2}\Psi_t(\zeta, X_t) = \lambda e^C\left\{(\beta_j - \Sigma_j^J\zeta)\left((\beta_i - \Sigma_i^J\zeta)^2 - \nu_i^2\right) - 2\nu_j\nu_i(\beta_i - \Sigma_i^J\zeta)\right\}$$

$$\partial_{\zeta_j\zeta_i^3}\Psi_t(\zeta, X_t) = \lambda e^C\left\{(\beta_j - \Sigma_j^J\zeta)(\beta_i - \Sigma_i^J\zeta)\left((\beta_i - \Sigma_i^J\zeta)^2 - 3\nu_i^3\right) - 3\nu_j\nu_i\left((\beta_i - \Sigma_i^J\zeta)^2 - \nu_i^2\right)\right\}$$

The cumulants can be found with the aid of

$$\kappa_L = i^{-L}\frac{\partial^L \Psi_t(\zeta)}{\partial\zeta_1^{l_1}, \partial\zeta_2^{l_2}, \ldots, \partial\zeta_N^{l_N}} \qquad \text{and} \qquad L = \sum_{j=1}^{N} l_j$$

Then,

$$\begin{aligned}
\kappa_1 &= i^{-1}\partial_{\zeta_j}\Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \alpha_j + \lambda\eta_j + X_{jt} \\
\kappa_2 &= -\partial_{\zeta_j\zeta_i}\Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \sigma_{ji} + \lambda(\eta_j\eta_i + \nu_j\nu_i) \\
\kappa_3 &= -i^{-1}\partial_{\zeta_j\zeta_i^2}\Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \lambda\left(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2)\right) \\
\kappa_4 &= \partial_{\zeta_j\zeta_i^3}\Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \lambda\left(\eta_j\eta_i(\eta_i^2 + 3\nu_i^2) + 3\nu_j\nu_i(\eta_i^2 + \nu_i^2)\right) \\
&= \lambda\left(\eta_j\eta_i^3 + 3\eta_j\eta_i\nu_i^2 + 3\nu_j\nu_i\eta_i^2 + 3\nu_j\nu_i^3\right)
\end{aligned}$$

In these expressions κ_1 is the first non-central moment, and κ_2 is the covariance between the returns of assets j and i. The standardized central moments are given by the expressions,

$$\begin{aligned}
\text{coskewness}_{ji} &= \frac{\kappa_3}{\sigma_j\sigma_i^2} = \frac{\lambda\left(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2)\right)}{\sigma_j\sigma_i^2} \\
\text{excess kurtosis}_{jj} &= \frac{\kappa_4}{\sigma_j^4} = \frac{\lambda\left(\eta_j^4 + 6\eta_j^2\nu_j^2 + 3\nu_j^4\right)}{\sigma_j^4}
\end{aligned}$$

A.2. **Central moments using the derivatives of the characteristic function with respect to variables j and i.** [10]

The characteristic function is given by

$$\Phi_t(\tau, \zeta, X_t) = \exp\left(\left[i\zeta'\alpha - \frac{1}{2}\zeta'\Sigma\zeta + \lambda\left(\exp\left(i\zeta\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) - 1\right)\right]\tau + i\zeta X_t\right)$$

Setting $\tau = 1$, we can write

$$\Phi_t(\zeta, X) = \Phi_t(\tau = 1, \zeta, X_t) = e^{i\zeta'(\alpha+X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)},$$

[10] This is the method suggested by Das and Uppal (2004).

where $C = \zeta'\beta - \frac{1}{2}\zeta'\Sigma^J\zeta$ and $\beta = i\eta$. It should be noted that $\Phi_t(0, X_t) = 1$ and $e^C\Big|_{\zeta=0} = 1$. The first non-central moment is given by

$$\bar{m}_1 = \frac{1}{i}\partial_\zeta \Phi_t(\zeta, X_t)\Big|_{\zeta=0}$$

Therefore,

$$\begin{aligned}
E[X_t] = \frac{1}{i}\partial_\zeta \Phi_t\Big|_{\zeta=0} &= \frac{1}{i}\Phi_t\partial_\zeta\left(i\zeta'(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)\right)\Bigg|_{\zeta=0} \\
&= \frac{1}{i}\Phi_t\partial_\zeta\left(i(\alpha + X_t) - \frac{1}{2}\Sigma\zeta + \lambda e^C(\beta - \Sigma^J\zeta)\right)\Bigg|_{\zeta=0} \\
&= \alpha + \lambda\eta + X_t
\end{aligned}$$

On the other hand, the $L-$central moment is given by the following expression:

$$m_L = i^{-L}\frac{\partial^L \Phi_t(\tau, \zeta, X_t - E[X_t])}{\partial\zeta_1^{l_1}, \partial\zeta_2^{l_2}, \ldots, \partial\zeta_N^{l_N}} \qquad \text{and} \qquad L = \sum_{j=1}^{N} l_j$$

The modified characteristic function takes the following form:

$$\begin{aligned}
\Psi_t(\zeta) &= e^{i\zeta'(\alpha+X_t)-\frac{1}{2}\zeta'\Sigma\zeta+\lambda(e^C-1)}e^{-i\zeta' E[X_t]} \\
&= e^{i\zeta'(\alpha+X_t)-\frac{1}{2}\zeta'\Sigma\zeta+\lambda(e^C-1)}e^{-i\zeta'(\alpha+\lambda\eta+X_t)} \\
&= e^{-\zeta'\lambda\beta-\frac{1}{2}\zeta'\Sigma\zeta+\lambda(e^C-1)}
\end{aligned}$$

In what follows we will use the following derivatives:

$$\begin{aligned}
F_k &= \partial_{\zeta_k}\left(-\zeta'\lambda\beta - \frac{1}{2}\zeta'\Sigma\zeta + \lambda\underbrace{(e^C - 1)}_{M}\right) = -\beta_k - \Sigma_k\zeta + \lambda\partial_{\zeta_k}M \\
H_{kl} &= \partial_{\zeta_l}F_k = \partial_{\zeta_l}(-\beta_k - \Sigma_k\zeta + \lambda\partial_{\zeta_k}M) = -\sigma_{kl} + \lambda\partial_{\zeta_k\zeta_l}M \\
\partial_{\zeta_k}C &= \partial_{\zeta_k}\left(\zeta'\beta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) = \beta_k - \Sigma_k^J\zeta
\end{aligned}$$

The partial derivatives of the characteristic function $\Psi_t(\zeta)$ with respect to variable ζ_j and ζ_i are,

$$\partial_{\zeta_j}\Psi_t(\zeta) = \Psi_t F_j$$

$$\partial_{\zeta_j\zeta_i}\Psi_t(\zeta) = \partial_{\zeta_j}(\Psi_t F_j) = \Psi_t\left\{F_jF_i + H_{ji}\right\}$$

$$\partial_{\zeta_j\zeta_i^2}\Psi_t(\zeta) = \partial_{\zeta_j}\left\{\Psi_t\left\{F_jF_i + H_{ji}\right\}\right\} = \Psi_t\left\{F_i(F_jF_i + H_{ji}) + H_{ji}F_i + F_jH_{ii} + \partial_{\zeta_i}H_{ji}\right\}$$

$$\begin{aligned}\partial_{\zeta_j\zeta_i^3}\Psi_t(\zeta) &= \partial_{\zeta_j}\Psi_t\left\{F_i(F_jF_i + H_{ji}) + H_{ji}F_i + F_jH_{ii} + \partial_{\zeta_i}H_{ji}\right\}\\ &= \Psi_t\left\{F_i^2(F_jF_i + H_{ji}) + H_{ji}F_i^2 + F_jF_iH_{ii} + F_i\partial_{\zeta_i}H_{ji} + H_{ii}(F_jF_i + H_{ji})\right.\\ &\qquad \left. F_i(H_{ji}F_i + F_jH_{ii} + \partial_{\zeta_i}H_{ji}) + \partial_{\zeta_i}H_{ji}F_i + 2H_{ji}H_{ii} + F_j\partial_{\zeta_i}H_{ii} + \partial_{\zeta_i^2}H_{ji}\right\}\end{aligned}$$

Taking into account that

$$M = e^C - 1$$

$$N = e^C$$

$$\partial_{\zeta_j}M = N(\beta_j - \Sigma_j^J\zeta)$$

$$\partial_{\zeta_j\zeta_i}M = N\left\{(\beta_j - \Sigma_j^J\zeta)(\beta_i - \Sigma_i^J\zeta) - \nu_j\nu_i\right\}$$

$$\partial_{\zeta_j\zeta_i^2}M = N\left\{(\beta_j - \Sigma_j^J\zeta)\left((\beta_i - \Sigma_i^J\zeta)^2 - \nu_i^2\right) - 2\nu_j\nu_i(\beta_i - \Sigma_i^J\zeta)\right\}$$

$$\partial_{\zeta_j\zeta_i^3}M = N\left\{(\beta_j - \Sigma_j^J\zeta)(\beta_i - \Sigma_i^J\zeta)\left((\beta_i - \Sigma_i^J\zeta)^2 - 3\nu_i^2\right) - 3\nu_j\nu_i\left((\beta_i - \Sigma_i^J\zeta)^2 - \nu_i^2\right)\right\}$$

And,

$$\partial_{\zeta_i}H_{ji} = \lambda\partial_{\zeta_j\zeta_i^2}M$$

$$\partial_{\zeta_i^2}H_{ji} = \lambda\partial_{\zeta_j\zeta_i^3}M$$

Evaluating the following expressions at $\zeta = 0$,

$$F_k|_{\zeta=0} = M|_{\zeta=0} = 0; \qquad N|_{\zeta=0} = 1$$

$$\partial_{\zeta_j}M\Big|_{\zeta=0} = i\eta_j$$

$$\partial_{\zeta_j\zeta_i}M\Big|_{\zeta=0} = -(\eta_j\eta_i + \nu_j\nu_i)$$

$$\partial_{\zeta_j\zeta_i^2}M\Big|_{\zeta=0} = -i\left(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2)\right)$$

$$\partial_{\zeta_j\zeta_i^3}M\Big|_{\zeta=0} = \eta_j\eta_i(\eta_i^2 + \nu_i^2) + 2\nu_j\nu_i\eta_i^2 + \nu_j\nu_i(\eta_i^2 + \nu_i^2) + 2\nu_i^2\eta_j\eta_i + 2\nu_j\nu_i^3$$

$$= \eta_j\eta_i^3 + 3\eta_j\eta_i\nu_i^2 + 3\nu_j\nu_i\eta_i^2 + 3\nu_j\nu_i^3$$

It implies that,

$$H_{ji}\big|_{\zeta=0} = -\sigma_{ji} - \lambda(\eta_j\eta_i + \nu_j\nu_i)$$

$$\partial_{\zeta_i} H_{ji}\big|_{\zeta=0} = -i\lambda\left(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2)\right)$$

$$\partial_{\zeta_i^2} H_{ji}\Big|_{\zeta=0} = \lambda\left(\eta_j\eta_i^3 + 3\eta_j\eta_i\nu_i^2 + 3\nu_j\nu_i\eta_i^2 + 3\nu_j\nu_i^3\right)$$

Therefore, the standardized central moments for assets j and i are given by,

$$\text{Covariance}_{ji} = i^{-2}\partial_{\zeta_j\zeta_i}\Psi_t(\zeta)\Big|_{\zeta=0} = -\,H_{ji}\big|_{\zeta=0} = \sigma_{ji} + \lambda(\eta_j\eta_i + \nu_j\nu_i)$$

$$\text{Coskewness}_{ji} = \frac{-i^{-1}\partial_{\zeta_j\zeta_i^2}\Psi_t(\zeta)\Big|_{\zeta=0}}{\sigma_j\sigma_i^2} = \frac{-i^{-1}\partial_{\zeta_i}H_{ji}\big|_{\zeta=0}}{\sigma_j\sigma_i^2} = \frac{\lambda(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2))}{\sigma_j\sigma_i^2}$$

$$\text{Excess kurtosis}_{jj} = \frac{\partial_{\zeta_j^4}\Psi_t(\zeta)\Big|_{\zeta=0}}{\sigma_j^4} - 3 = \frac{3H_{jj}^2 + \partial_{\zeta_i^2}H_{ji}\Big|_{\zeta=0}}{\sigma_j^4} - 3 = \frac{\lambda(\eta_j^4 + 6\eta_j^2\nu_j^2 + 3\nu_j^4)}{\sigma_j^4}$$

where the last step comes from the fact that H_{jj}^2 is the square of the variance.

A.3. **Central moments using the derivatives of the cumulant function in vector form.** The characteristic function is given by

$$\Phi_t(\tau, \zeta, X_t) = \exp\left(\left[i\zeta'\alpha - \frac{1}{2}\zeta'\Sigma\zeta + \lambda\left(\exp\left(i\zeta\eta - \frac{1}{2}\zeta'\Sigma^J\zeta\right) - 1\right)\right]\tau + i\zeta X_t\right)$$

Setting $\tau = 1$, we can write

$$\Phi_t(\zeta, X) = \Phi_t(\tau = 1, \zeta, X_t) = e^{i\zeta'(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)},$$

where $C = \zeta'\beta - \frac{1}{2}\zeta'\Sigma^J\zeta$ and $\beta = i\eta$. Once again, we define the cumulant generating function as follows,

$$\Psi_t(\zeta, X_t) = ln(\Phi_t(\zeta, X_t)) = i\zeta'(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)$$

Its derivatives are as follows,

$$\begin{aligned}
\partial_\zeta \Psi_t(\zeta, X_t) &= i(\alpha + X_t) - \Sigma\zeta + \lambda e^C(\beta - \Sigma^J\zeta) = F \\
\partial_{\zeta\zeta'} \Psi_t(\zeta, X_t) &= \partial_\zeta(\partial_{\zeta'} \Psi_t(\zeta, X_t)) \\
&= \partial_\zeta \left(\partial_{\zeta'} \left(\zeta' i(\alpha + X_t) - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1) \right) \right) \\
&= \partial_\zeta F' = -\Sigma + \lambda e^C(GG' - \Sigma^J) = H
\end{aligned}$$

where, for convenience, we define $G = (\beta - \Sigma^J\zeta)$. The third and fourth moments are calculated as,

$$\begin{aligned}
m_3 &= E[(X - \bar{m}_1)(X - \bar{m}_1)' \otimes (X - \bar{m}_1)'] \\
m_4 &= E[(X - \bar{m}_1)(X - \bar{m}_1)' \otimes (X - \bar{m}_1)' \otimes (X - \bar{m}_1)']
\end{aligned}$$

where $\otimes$ is the Kronecker product.

As long as we will use only the terms that accounts for the combination of two different variables, we are interested in elements that are the results of taking partial derivatives twice. The first derivative will be with respect to ζ_j, and the other with respect to ζ_i. For that purpose, we find first the expressions for the derivative with respect to the vector ζ and then with respect to one variable alone, say ζ_k, and finally construct the moments. To that end, we use the single entry vector defined as

$$\Gamma^{k1} = \begin{pmatrix} 0 \\ \vdots \\ (1)_k \\ \vdots \\ 0 \end{pmatrix}, \quad \Gamma^{1k} = (0 \ldots, (1)_k \ldots 0)$$

Some properties of the single entry vector are,

$$\Gamma^{1k} a = a'\Gamma^k$$

$$(\Gamma^{1k} a)a = a(a'\Gamma^k) = (aa')\Gamma^k$$

$$(\Gamma^{1k}a)a' = \Gamma^{1k}(aa')$$

These vectors are useful when dealing with partial derivatives:

$$\partial_{\zeta_k}\zeta = \Gamma^k$$

$$\partial_{\zeta_k}\zeta' = \Gamma^{1k}$$

$$\partial_{\zeta_k}\zeta\zeta' = 2\zeta'\Gamma^k = 2\Gamma^{1k}\zeta$$

$$\partial_{\zeta_k}A\zeta = A\Gamma^k$$

$$\partial_{\zeta_k}\zeta' A = \Gamma^{1k}A$$

$$\partial_{\zeta_k}\zeta' A\zeta = \Gamma^{1k}(A + A')\zeta$$

$$\partial_{\zeta_k}AB = (\partial_{\zeta_k}A)B + A\partial_{\zeta_k}B$$

The following calculations will be used in the derivation,

$$\partial_{\zeta_k}G = -\Sigma^J\Gamma^k$$

$$\partial_{\zeta_k}G' = -\Gamma^{1k}\Sigma^J$$

$$\partial_{\zeta_k}GG' = -(\Sigma^J\Gamma^k G' + G\Gamma^{1k}\Sigma^J)$$

$$\partial_{\zeta_k\zeta_k}GG' = 2\Sigma^J\Gamma^k\Gamma^{1k}\Sigma^J$$

$$\partial_{\zeta_k}\lambda e^C = \lambda e^C\Gamma^{1k}G$$

$$\partial_{\zeta_k}\lambda e^C G = \lambda e^C(GG')\Gamma^k$$

$$\partial_{\zeta_k}\lambda e^C G' = \lambda e^C\Gamma^{1k}(GG')$$

Also,

$$H = -\Sigma + \lambda e^C(GG' - \Sigma^J)$$

$$H|_{\zeta=0} = -\Sigma - \lambda(\eta\eta' + \Sigma^J)$$

$$\partial_{\zeta_k}H = \lambda e^C\left(\Gamma^{1k}G(GG' - \Sigma^J) - \Sigma^J\Gamma^k G' - G\Gamma^{1k}\Sigma^J\right)$$

$$\partial_{\zeta_k}H|_{\zeta=0} = -i\lambda\left(\Gamma^{1k}\eta(\eta\eta' + \Sigma^J) + \Sigma^J\Gamma^k\eta' + \eta\Gamma^{1k}\Sigma^J\right)$$

$$\partial_{\zeta_k\zeta_k} H = \lambda e^C \left((\Gamma^{1k} G \Gamma^{1k} G - \Gamma^{1k} \Sigma^J \Gamma^k)(GG' - \Sigma^J) - 2\Gamma^{1k} G (\Sigma^J \Gamma^k G' + G \Gamma^{1k} \Sigma^J) \right.$$
$$\left. +2\Sigma^J \Gamma^k \Gamma^{1k} \Sigma^J \right)$$
$$\partial_{\zeta_k\zeta_k} H|_{\zeta=0} = \lambda \left((\Gamma^{1k} \eta \Gamma^{1k} \eta + \Gamma^{1k} \Sigma^J \Gamma^k)(\eta\eta' + \Sigma^J) - 2\Gamma^{1k} \eta (\Sigma^J \Gamma^k \eta' + \eta \Gamma^{1k} \Sigma^J) \right.$$
$$\left. +2\Sigma^J \Gamma^k \Gamma^{1k} \Sigma^J \right)$$

Then, the third and fourth derivatives are

$$\partial_{\zeta_k\zeta\zeta'} \Psi_t = \partial_{\zeta_k} H = \lambda e^C \left(\Gamma^{1k} G (GG' - \Sigma^J) - \Sigma^J \Gamma^k G' - G \Gamma^{1k} \Sigma^J \right)$$
$$\partial_{\zeta_k\zeta_k\zeta\zeta'} \Psi_t = \partial_{\zeta_k} (\partial_{\zeta_k\zeta\zeta'} \Psi_t) = \partial_{\zeta_k\zeta_k} H$$
$$= \lambda e^C \left((\Gamma^{1k} G \Gamma^{1k} G - \Gamma^{1k} \Sigma^J \Gamma^k)(GG' - \Sigma^J) - 2\Gamma^{1k} G (\Sigma^J \Gamma^k G' + G \Gamma^{1k} \Sigma^J) \right.$$
$$\left. +2\Sigma^J \Gamma^k \Gamma^{1k} \Sigma^J \right)$$

The cumulants are

$$\kappa_1 = i^{-1} \partial_\zeta \Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \alpha + \lambda\eta + X_t$$
$$\kappa_2 = -\partial_{\zeta\zeta'} \Psi_t(\zeta, X_t)|_{\zeta=0} = \Sigma + \lambda(\eta\eta' + \Sigma^J)$$
$$\kappa_3 = -i^{-1} \partial_{\zeta_k\zeta\zeta'} \Psi_t(\zeta, X_t)\Big|_{\zeta=0} = \lambda \left(\Gamma^{1k} \eta (\eta\eta' + \Sigma^J) + \Sigma^J \Gamma^k \eta' + \eta \Gamma^{1k} \Sigma^J \right)$$
$$\kappa_4 = \partial_{\zeta_k^2\zeta\zeta'} \Psi_t(\zeta, X_t)\Big|_{\zeta=0}$$
$$= \lambda \left((\Gamma^{1k} \eta \Gamma^{1k} \eta + \Gamma^{1k} \Sigma^J \Gamma^k)(\eta\eta' + \Sigma^J) + 2\Gamma^{1k} \eta (\Sigma^J \Gamma^k \eta' + \eta \Gamma^{1k} \Sigma^J) + 2\Sigma^J \Gamma^k \Gamma^{1k} \Sigma^J \right)$$

Cumulants κ_1 and κ_2 account for the mean and covariance. Regarding coskewness and cokurtosis, we note that in the expressions for κ_3 and κ_4 there is a $N \times N$ matrix for each value of k. We have to choose column $k = i$. Then, the column i of the cumulants matrices is:

$$\kappa_3^i = \lambda \left(\Gamma^{1i} \eta (\eta\eta' + \Sigma^J) + \Sigma^J \Gamma^i \eta' + \eta \Gamma^{1i} \Sigma^J \right) \Gamma^i = \lambda \left(\eta_i (\eta_i \eta + (\Sigma^J)^i) + \eta_i (\Sigma^J)^i + \eta \nu_i^2 \right)$$
$$\kappa_4^i = \lambda \left((\Gamma^{1i} \eta \Gamma^{1i} \eta + \Gamma^{1i} \Sigma^J \Gamma^i)(\eta\eta' + \Sigma^J) + 2\Gamma^{1i} \eta (\Sigma^J \Gamma^i \eta' + \eta \Gamma^{1i} \Sigma^J) + 2\Sigma^J \Gamma^i \Gamma^{1i} \Sigma^J \right) \Gamma^i$$
$$= \lambda \left((\eta_i^2 + \nu_i^2)(\eta_i \eta + (\Sigma^J)^i) + 2\eta_i (\eta_i (\Sigma^J)^i + \eta \Sigma_i^J) + 2(\Sigma^J)^i \Sigma_i^J \right)$$

The skewness and kurtosis are standardized using the standard deviation. Therefore, in the expression above, in order to find any element ji we have to select row j and column i with $k = i$. The matrix components for each moment are,

$$\text{Covariance}_{ji} = (\kappa_2)_{ji} = \sigma_{ji} + \lambda(\eta_j\eta_i + \nu_j\nu_i)$$

$$\text{Coskewness}_{ji} = \frac{(\kappa_3)_{ji}}{\sigma_j\sigma_i^2} = \frac{\lambda\left(\eta_i(\eta_i\eta' + \Sigma^J) + \Sigma^J\Gamma^i\eta' + \eta\Gamma^{1i}\Sigma^J\right)_j}{\sigma_j\sigma_i^2}$$

$$= \frac{\lambda(\eta_i(\eta_j\eta_i + \nu_j\nu_i) + \nu_j\nu_i\eta_i + \eta_j\nu_i^2)}{\sigma_j\sigma_i^2} = \frac{\lambda(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2))}{\sigma_j\sigma_i^2}$$

$$\text{Excess cokurtosis}_{ji} = \frac{(\kappa_4)_{ji}}{\sigma_j\sigma_i^3} = \frac{\lambda\left((\eta_i^2 + \nu_i^2)(\eta_i\eta + (\Sigma^J)^i) + 2\eta_i(\eta_i(\Sigma^J)^i + \eta\Sigma_i^J) + 2(\Sigma^J)^i\Sigma_i^J\right)_j}{\sigma_j\sigma_i^3}$$

$$= \frac{((\eta_i^2 + \nu_i^2)(\eta_j\eta_i + \nu_j\nu_i) + 2\eta_i(\nu_j\nu_i\eta_i + \eta_i\nu_j\nu_i) + 2\nu_j\nu_i^3)}{\sigma_j\sigma_i^3}$$

$$= \frac{\lambda(\eta_j\eta_i^3 + 3\nu_j\nu_i\eta_i^2 + 3\eta_j\eta_i\nu_i^2 + 3\nu_j\nu_i^3)}{\sigma_j\sigma_i^3}$$

For $j = i$.

$$\text{Excess kurtosis}_{jj} = \frac{\lambda(\eta_j^4 + 6\eta_j^2\nu_j^2 + 3\nu_j^4)}{\sigma_j^4}$$

A.4. **Central moments using the derivatives of the characteristic function in vector form.** Given the characteristic function, the $L-$central moments, and the modified characteristic function given at the beginning of sub-section (A.2) in this appendix, the second central moments is

$$m_2 = -\partial_{\zeta\zeta'}\Psi_t(\zeta, X_t)\big|_{\zeta=0}$$

The derivative is

$$-\partial_{\zeta\zeta'}\Psi_t = -\partial_\zeta(\partial_{\zeta'}\Psi_t)$$

$$= -\partial_\zeta\left\{\partial_{\zeta'}e^{-\zeta'\lambda\beta - \frac{1}{2}\zeta'\Sigma\zeta + \lambda(e^C - 1)}\right\}$$

$$= -\partial_\zeta(\Psi_t(-\lambda\beta' - \zeta'\Sigma + \lambda e^C G))$$

$$= -\Psi_t FF' - \Psi_t(-\Sigma + \lambda e^C(GG' - \Sigma^J))$$

Recall that for convenience,

$$F = -\lambda\beta - \Sigma\zeta + \lambda e^{C} G$$
$$G = \beta - \Sigma^{J}\zeta$$

Note that

$$\Psi_t|_{\zeta=0} = 1$$
$$G|_{\zeta=0} = \beta$$
$$G'|_{\zeta=0} = \beta'$$
$$F|_{\zeta=0} = F'|_{\zeta=0} = 0$$

Therefore,

$$m_2 = \Sigma + \lambda(\eta\eta' + \Sigma^{J})$$

From sub-section (A.3) we know how to calculate the third and fourth moments. We again employ the single entry vectors also defined in the same sub-section and its properties. Then, we recall the calculations needed in derivation,

$$\partial_{\zeta_k} G = -\Sigma^{J}\Gamma^{k}$$
$$\partial_{\zeta_k} G' = -\Gamma^{1k}\Sigma^{J}$$
$$\partial_{\zeta_k} GG' = -(\Sigma^{J}\Gamma^{k}G' + G\Gamma^{1k}\Sigma^{J})$$
$$\partial_{\zeta_k\zeta_k} GG' = 2\Sigma^{J}\Gamma^{k}\Gamma^{1k}\Sigma^{J}$$
$$\partial_{\zeta_k} \lambda e^{C} = \lambda e^{C}\Gamma^{1k}G$$
$$\partial_{\zeta_k} \lambda e^{C} G = \lambda e^{C}(GG')\Gamma^{k}G$$
$$\partial_{\zeta_k} \lambda e^{C} G' = \lambda e^{C}\Gamma^{1k}(GG')$$
$$\partial_{\zeta_k} F = (-\Sigma + \lambda e^{C} GG')\Gamma^{k}$$
$$\partial_{\zeta_k\zeta_k} F = \lambda e^{C}(\Gamma^{1k}G(GG')\Gamma^{k} + 2\Sigma^{J}\Gamma^{k}\Gamma^{1k}\Sigma^{J})$$
$$\partial_{\zeta_k} \Psi_t = \Psi_t\Gamma^{1k}F$$

And,

$$H = -\Sigma + \lambda e^C (GG' - \Sigma^J)$$

$$H|_{\zeta=0} = -\Sigma - \lambda(\eta\eta' + \Sigma^J)$$

$$\partial_{\zeta_k} H = \lambda e^C \left(\Gamma^{1k} G(GG' - \Sigma^J) - \Sigma^J \Gamma^k G' - G\Gamma^{1k}\Sigma^J\right)$$

$$\partial_{\zeta_k} H|_{\zeta=0} = -i\lambda \left(\Gamma^{1k}\eta(\eta\eta' + \Sigma^J) + \Sigma^J\Gamma^k\eta' + \eta\Gamma^{1k}\Sigma^J\right)$$

$$\partial_{\zeta_k\zeta_k} H_{ji} = \lambda e^C \left((\Gamma^{1k}G\Gamma^{1k}G - \Gamma^{1k}\Sigma^J\Gamma^k)(GG' - \Sigma^J) - 2\Gamma^{1k}G(\Sigma^J\Gamma^kG' + G\Gamma^{1k}\Sigma^J) + 2\Sigma^J\Gamma^k\Gamma^{1k}\Sigma^J\right)$$

$$\partial_{\zeta_k\zeta_k} H_{ji}|_{\zeta=0} = \lambda \left((\Gamma^{1k}\eta\Gamma^{1k}\eta + \Gamma^{1k}\Sigma^J\Gamma^k)(\eta\eta' + \Sigma^J) + 2\Gamma^{1k}\eta(\Sigma^J\Gamma^k\eta' + \eta\Gamma^{1k}\Sigma^J) + 2\Sigma^J\Gamma^k\Gamma^{1k}\Sigma^J\right)$$

Then, the third moment is given by

$$\begin{aligned}
-i^{-1}\partial_{\zeta_k\zeta\zeta'}\Psi_t\Big|_{\zeta=0} &= -i^{-1}\partial_{\zeta_k}\partial_{\zeta\zeta'}\Psi_t\Big|_{\zeta=0} \\
&= -i^{-1}\partial_{\zeta_k}\Psi_t(FF' + H)\Big|_{\zeta=0} \\
&= -i^{-1}\Psi_t\left\{\Gamma^{1k}F(FF' + H) + \partial_{\zeta_k}FF' + F\partial_{\zeta_k}F' + \partial_{\zeta_k}H\right\}\Big|_{\zeta=0} \\
&= \lambda\left(\Gamma^{1k}\eta(\eta\eta' + \Sigma^J) + \Sigma^J\Gamma^k\eta' + \eta\Gamma^{1k}\Sigma^J\right)
\end{aligned}$$

This is an $N \times N$ matrix for each value of k. In order to find element $(m_3)_{ji}$ of the coskewness matrix, we must select the row j, column i with $k = i$ in the above expression. Hence,

$$\begin{aligned}
(m_3)_{ji} &= \lambda(\eta_i(\eta\eta' + \Sigma^J) + \Sigma^J\Gamma^i\eta' + \eta\Gamma^{1i}\Sigma^J)_{ji} \\
&= \lambda(\eta_i(\eta_j\eta_i + \nu_j\nu_i) + \nu_j\nu_i\eta_i + \eta_j\nu_i^2) \\
&= \lambda(2\nu_j\nu_i\eta_i + \eta_j(\eta_i^2 + \nu_i^2))
\end{aligned}$$

And the fourth moment is,

$$\begin{aligned}
\partial_{\zeta_k\zeta_k\zeta\zeta'}\Psi_t|_{\zeta=0} &= \partial_{\zeta_k}\Psi_t\left\{\Gamma^{1k}F(FF' + H) + \partial_{\zeta_k}FF' + F\partial_{\zeta_k}F' + \partial_{\zeta_k}H\right\}\Big|_{\zeta=0} \\
&= \Psi_t\left\{\Gamma^{1k}F(\Gamma^{1k}F(FF' + H) + \partial_{\zeta_k}FF' + F\partial_{\zeta_k}F' + \partial_{\zeta_k}H) + \Gamma^{1k}H\Gamma^k(FF' + H)\right. \\
&\quad + \Gamma^{1k}F(\partial_{\zeta_k}FF' + F\partial_{\zeta_k}F' + \partial_{\zeta_k}H) + \partial_{\zeta_k\zeta_k}FF' + 2\partial_{\zeta_k}F\partial_{\zeta_k}F' + F\partial_{\zeta_k\zeta_k}F'
\end{aligned}$$

$$+\partial_{\zeta_k\zeta_k}H\}|_{\zeta=0}$$

$$\partial_{\zeta_k\zeta_k\zeta\zeta'}\Psi_t|_{\zeta=0} = \Gamma^{1k}H\Big|_{\zeta=0}\Gamma^kH\Big|_{\zeta=0} + 2\,H|_{\zeta=0}\,\Gamma^k\Gamma^{1k}H\Big|_{\zeta=0} + \partial_{\zeta_k\zeta_k}H|_{\zeta=0}$$

The first two terms account for 3 times the covariance. Then, the excess kurtosis must be given by

$$\begin{aligned} m_4 &= \partial_{\zeta_k\zeta_k}H|_{\zeta=0} \\ &= \lambda\left((\Gamma^{1k}\eta\Gamma^{1k}\eta + \Gamma^{1k}\Sigma^J\Gamma^k)(\eta\eta' + \Sigma^J) + 2\Gamma^{1k}\eta(\Sigma^J\Gamma^k\eta' + \eta\Gamma^{1k}\Sigma^J) + 2\Sigma^J\Gamma^k\Gamma^{1k}\Sigma^J\right) \\ &= \lambda\left((\eta_k^2 + \nu_k^2)(\eta\eta' + \Sigma^J) + 2\eta_k((\Sigma^J)^k\eta' + \eta\Sigma_k^J) + 2(\Sigma^J)^k\Sigma_k^J\right) \end{aligned}$$

This is a $N \times N$ matrix for each value of k, in order to find element $(m_4)_{ji}$ of the cokurtosis matrix we must select the row j, column i with $k = i$ in the above expression, hence:

$$\begin{aligned} (m_4)_{ji} &= \lambda\left((\eta_k^2 + \nu_k^2)(\eta\eta' + \Sigma^J) + 2\eta_k((\Sigma^J)^k\eta' + \eta\Sigma_k^J) + 2(\Sigma^J)^k\Sigma_k^J\right)_{ji} \\ &= \lambda\left((\eta_i^2 + \nu_i^2)(\eta_j\eta_i + \nu_j\nu_i) + 2\eta_i(\nu_j\nu_i\eta_i + \eta_i\nu_j\nu_i) + 2\nu_j\nu_i^3\right) \\ &= \lambda(\eta_j\eta_i^3 + 3\eta_j\eta_i\nu_i^2 + 3\nu_j\nu_i\eta_i^2 + 3\nu_j\nu_i^3) \end{aligned}$$

Therefore, the standardized central moments for assets j and i are given by

$$\begin{aligned} \text{covariance}_{ji} &= \sigma_{ji} + \lambda(\eta_j\eta_i + \nu_j\nu_i) \\ \text{coskewness}_{ji} &= \frac{\lambda(2\eta_i\nu_j\nu_i + \eta_j(\eta_i^2 + \nu_i^2))}{\sigma_j\sigma_i^2} \\ \text{excess kurtosis}_{jj} &= \frac{\lambda(\eta_j^4 + 6\eta_j^2\nu_j^2 + 3\nu_j^4)}{\sigma_j^4} \end{aligned}$$

Table 1

Descriptive Statistics for the Fama-French Growth and Value Portfolios

Panel A of this table reports the first four moments of the monthly returns for the growth and value portfolios constructed by Fama-French using their ten book-to-market sorted portfolios, where the first portfolio is the growth portfolio, portfolio BE-ME 5 is the intermediate portfolio, and the last portfolio is the value portfolio. The data for the full period goes from January 1982 to October 2010, while the sub-periods contain data from January 1982 to February 1997 and from March 1998 to October 2010. Panel B of the table gives the correlation coefficients among the monthly returns for the growth, intermediate, and value portfolios.

Panel A. Moments of the Monthly Returns

	Jan 82-Oct 10			Mar 97-Oct 10			Jan 82-Feb 97		
	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value
Mean %	0.493	1.306	1.813	0.537	1.113	1.618	0.454	1.479	1.989
Volatility %	7.681	5.356	6.314	9.053	6.046	7.452	6.217	4.657	5.091
Skewness	0.043	−0.768	−0.006	0.250	−0.438	−0.033	−0.560	−1.298	0.190
Kurtosis	3.508	3.898	4.376	2.478	1.805	2.306	4.071	8.037	8.716

Panel B. Correlation Coefficients

	Jan 82-Oct 10			Mar 97-Oct 10			Jan 82-Feb 97		
	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value
Growth	1.000	0.892	0.819	1.000	0.868	0.802	1.000	0.938	0.855
BE-ME 5		1.000	0.890		1.000	0.899		1.000	0.879
Value			1.000			1.000			1.000

Table 2

Parameter Estimates for the Returns Processes with Jumps

Panel A of this table reports estimates of the parameter for the jump-diffusion portfolio returns obtained by minimizing the square of the difference between the theoretical moment conditions and the moments implied by the data. η_j, for $j = 1, 2, 3$, refers to the mean of the jump size for the growth, intermediate, and value portfolios respectively. ν_j, for j = 1, 2, 3, is the volatility of the jump size for the growth, intermediate, and value portfolios respectively. These estimates are given in percentage terms. *Avg* gives the average magnitude of the average and volatility jump sizes across portfolios, λ represents the frequency of the jumps, and *Year (s)* is the number of years necessary to observe a jump according to the jump-diffusion process. Panel B contains the reconstructed moments that are obtained by substituting the parameter estimates in the theoretical model, and the sample moments.

Panel A. Parameter Estimates for the Jump Process

Period	η_1	η_2	η_3	Avg	ν_1	ν_2	ν_3	Avg	λ	Year (s)
Jan 82-Oct 10	−0.035	−3.479	−0.409	−1.307	12.804	8.519	11.117	10.814	0.152	0.55
Mar 97-Oct 10	0.768	−1.218	−0.208	−0.243	8.761	5.292	7.091	7.048	0.938	0.09
Jan 82-Feb 97	−7.112	−10.088	0.529	−5.559	14.506	10.859	15.891	13.752	0.031	2.72

Panel B. Comparison between Higher Order Sample Moments of Returns and the Jumps Process Theoretical Higher Order Moments

	Jan 82-Oct 10			Mar 97-Oct 10			Jan 82-Feb 97		
	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value	Growth	BE-ME 5	Value
Sample Skewness	0.043	−0.768	−0.006	0.250	−0.438	−0.033	−0.560	−1.298	0.190
Model Skewness	−0.006	−0.787	−0.091	0.226	−0.438	−0.093	−0.616	−1.390	0.091
Sample Kurtosis	3.508	3.898	4.376	2.478	1.805	2.306	4.071	8.037	8.716
Model Kurtosis	3.498	3.894	4.359	2.512	1.817	2.287	4.066	8.045	8.674

Table 3

Portfolio Weights

Panel A of this table gives the portfolio weights for an investor who selects investments in three equity portfolios (growth, intermediate, value) and the riskless asset to maximize expected power utility of terminal wealth with constant relative risk aversion. The investor optimizes expected utility ignoring systemic jumps and assumes a pure diffusion process for portfolio returns. γ is the relative risk aversion coefficient, G, 5, and V indicate the growth, intermediate portfolio, and value portfolios respectively. Risky is the total weight given to all three equity portfolios. Panel B reports the optimal weights when the investor recognized systemic jumps, and Panel C contains the differences between the optimal weights for pure diffusion and the weights for the jump-diffusion process.

Panel A. Diffusion Weights

γ	G	5	V	Riskless	Risky
1	−9.27	7.39	6.95	−4.07	5.07
2	−4.64	3.69	3.48	−1.54	2.54
3	−3.09	2.46	2.32	−0.69	1.69
4	−2.32	1.85	1.74	−0.27	1.27
5	−1.85	1.48	1.39	−0.01	1.01
6	−1.55	1.23	1.16	0.15	0.85
7	−1.32	1.06	0.99	0.28	0.72
8	−1.16	0.92	0.87	0.37	0.63
9	−1.03	0.82	0.77	0.44	0.56
10	−0.93	0.74	0.70	0.49	0.51

Jan 82-Oct 10

γ	G	5	V	Riskless	Risky
1	−3.96	1.83	4.55	−1.42	2.42
2	−1.98	0.91	2.28	−0.21	1.21
3	−1.32	0.61	1.52	0.19	0.81
4	−0.99	0.46	1.14	0.39	0.61
5	−0.79	0.37	0.91	0.52	0.48
6	−0.66	0.30	0.76	0.60	0.40
7	−0.57	0.26	0.65	0.65	0.35
8	−0.49	0.23	0.57	0.70	0.30
9	−0.44	0.20	0.51	0.73	0.27
10	−0.40	0.18	0.46	0.76	0.24

Mar 97-Oct 10

γ	G	5	V	Riskless	Risky
1	−30.03	28.68	14.02	−11.68	12.68
2	−15.01	14.34	7.01	−5.34	6.34
3	−10.01	9.56	4.67	−3.23	4.23
4	−7.51	7.17	3.51	−2.17	3.17
5	−6.01	5.74	2.80	−1.54	2.54
6	−5.00	4.78	2.34	−1.11	2.11
7	−4.29	4.10	2.00	−0.81	1.81
8	−3.75	3.59	1.75	−0.58	1.58
9	−3.34	3.19	1.56	−0.41	1.41
10	−3.00	2.87	1.40	−0.27	1.27

Jan 82-Feb 97

Panel B. Jump-diffusion (Systemic) Weights

γ	G	5	V	Riskless	Risky
1	−8.88	5.87	7.17	−3.16	4.16
2	−4.49	3.14	3.56	−1.21	2.21
3	−3.01	2.14	2.36	−0.50	1.50
4	−2.26	1.62	1.77	−0.13	1.13
5	−1.81	1.31	1.42	0.09	0.91
6	−1.51	1.09	1.18	0.24	0.76
7	−1.29	0.94	1.01	0.35	0.65
8	−1.13	0.82	0.88	0.43	0.57
9	−1.01	0.73	0.79	0.49	0.51
10	−0.91	0.66	0.71	0.54	0.46

Jan 82-Oct 10

γ	G	5	V	Riskless	Risky
1	−3.93	1.66	4.54	−1.26	2.26
2	−1.97	0.85	2.27	−0.15	1.15
3	−1.31	0.57	1.51	0.23	0.77
4	−0.99	0.43	1.14	0.42	0.58
5	−0.79	0.35	0.91	0.53	0.47
6	−0.66	0.29	0.76	0.61	0.39
7	−0.56	0.25	0.65	0.67	0.33
8	−0.49	0.22	0.57	0.71	0.29
9	−0.44	0.19	0.50	0.74	0.26
10	−0.39	0.17	0.45	0.77	0.23

Mar 97-Oct 10

γ	G	5	V	Riskless	Risky
1	−26.70	17.44	18.50	−8.24	9.24
2	−13.74	10.09	8.60	−3.94	4.94
3	−9.26	7.06	5.58	−2.38	3.38
4	−6.98	5.43	4.13	−1.57	2.57
5	−5.60	4.41	3.28	−1.08	2.08
6	−4.68	3.71	2.71	−0.74	1.74
7	−4.02	3.20	2.32	−0.50	1.50
8	−3.52	2.81	2.02	−0.32	1.32
9	−3.13	2.51	1.79	−0.17	1.17
10	−2.82	2.27	1.61	−0.06	1.06

Jan 82-Feb 97

Panel C. Diffusion - Jump Weight Differences

γ	G	5	V	Riskless	Risky
1	−0.394	1.518	−0.216	−0.908	0.908
2	−0.143	0.551	−0.078	−0.330	0.330
3	−0.084	0.323	−0.046	−0.193	0.193
4	−0.059	0.226	−0.032	−0.135	0.135
5	−0.045	0.173	−0.024	−0.103	0.103
6	−0.036	0.139	−0.020	−0.083	0.083
7	−0.030	0.117	−0.017	−0.070	0.070
8	−0.026	0.101	−0.014	−0.060	0.060
9	−0.023	0.088	−0.012	−0.053	0.053
10	−0.020	0.078	−0.011	−0.047	0.047

Jan 82-Oct 10

γ	G	5	V	Riskless	Risky
1	−0.021	0.167	0.015	−0.161	0.161
2	−0.008	0.062	0.006	−0.060	0.060
3	−0.005	0.037	0.003	−0.035	0.035
4	−0.003	0.026	0.002	−0.025	0.025
5	−0.002	0.020	0.002	−0.019	0.019
6	−0.002	0.016	0.001	−0.016	0.016
7	−0.002	0.013	0.001	−0.013	0.013
8	−0.001	0.012	0.001	−0.011	0.011
9	−0.001	0.010	0.001	−0.010	0.010
10	−0.001	0.009	0.001	−0.009	0.009

Mar 97-Oct 10

γ	G	5	V	Riskless	Risky
1	−3.327	11.241	−4.478	−3.437	3.437
2	−1.275	4.255	−1.585	−1.394	1.394
3	−0.752	2.497	−0.905	−0.840	0.840
4	−0.526	1.743	−0.622	−0.594	0.594
5	−0.402	1.331	−0.471	−0.458	0.458
6	−0.325	1.074	−0.377	−0.371	0.371
7	−0.272	0.899	−0.315	−0.312	0.312
8	−0.234	0.773	−0.269	−0.269	0.269
9	−0.205	0.677	−0.235	−0.236	0.236
10	−0.183	0.602	−0.209	−0.211	0.211

Jan 82-Feb 97

Table 4

Composition of the Risky Portfolios

Panel A of this table contains the composition of the risky portfolios for alternative levels of relative risk aversion, which is obtained by dividing the weight for each portfolio by the total amount invested in risky portfolios. It assumes the jump-diffusion process. Panel B contains the same results assuming the pure diffusion process. It should be noted that the weights do not depend on the level of relative risk aversion when the simple diffusion process is employed in the estimation.

Panel A. Jump-diffusion (Systemic) Weights

Mar 97-Oct 10

γ	Growth	BE-ME 5	Value	Spread
1	−1.73887	0.73449	2.00438	3.74324
2	−1.71023	0.73976	1.97047	3.68070
3	−1.70098	0.74148	1.95950	3.66048
4	−1.69639	0.74241	1.95398	3.65037
5	−1.69365	0.74290	1.95075	3.64440
6	−1.69186	0.74311	1.94875	3.64061
7	−1.69055	0.74349	1.94706	3.63761
8	−1.68959	0.74359	1.94600	3.63559
9	−1.68886	0.74384	1.94502	3.63387
10	−1.68822	0.74396	1.94427	3.63249

Jan 82-Feb 97

γ	Growth	BE-ME 5	Value	Spread
1	−2.89047	1.88793	2.00254	4.89302
2	−2.77956	2.04046	1.73909	4.51865
3	−2.73519	2.08690	1.64829	4.38348
4	−2.71186	2.10849	1.60337	4.31523
5	−2.69756	2.12084	1.57672	4.27428
6	−2.68791	2.12879	1.55912	4.24703
7	−2.68097	2.13433	1.54664	4.22761
8	−2.67574	2.13840	1.53734	4.21308
9	−2.67166	2.14153	1.53013	4.20179
10	−2.66839	2.14399	1.52439	4.19278

Panel B. Diffusion Weights

Period	Growth	BE-ME 5	Value	Spread
Mar 97-Oct 10	−1.63194	0.75446	1.87748	3.50942
Jan 82-Feb 97	−2.36914	2.26287	1.10627	3.47541

Table 5

Certainty Equivalent Cost of Ignoring Systemic Jumps

This table reports the certainty equivalent costs (CEQ) of ignoring systemic jumps calculated as the additional wealth per $1,000 of investment needed to raise the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal investment strategy. The table contains the CEQ for investment horizons of 1 to 5 years, and for levels of relative risk aversion (γ) from 2 to 10.

Jan 82-Oct 10

γ	One year	Two years	Three years	Four years	Five years
2	0.40	0.79	1.19	1.59	1.99
3	0.19	0.38	0.58	0.77	0.96
4	0.12	0.24	0.36	0.49	0.61
5	0.09	0.17	0.26	0.35	0.44
6	0.07	0.14	0.20	0.27	0.34
7	0.06	0.11	0.17	0.22	0.28
8	0.05	0.09	0.14	0.19	0.23
9	0.04	0.08	0.12	0.16	0.20
10	0.04	0.07	0.11	0.14	0.18

Mar 97-Oct 10

γ	One year	Two years	Three years	Four years	Five years
2	0.01	0.03	0.04	0.06	0.07
3	0.01	0.01	0.02	0.03	0.04
4	0.00	0.01	0.01	0.02	0.02
5	0.00	0.01	0.01	0.01	0.02
6	0.00	0.01	0.01	0.01	0.01
7	0.00	0.00	0.01	0.01	0.01
8	0.00	0.00	0.01	0.01	0.01
9	0.00	0.00	0.00	0.01	0.01
10	0.00	0.00	0.00	0.01	0.01

Jan 82-Feb 97

γ	One year	Two years	Three years	Four years	Five years
2	48.40	99.14	152.34	208.12	266.59
3	11.50	23.12	34.88	46.78	58.81
4	6.01	12.06	18.14	24.26	30.42
5	3.94	7.90	11.88	15.87	19.88
6	2.89	5.80	8.71	11.62	14.55
7	2.27	4.54	6.82	9.10	11.39
8	1.86	3.72	5.58	7.45	9.32
9	1.57	3.14	4.71	6.29	7.87
10	1.36	2.71	4.07	5.43	6.80

TABLE 6

Jump Diffusion Parameter and Weight Estimates March 1997-October 2010 for the Sample Average Annualized Riskless Rate of 3%

Panel A reports the estimates of the parameters for the jump-diffusion portfolio returns for an annualized riskless rate of 3%. Panel B contains the reconstructed moments and the sample moments. Panel C these tables gives the portfolio weights for an investor who ignoring systemic jumps, the optimal weights when the investor recognized systemic jumps and the certainty equivalent costs (CEQ) of ignoring systemic jumps calculated as the additional wealth per $1,000 of investment needed to raise the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal investment strategy.

Panel A. Parameter Estimates for the Jump Process

Parameter	η_1	η_2	η_3	Avg	ν_1	ν_2	ν_3	Avg	λ	Year (s)
Value	0.707	−1.033	−0.211	−0.179	8.161	4.953	6.618	6.577	1.231	0.07

Panel B. Comparison between Higher Order Sample Moments of Returns and the Jumps Process Theoretical Higher Order Moments

	Growth	BE-ME 5	Value
Sample Skewness	0.250	−0.438	−0.033
Model Skewness	0.235	−0.429	−0.082
Sample Kurtosis	2.478	1.805	2.306
Model Kurtosis	2.476	1.808	2.301

Panel C. Diffusion Weights, Jump-diffusion Weights and Certainty Equivalent Costs (CEQ)

Diffusion weights

γ	G	5	V	Riskless	Risky
1	−4.45	3.22	4.45	−2.22	3.22
2	−2.23	1.61	2.23	−0.61	1.61
3	−1.48	1.07	1.48	−0.07	1.07
4	−1.11	0.80	1.11	0.20	0.80
5	−0.89	0.64	0.89	0.36	0.64
6	−0.74	0.54	0.74	0.46	0.54
7	−0.64	0.46	0.64	0.54	0.46
8	−0.56	0.40	0.56	0.60	0.40
9	−0.49	0.36	0.49	0.64	0.36
10	−0.45	0.32	0.45	0.68	0.32

Jump Diffusion Weights

γ	G	5	V	Riskless	Risky
1	−4.42	2.95	4.42	−1.95	2.95
2	−2.22	1.51	2.21	−0.51	1.51
3	−1.48	1.01	1.48	−0.01	1.01
4	−1.11	0.76	1.11	0.24	0.76
5	−0.89	0.61	0.89	0.39	0.61
6	−0.74	0.51	0.74	0.49	0.51
7	−0.63	0.44	0.63	0.56	0.44
8	−0.55	0.38	0.55	0.62	0.38
9	−0.49	0.34	0.49	0.66	0.34
10	−0.44	0.31	0.44	0.69	0.31

Certainty Equivalent Cost

γ	One year	Two years	Three years	Four years	Five years
2	0.04	0.08	0.12	0.17	0.21
3	0.02	0.04	0.06	0.09	0.11
4	0.01	0.03	0.04	0.06	0.07
5	0.01	0.02	0.03	0.04	0.05
6	0.01	0.02	0.02	0.03	0.04
7	0.01	0.01	0.02	0.03	0.03
8	0.01	0.01	0.02	0.02	0.03
9	0.00	0.01	0.01	0.02	0.02
10	0.00	0.01	0.01	0.02	0.02

Table 7

Descriptive Statistics for the MSCI Growth and Value Portfolios across Geographic Regions

Panel A of this table reports the first four moments of the monthly returns for the growth and value portfolios constructed by MSCI across three geographic world regions, Europe, North America, and the Pacific. The data for the full period goes from January 1982 to September 2010, while the sub-periods contain data from January 1982 to February 1997 and from March 1998 to September 2010. Panel B of the table gives the correlation coefficients among the monthly returns for the value and growth portfolios across the three geographic regions.

Panel A. Moments of the Monthly Returns

	Value			Growth		
	Europe	North America	Pacific	Europe	North America	Pacific
Jan 82 - Sep 10						
Mean %	0.719	0.572	0.627	0.660	0.712	0.312
Volatility %	5.430	4.513	5.849	5.042	5.034	6.715
Skewness	−0.918	−1.004	−0.002	−0.817	−0.908	−0.139
Kurtosis	2.805	3.570	0.593	2.274	2.921	0.830
Mar 97 - Sep 10						
Mean %	0.299	0.204	0.156	0.217	0.320	−0.088
Volatility %	6.204	4.983	5.406	5.418	5.559	6.032
Skewness	−0.882	−0.886	−0.039	−0.766	−0.782	−0.629
Kurtosis	2.288	1.805	1.003	1.569	1.006	0.449
Jan 82 - Feb 97						
Mean %	1.082	0.926	1.076	1.044	1.099	0.691
Volatility %	4.632	4.031	6.206	4.673	4.487	7.281
Skewness	−0.740	−1.084	−0.041	−0.802	−1.010	0.056
Kurtosis	2.373	6.605	0.341	3.158	6.416	0.766

Panel B. Correlation Coefficients

	Value			Growth		
	Europe	North America	Pacific	Europe	North America	Pacific
Jan 82 - Sep 10						
Europe	1.000	0.754	0.608	1.000	0.716	0.581
North America		1.000	0.455		1.000	0.472
Pacific			1.000			1.000
Mar 97 - Sep 10						
Europe	1.000	0.858	0.655	1.000	0.814	0.725
North America		1.000	0.631		1.000	0.728
Pacific			1.000			1.000
Jan 82 - Feb 97						
Europe	1.000	0.605	0.586	1.000	0.595	0.470
North America		1.000	0.291		1.000	0.249
Pacific			1.000			1.000

Table 8

Parameter Estimates for the Returns Processes with Jumps

Panel A of this table reports estimates of the parameter for the jump-diffusion portfolio returns obtained by minimizing the square of the difference between the theoretical moment conditions and the moments implied by the data. For the value and growth portfolios separately, η_j , for $j = 1, 2, 3$, refers to the mean of the jump size for Europe, North America, and the Pacific respectively. ν_j , for $j = 1, 2, 3$, is the volatility of the jump size for Europe, North America, and the Pacific respectively. These estimates are given in percentage terms. *Avg* gives the average magnitude of the average and volatility jump sizes across the three regions, λ represents the frequency of the jumps, and *Year (s)* is the number of years necessary to observe a jump according to the jump-diffusion process. Panel B contains the reconstructed moments that are obtained by substituting the parameter estimates in the theoretical model, and the sample moments.

Panel A. Parameter Estimates for the Jump Process

Series	Period	η_1	η_2	η_3	Avg	ν_1	ν_2	ν_3	Avg	λ	Year (s)
Value	Jan 82 - Sep 10	−5.487	−4.269	−1.244	−3.667	6.746	6.281	6.097	6.375	0.159	0.52
	Mar 97 - Sep 10	−3.810	−3.417	−0.661	−2.629	5.984	4.243	4.946	5.058	0.470	0.18
	Jan 82 - Feb 97	−9.280	−5.130	−2.436	−5.615	4.936	9.139	7.204	7.093	0.050	1.66
Growth	Jan 82 - Sep 10	−5.138	−4.353	−1.204	−3.565	5.445	6.448	7.344	6.412	0.183	0.46
	Mar 97 - Sep 10	−4.680	−6.250	−5.658	−5.529	3.699	1.298	1.852	2.283	0.483	0.17
	Jan 82 - Feb 97	−6.803	−4.659	2.983	−2.826	7.181	9.912	9.900	8.997	0.062	1.35

Panel B. Comparison between Higher Order Sample Moments of Returns and the Jumps Process Theoretical Higher Order Moments

	Value			Growth		
	Europe	North America	Pacific	Europe	North America	Pacific
Jan 82 - Sep 10						
Mean %	−0.918	−1.004	−0.002	−0.817	−0.908	−0.139
Volatility %	−0.908	−1.008	−0.112	−0.845	−0.896	−0.119
Skewness	2.805	3.570	0.593	2.274	2.921	0.830
Kurtosis	2.804	3.568	0.610	2.271	2.924	0.827
Mar 97 - Sep 10						
Mean %	−0.882	−0.886	−0.039	−0.766	−0.782	−0.629
Volatility %	−0.915	−0.853	−0.145	−0.895	−0.776	−0.527
Skewness	2.288	1.805	1.003	1.569	1.006	0.449
Kurtosis	2.277	1.807	1.023	1.593	0.976	0.628
Jan 82 - Feb 97						
Mean %	−0.740	−1.084	−0.041	−0.802	−1.010	0.056
Volatility %	−0.745	−1.087	−0.083	−0.825	−1.006	0.144
Skewness	2.373	6.605	0.341	3.158	6.416	0.766
Kurtosis	2.372	6.606	0.337	3.157	6.419	0.748

Table 9

Portfolio Weights

Panel A of this table gives the portfolio weights from March 1997 to September 2010 for an investor who selects investments in equity portfolios (either value or growth) across three geographic regions and the riskless asset to maximize expected power utility of terminal wealth with constant relative risk aversion. The exercise is performed separately for the value and growth portfolios. The investor optimizes expected utility ignoring systemic jumps and assumes a pure diffusion process for portfolio returns. γ is the relative risk aversion coefficient. Risky is the total weight given to all three equity portfolios. Panel B reports the optimal weights when the investor recognized systemic jumps, and Panel C contains the differences between the optimal weights for pure diffusion and the weights for the jump-diffusion process.

Panel A. Diffusion Weights

γ	Europe	North America	Pacific	Riskless	Risky
1	1.56	−2.08	−1.14	2.66	−1.66
2	0.78	−1.04	−0.57	1.83	−0.83
3	0.52	−0.69	−0.38	1.55	−0.55
4	0.39	−0.52	−0.29	1.41	−0.41
5	0.31	−0.42	−0.23	1.33	−0.33
6	0.26	−0.35	−0.19	1.28	−0.28
7	0.22	−0.30	−0.16	1.24	−0.24
8	0.19	−0.26	−0.14	1.21	−0.21
9	0.17	−0.23	−0.13	1.18	−0.18
10	0.16	−0.21	−0.11	1.17	−0.17

Value

γ	Europe	North America	Pacific	Riskless	Risky
1	−0.31	1.66	−2.53	2.18	−1.18
2	−0.15	0.83	−1.26	1.59	−0.59
3	−0.10	0.55	−0.84	1.39	−0.39
4	−0.08	0.41	−0.63	1.29	−0.29
5	−0.06	0.33	−0.51	1.24	−0.24
6	−0.05	0.28	−0.42	1.20	−0.20
7	−0.04	0.24	−0.36	1.17	−0.17
8	−0.04	0.21	−0.32	1.15	−0.15
9	−0.03	0.18	−0.28	1.13	−0.13
10	−0.03	0.17	−0.25	1.12	−0.12

Growth

Panel B. Jump-diffusion (Systemic) Weights

γ	Europe	North America	Pacific	Riskless	Risky
1	1.54	−2.10	−1.14	2.69	−1.69
2	0.77	−1.05	−0.57	1.84	−0.84
3	0.52	−0.70	−0.38	1.56	−0.56
4	0.39	−0.52	−0.28	1.42	−0.42
5	0.31	−0.42	−0.23	1.34	−0.34
6	0.26	−0.35	−0.19	1.28	−0.28
7	0.22	−0.30	−0.16	1.24	−0.24
8	0.19	−0.26	−0.14	1.21	−0.21
9	0.17	−0.23	−0.13	1.19	−0.19
10	0.16	−0.21	−0.11	1.17	−0.17

Value

γ	Europe	North America	Pacific	Riskless	Risky
1	−0.35	1.65	−2.53	2.24	−1.24
2	−0.17	0.83	−1.27	1.61	−0.61
3	−0.11	0.55	−0.84	1.41	−0.41
4	−0.08	0.41	−0.63	1.30	−0.30
5	−0.07	0.33	−0.51	1.24	−0.24
6	−0.06	0.28	−0.42	1.20	−0.20
7	−0.05	0.24	−0.36	1.17	−0.17
8	−0.04	0.21	−0.32	1.15	−0.15
9	−0.04	0.18	−0.28	1.13	−0.13
10	−0.03	0.17	−0.25	1.12	−0.12

Growth

Panel C. Diffusion Jump Weight Differences

γ	Europe	North America	Pacific	Riskless	Risky
1	0.018	0.021	−0.007	−0.032	0.032
2	0.007	0.008	−0.003	−0.012	0.012
3	0.004	0.005	−0.002	−0.007	0.007
4	0.003	0.003	−0.001	−0.005	0.005
5	0.002	0.002	−0.001	−0.004	0.004
6	0.002	0.002	−0.001	−0.003	0.003
7	0.001	0.002	−0.001	−0.003	0.003
8	0.001	0.001	−0.000	−0.002	0.002
9	0.001	0.001	−0.000	−0.002	0.002
10	0.001	0.001	−0.000	−0.002	0.002

Value

γ	Europe	North America	Pacific	Riskless	Risky
1	0.043	0.010	0.007	−0.061	0.061
2	0.016	0.004	0.003	−0.023	0.023
3	0.010	0.002	0.002	−0.014	0.014
4	0.007	0.002	0.001	−0.010	0.010
5	0.005	0.001	0.001	−0.007	0.007
6	0.004	0.001	0.001	−0.006	0.006
7	0.004	0.001	0.001	−0.005	0.005
8	0.003	0.001	0.000	−0.004	0.004
9	0.003	0.001	0.000	−0.004	0.004
10	0.002	0.001	0.000	−0.003	0.003

Growth

TABLE 10

Certainty Equivalent Cost of Ignoring Systemic Jumps

This table reports the certainty equivalent costs (CEQ) of ignoring systemic jumps calculated as the additional wealth per $1,000 of investment needed to raise the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal investment strategy. The table contains the CEQ for investment horizons of 1 to 5 years, and for levels of relative risk aversion (γ) from 2 to 10. The sample period goes from March 1997 to September 2010.

γ	One year	Two years	Three years	Four years	Five years
2	0.000	0.001	0.001	0.002	0.002
3	0.000	0.000	0.001	0.001	0.001
4	0.000	0.000	0.000	0.001	0.001
5	0.000	0.000	0.000	0.000	0.001
6	0.000	0.000	0.000	0.000	0.000
7	0.000	0.000	0.000	0.000	0.000
8	0.000	0.000	0.000	0.000	0.000
9	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000

Value

γ	One year	Two years	Three years	Four years	Five years
2	0.001	0.003	0.004	0.005	0.007
3	0.001	0.001	0.002	0.003	0.004
4	0.000	0.001	0.001	0.002	0.002
5	0.000	0.001	0.001	0.001	0.002
6	0.000	0.001	0.001	0.001	0.001
7	0.000	0.000	0.001	0.001	0.001
8	0.000	0.000	0.001	0.001	0.001
9	0.000	0.000	0.000	0.001	0.001
10	0.000	0.000	0.000	0.001	0.001

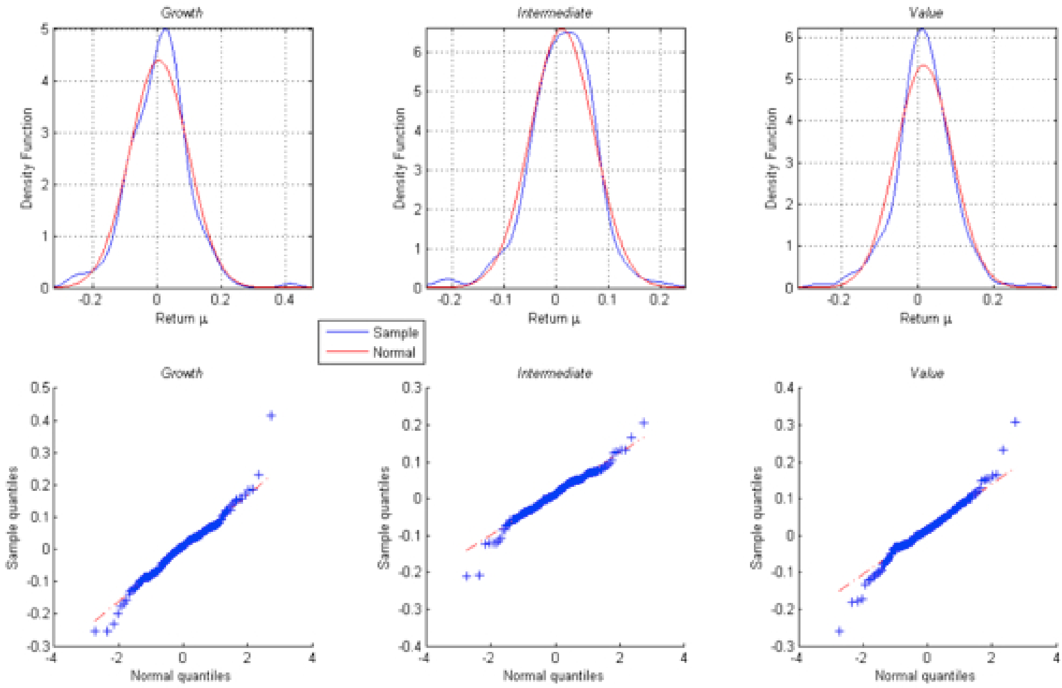
Growth

FIGURE 1

Density Functions and QQ Plots

This figure shows the density functions of the Fama-French growth, intermediate, and value portfolios and the QQ plots to assess the appropriateness of non-normal distributions.

Panel A. March 1997-October 2010



Panel B. January 1982-February 1997

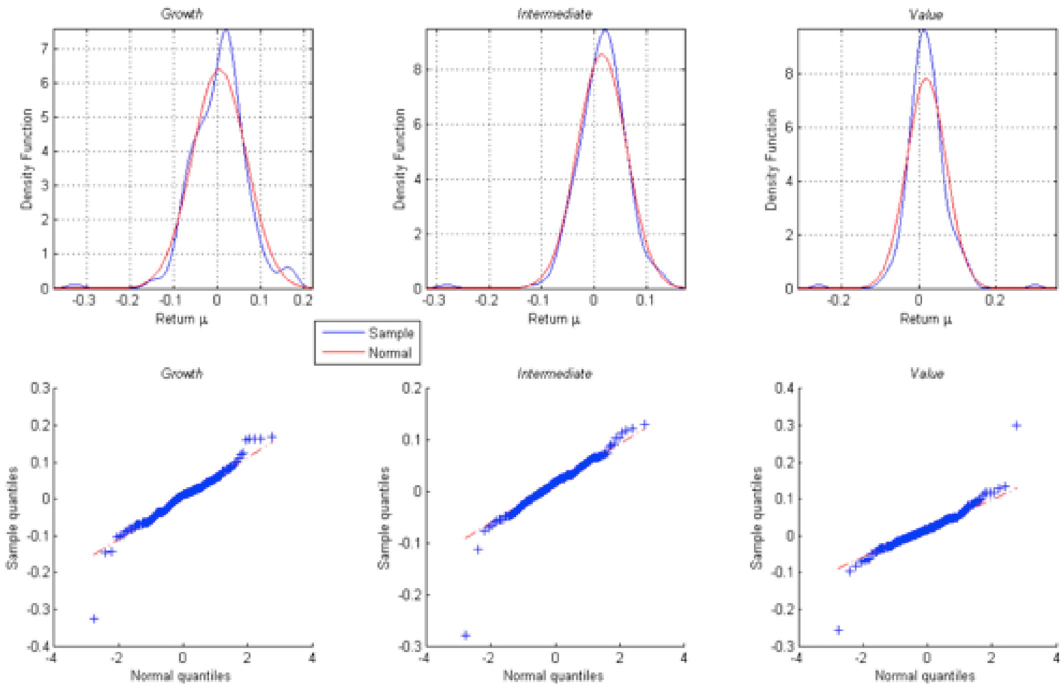
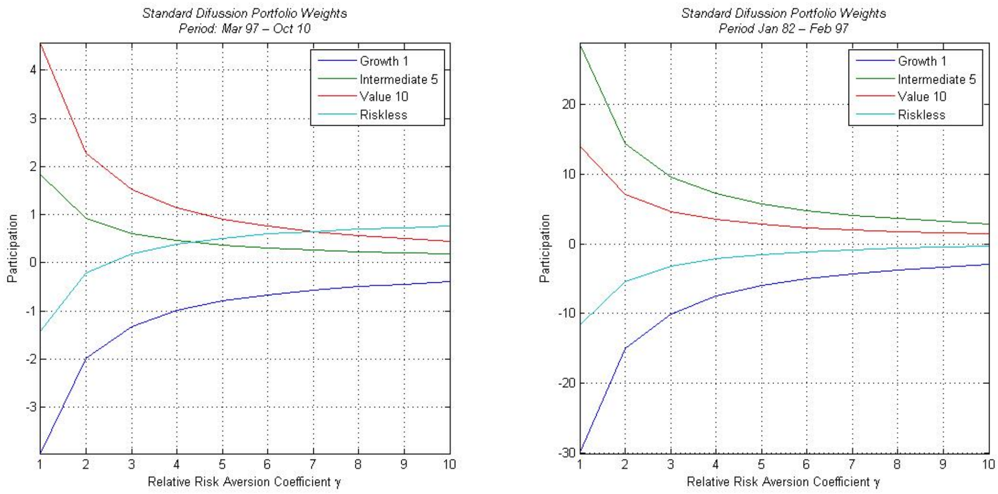


FIGURE 2

Portfolio Weights and Relative Risk Aversion

This figure shows the portfolio weights for the growth, intermediate and value Fama-French portfolios for alternative levels of relative risk aversion. Panel A contains the weights for the pure-diffusion process, while Panel B gives the weights for the jump-diffusion process. The first figure of each panel corresponds to the last sample period from March 1997 to October 2010, and the second figure of each panel corresponds to the first sample period from January 1982 to February 1997.

Panel A. Pure-diffusion Process



Panel B. Jump-diffusion Process

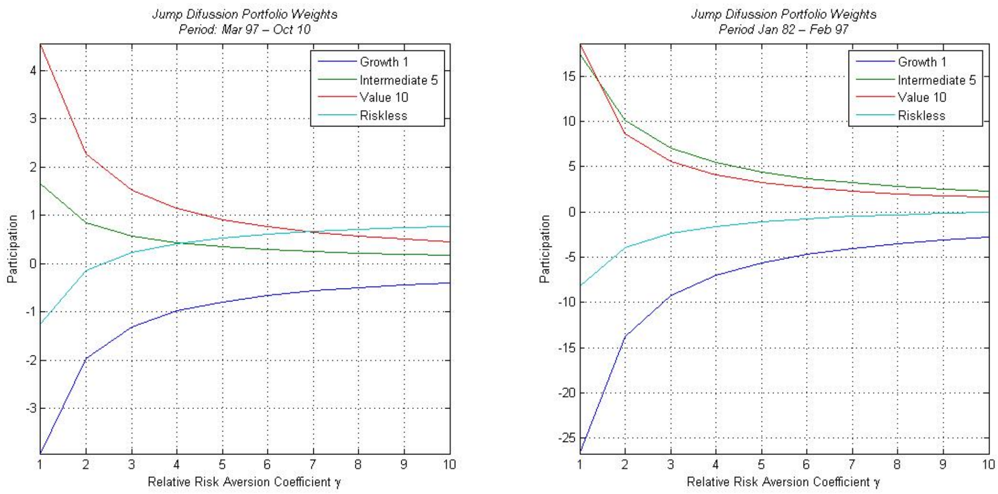
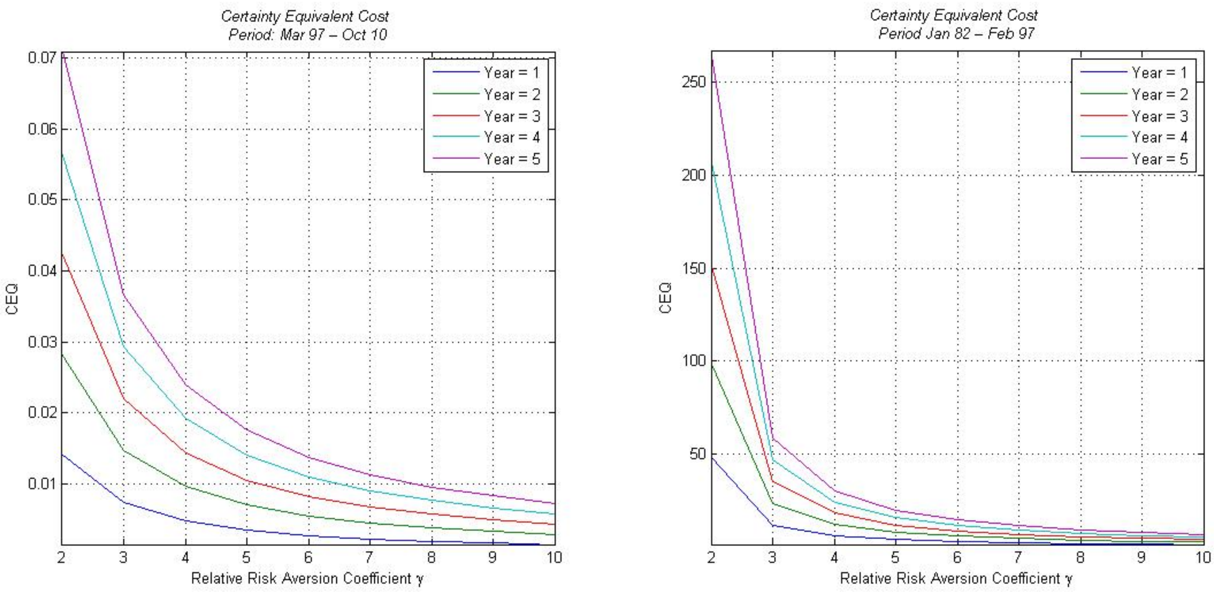


FIGURE 3

Certainty Equivalent Cost of Ignoring Systemic Jumps

This figure shows the certainty equivalent costs (CEQ) of ignoring systemic jumps calculated as the additional wealth per \$1,000 of investment needed to raised the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal investment strategy. The figure contains the CEQ for investment horizons of 1 to 5 years, and for levels of relative risk aversion (γ) from 2 to 10. The first figure corresponds to the last sample period from March 1997 to October 2010, and the second figure of each panel corresponds to the first sample period from January 1982 to February 1997.



A MARTINGALE APPROACH FOR PORTFOLIO ALLOCATION WITH STOCHASTIC VOLATILITY AND JUMPS

ABSTRACT. A market model composed of a risky asset and a riskless bond is considered. The risky security satisfies a stochastic differential equation which includes a jump component with lognormal amplitude change. Volatility is assumed stochastic and following a mean reverting process. The investor objective is to maximize the expected utility on terminal wealth, to this end, the optimal allocation rule is derived through the use of martingale and duality techniques. The weights on assets are found along with the expressions for market price of risk, market price of volatility risk and the market price for jump risk. The results are applied to market data, therefore, the conditional characteristic function associated with the market model is calculated and the first four cumulants are derived. Then, the exact expressions for the mean, standard deviation, skewness and excess kurtosis are obtained. Finally, the model parameters are estimated by means of a distance minimization using a discretization of the empirical characteristic function.

1. INTRODUCTION

One of the main concerns of investors, and a current topic of research in mathematical finance, is how to decide where to allocate their funds among the many available choices in the market. When facing such a decision, an investor should take into account all the relevant sources of risk. The studies on market return data have shown that the probability distribution of the returns changes over time, and in particular the volatility reveals itself to be stochastic. Besides modeling the price as a diffusion process, it is also desirable to include the impact of sudden, and sometimes, high shocks to price levels that have been observed in the market.

The importance of jump events and stochastic volatility in modeling the return series has been studied by Bates (1996). Authors such as Andersen, Benzoni, and

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Lund (2002) have reported the statistical importance of these factors in the context of the U.S. market returns. González, Novales, and Rubio (2011) show the relevance of jumps in European returns, characterizing the French, German and Spanish equity indexes. Estimation and comparison of different specifications has given similar results, see for example Bakshi, Cao, and Chen (1997), Chacko and Viceira (2003), Chernov, Gallant, Ghysels, and Tauchen (2003), and Eraker, Johannes, and Polson (2003). All these models aim to explain a number of statistical facts observed in the return series such as, among others, high negative skewness and high kurtosis (leptokurtic behavior). These factors are of importance given their influence on the investor's decisions at the moment of buying risky assets, so much so that portfolios including these two features are less loaded with risky securities.

The aforementioned stylized facts have had an important influence on the literature on portfolio allocation. In this sense, Das and Uppal (2004) consider a model for a market with a riskless security and multiple risky assets including systemic jumps. Chacko and Viceira (2005) solve the optimal consumption problem by assuming stochastic volatility but, for mathematical convenience, they specify a mean reverting precision process[1]. This leads to an optimal portfolio rule depending linearly on the precision. In this line, Liu, Longstaff, and Pan (2003) address the problem of optimal allocation in the presence of stochastic volatility and in that of event risk in stock prices and volatility. They assume that the risk premium is proportional to the instantaneous volatility, thus the portfolio weight does not depend on volatility.

In this paper we solve the optimal allocation problem for an investor who takes into account the occurrence of unexpected and potentially high events and whose investment set changes due to stochastic volatility. We propose a market model made up of two securities: a riskless asset and a risky asset. The riskless asset, which can be regarded as a bank account or a riskless bond, grows at a constant interest rate. The risky asset satisfies a stochastic differential equation bearing a diffusion term, a Brownian motion, and a jump term, a Poisson process. On the occurrence of

[1]The precision is the inverse of volatility.

an unexpected event, the asset price changes by a lognormally distributed amount. The stochastic volatility follows a mean reverting square root process, where the diffusion innovations are correlated with asset innovations, i.e., changes in volatility are correlated with changes in asset returns. The goal of the investor is to maximize the expected utility of terminal wealth, and as a result, the optimal allocation rule is obtained along with explicit expressions for market price of risk, market price of volatility risk and market price of jump risk. Moreover, the terminal wealth is found.

The studies mentioned above use a dynamic programming approach in solving the allocation problem. By contrast, and in order to deal with the market incompleteness caused by the discontinuities in asset returns and stochastic volatility, we approach the problem using martingale and duality techniques. These methods have long been used to resolve portfolio optimization problems. Jeanblanc and Pontier (1990) consider the consumption - investment problem for prices evolving stochastically and with jump components. They solve the portfolio problem for three assets and establish a Black - Scholes formula for option pricing. Karatzas, Lehoczky, Shreve, and Xu (1991) solve the problem of maximizing expected utility on terminal wealth in incomplete markets by introducing *fictitious* stocks to complete them. He and Pearson (1991) study the consumption - terminal wealth portfolio problem for several stocks in an incomplete market and also add a set of state variables on which the security prices depend. Bardhan and Chao (1996) address the market incompleteness that arises when introducing jumps in the securities. They construct a financial policy for consumption plans and apply it to give bounds to the fair price of contingent claims. Bellamy (2001) considers a small investor in an incomplete market with jumps who maximizes the expected utility on terminal wealth. More recently Callegaro and Vargiolu (2009) analyze a pure jump incomplete market, and Buraschi, Porchia, and Trojani (2010) develop a multivariate portfolio problem in an incomplete market with stochastic volatility.

As an application of the mathematical relations found, we calculate the weights of a portfolio composed by a risky asset and a riskless bond. Three indexes are used as risky securities, the Standard and Poor's Composite Index (S&P500 hereafter), the lowest Book to Market (Growth stocks) portfolio and the largest Book to Market (Value stocks) portfolio series constructed by Fama and French (French (2012)). Additionally, the allocation for a pure diffusion model is calculated for comparison purposes. According to the results, an investor who recognizes the occurrence of unexpected large events and, moreover, regards volatility as stochastic, will invests less in the risky asset. For the particular set of chosen series we find that the allocation in Value series is higher than in the other two series. In addition, the weights for this portfolio are less sensitive to interest rate shifts. On the other hand, the allocation in the S&P500 index is the lowest and exhibits the highest sensitivity.

Estimation of the model parameters is an important point to deal with in itself. The common procedure used for model estimation is the method of maximum likelihood, given that it is consistent, that the estimates of the distribution parameters are asymptotic normal, and it is efficient asymptotically. However, it has no optimal properties for finite samples (see Pfanzagl and Hambker (1994)), its implementation can be troublesome because it is sensitive to initial conditions, and for some specific problems the probability density function is not available or is very complex to estimate. Moreover, the estimator can be unbounded.

An alternative to this procedure derives from the use of the characteristic function (CF hereafter), which is basically the Fourier transform of the density function. The CF has properties that make it appealing for parameter estimation as it is always bounded and can be found for real valued random variables even though there is no density function. Additionally, the mathematical expressions are often simpler than those of the associated density functions. This technique has been the subject of research by several authors. Paulson, Holcomb, and Leitch (1975) investigate how to estimate the density parameters for stable laws, while Heathcote (1977) generalizes the results obtained previously by Paulson *et al.*, Feuerverger and Mureika (1977), Feuerverger and McDunnough (1981), and Feuerverger (1990) investigate and extend the properties of the empirical characteristic function (ECF) so as to solve problems in statistical inference. Later on, Singleton (2001) uses the conditional characteristic function (CCF) to construct an asymptotically efficient estimator and devise a procedure to make it feasible to implement. Chacko and Viceira (2003) propose using a discretization of the ECF in the generalized method of moments (GMM) to estimate the model parameters. They use the unconditional characteristic function (UCF) to calibrate processes with latent variables, thus avoiding bias estimation but losing efficiency. Carrasco and Florens (2002) propose a continuum of moment conditions of the ECF as a way to construct an asymptotically efficient estimator.

The approach in this paper is to minimize a distance measure between the theoretical characteristic function and the ECF. In order to achieve it, a discrete set of points is selected, say N, leading to the moment conditions, and the distance in the space $\mathbb{R}^N$ is minimized. By contrast to the works of Singleton (2001) and Chacko and Viceira (2003), we impose the restriction that the first four sampling moments must match the theoretical moments. Further improvements in the optimum are looked for by using Garch estimates of the long run variance.

The paper proceeds as follows. Section 2 establishes the set up for the market model and renders the wealth dynamics. In Section 3 we determine the set of equivalent martingale measures and restate the budget restriction. In Section 4 the problem of maximization the expected utility on terminal wealth is posed along with the dual problem, also the optimal martingale measure and portfolio allocation are characterized. The solution to the portfolio problem is presented in Section 5, along with explicit expressions for the market price of risk, market price of volatility risk and market price of jump risk. The model parameters estimation is performed in Section 6. To this end, the one step GMM is used with the characteristic function associated to the market model. In Section 7 the numerical results are presented, and finally the conclusions are drawn in Section 8.

2. Market Model

The market model is made up of a risky asset, denoted by $S = S_t$, and a riskless asset $B = B_t$. The securities are traded in the market with a time horizon T. The riskless asset may be regarded as a money market account or a riskless bond that have a positive constant rate of return r. Its dynamics is given by

$$dB = rBdt, B(0) = 1 \tag{2.1}$$

On the other hand, the risky security is a semimartingale in a complete filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ with dynamics

$$dS = S_{-}(\mu dt + \sqrt{V}dW_1 + \phi dN)$$

where μ is a bounded $\mathcal{F}_t-$predictable process, the *drift*, W_1 is a $\mathcal{F}_t-$Brownian motion, and V is a stochastic volatility process. The jump arrivals are modelled through the $\mathcal{F}_t-$Poisson process $N = N_t$ with intensity λ_t. The asset's percentage change, $\phi(J) = e^J - 1$, depends on the IID normally distributed random variable J, whose cumulative distribution function $\psi(z)$ has mean μ_J and variance ν^2. J is assumed to be independent of the Poisson process and the Brownian motion. In particular λ_t has the form $\lambda_t(dz) = \lambda\psi(dz)$ where λ, the *intensity*, is a positive constant and $\psi(dz)$ is a probability measure. Note that

$$\phi(J)dN_t = \int_{\mathbb{R}} \phi(z)N(dt, dz)$$

If we denote by $M = M_t$ the compensated $\mathcal{F}_t-$martingale, associated with N_t, i.e., $M(dt, dz) = N(dt, dz) - \lambda\psi(dz)$, we have (see Øksendal and Sulem (2005))

$$\begin{aligned}\int_{\mathbb{R}} \phi(t,z)M(dt,dz) &= \int_{\mathbb{R}} \phi(t,z)(N(dt,dz) - \lambda\psi(dz)dt) \\ &= \phi(J)dN_t - \lambda dt\int_{\mathbb{R}} \phi(t,z)\psi(dz) = \phi dN_t - \bar{\phi}\lambda dt\end{aligned}$$

The right-hand side integral in the last equation can be recognised as the expected value taken w.r.t. the random variable J, and is denoted as $\bar{\phi} = E^z[\phi(t,z)]$. Hence the risky asset dynamics becomes

$$dS = S_- \Big((\mu + \bar{\phi}\lambda)dt + \sqrt{V} dW_1 + \int_{\mathbb{R}} \phi(t,z) M(dt,dz) \Big) \tag{2.2}$$

For the stochastic volatility we follow Heston (1993) root square mean reverting volatility model

$$dV = \kappa(\vartheta - V)dt + \sigma\sqrt{V}(\rho dW_1 + \bar{\rho} dW_2) \tag{2.3}$$

where κ is the constant mean reversion recovery rate, ϑ is the constant long run volatility, σ is the constant variation coefficient of the volatility, and W_2 is a $\mathcal{F}_t-$Brownian motion independent from W_1. In this context the $\mathbb{R}-$Brownian motions W_1 and $\rho W_1 + \bar{\rho} W_2$, where $\bar{\rho} = \sqrt{1-\rho^2}$, are correlated and its coefficient of correlation is ρ.

The probability space is supposed to satisfy the condition of saturatedness. The filtration is right continuous and generated by the $\mathbb{R}^2-$Brownian motion $W = (W_1, W_2)'$ and the $\mathcal{F}_t-$Poisson process, that is $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N$. The Brownian motion and the compensated process are independent (see Revuz and Yor (1991)).

Let the $\mathcal{F}_t-$predictable, and $S-$integrable process $\pi = \pi_t$ be the number of units of the risky asset, and $\varpi = \varpi_t$ the number of units of the riskless asset held by the investor. These variables form a pair $\Pi = (\pi, \varpi)$ called the *portfolio*. In a self-financing strategy the portfolio changes are not due to funds' inflows or outflows, instead, they are the result of gains or losses realized in the market. Formally we express this as a pair (X_0, Π) such that the wealth process $X = X_t$ evolves as $dX_t = \pi_t dS_t + \varpi_t dB_t$. Following the standard convention we denote as $(\Pi \cdot \mathbf{S})_t$ the stochastic integration

$$(\Pi \cdot \mathbf{S})_t \doteq \int_0^t \Pi_u d\mathbf{S}_u$$

where $\mathbf{S}_t = (S_t, B_t)$. This relation allows us to write wealth dynamics in terms of the asset price dynamics as $X_t = X_0 + (\Pi \cdot \mathbf{S})_t$. In what follows we consider only admissible strategies, meaning that for $X_0 \geq 0$ we have $X_t \geq 0$ $\mathbf{P}-$a.s., for $t \in [0,T]$.

Instead of using the variables π and ϖ to formulate the portfolio problem, we introduce the portfolio weight $\alpha = \alpha_t$. It is the proportion of wealth invested in the risky asset, thus, $\pi_t S_t = \alpha_t X_t$ and $\varpi_t B_t = (1-\alpha_t)X_t$. As a result, the self-financing condition becomes

$$dX = [\alpha(\mu - r + \bar{\phi}\lambda) + r]X dt + \alpha\sqrt{V} X dW_1 + \alpha X_{-} \int_{\mathbb{R}} \phi(t,z) M(dt,dz) \tag{2.4}$$

which is referred to as the *budget constraint*. Starting with the bond process (2.1), the discount process is found to be $d\beta = -r\beta dt$. Applying Ito's lemma to $\bar{X}_t = \beta_t X_t$ we arrive at the discounted wealth process

$$d\bar{X} = \alpha(\mu - r + \bar{\phi}\lambda)\bar{X} dt + \alpha\sqrt{V}\bar{X} dW_1 + \alpha \bar{X}_{-} \int_{\mathbb{R}} \phi(t,z) M(dt,dz) \tag{2.5}$$

Applying Ito's lemma again to $\bar{S}_t = \beta_t S_t$ we get

$$d\bar{S} = (\mu - r + \bar{\phi}\lambda)\bar{S} dt + \sqrt{V}\bar{S} dW_1 + \bar{S}_{-} \int_{\mathbb{R}} \phi(t,z) M(dt,dz) \tag{2.6}$$

which is the discounted asset price process.

3. Equivalent Martingale Measures

A probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ is an equivalent martingale measure (EMM hereafter) if the discounted asset process $\bar{S}$ is a martingale under this measure. In what follows, we consider probability measures $\mathbf{Q}$ such that the Radon - Nikodym derivative, $\xi_t(\theta,\varphi) = d(\mathbf{Q}^{\theta,\varphi}\big|_{\mathcal{F}_t})/\, d(\mathbf{P}|_{\mathcal{F}_t})$, is given by[2]

$$\begin{aligned} \xi_t(\theta,\varphi) \doteq \exp\bigg\{ \int_0^t \bigg(\int_{\mathbb{R}} \lambda(\ln(\varphi(s,z)) + 1 - \varphi(s,z))\psi(dz) - \frac{|\theta|^2}{2} \bigg) ds \\ - \int_0^t \theta' dW_s + \int_0^t \int_{\mathbb{R}} \ln(\varphi(s,z)) M(ds,dz) \bigg\} \end{aligned} \tag{3.1}$$

where $\theta = (\theta_1, \theta_2)'$. Here θ_1, θ_2, and φ are $\mathcal{F}_t-$predictable processes that account for the market price of risk. The first is related to the asset's Brownian component, the second to the stochastic volatility's Brownian motion and the third to the asset's

[2]We assume that $E[\xi_T] = 1$, thus, ξ_T is a probability measure.

jump component. In order to preclude arbitrage opportunities it is necessary to impose an L-integrability condition to θ. The process followed by the risk neutral density $\xi = \xi_t$ can be obtained applying Ito's lemma,

$$d\xi = -\xi_{-}\Big(\theta' dW + \int_{\mathbb{R}} (1-\varphi) M(dt, dz)\Big) \tag{3.2}$$

The market price of risk processes θ and φ are chosen such that the process $\xi \bar{S}$ is a $\mathbf{P}-$ martingale, implying the following relationship

$$\mu - r - \theta_1 \sqrt{V} + \lambda \int_{\mathbb{R}} \phi(t,z)\varphi(t,z)\psi(dz) = 0 \tag{3.3}$$

Let $W^{\mathbf{Q}} = W^{\mathbf{Q}}_t$ be $W^{\mathbf{Q}} = (W^{\mathbf{Q}}_1, W^{\mathbf{Q}}_2)'$, and let us define

$$W^{\mathbf{Q}} \doteq W + \int_0^t \theta_s ds \tag{3.4a}$$

$$\int_0^t \int_{\mathbb{R}} M^{\mathbf{Q}}(ds,dz) \doteq \int_0^t \int_{\mathbb{R}} N(ds,dz) - \lambda \int_0^t \int_{\mathbb{R}} \varphi(z)\psi(dz) ds \tag{3.4b}$$

these processes are a $\mathbf{Q}^{\theta,\varphi}-$ Brownian motion and a $\mathbf{Q}^{\theta,\varphi}-$ martingale process respectively. By using them, the discounted wealth process, equation (2.5), can be written as

$$\bar{X}_t = X_0 + \int_0^t \alpha_s \sqrt{V_s} \bar{X}_s dW^{\mathbf{Q}}_1(s) + \int_0^t \int_{\mathbb{R}} \alpha_s \phi(s,z) \bar{X}_s M^{\mathbf{Q}}(ds,dz) \tag{3.5}$$

Since the right hand side is a $\mathbf{Q}^{\theta,\varphi}-$local martingale, and given that for admissible strategies the wealth is nonnegative, then, the discounted wealth is a $\mathbf{Q}^{\theta,\varphi}-$supermartingale. This implies

$$E^{\mathbf{Q}^{\theta,\varphi}}[\bar{X}_T] \leq X_0 \tag{3.6}$$

this is referred to as the *static* budget constraint. Furthermore, if we apply Ito's lemma to the process $\xi_t \overline{X}_t$ we arrive at

$$d\xi\overline{X} = (\alpha\sqrt{V} - \theta_1)\xi\overline{X} dW_1 - \theta_2 \xi \overline{X} dW_2 + \int_{\mathbb{R}} ((1+\alpha\phi)\varphi - 1)\xi_{-}\overline{X}_{-} M(dt,dz) \tag{3.7}$$

The drift disappears because the Radon-Nikodym ξ has been chosen such that the asset price is a martingale, and this fact is reflected in $\mathbf{P}$.

4. Utility Maximization

The posed problem starts with an agent endowed with a capital X_0. At time t he invests his wealth, allocating α_t percent in the risky asset and $1-\alpha_t$ in the riskless bond, and wishes to find out the optimal strategy to reach the maximum expected utility over terminal wealth. To be precise, if we denote $\mathcal{S}$ as the set of $S-$integrable, admissible strategies Π, we are interested in a portfolio strategy Π such that the function $F(X_0)$ is maximized

$$F(X_0) \doteq \sup_{\Pi\in\mathcal{S}} E[u(X_0 + (\Pi \cdot \mathbf{S})_T)] \tag{4.1}$$

Let $\mathcal{X}$ be the space of $\mathbf{P}-$ a.s. finite non-negative $\mathcal{F}_T-$measurable random variables X_T such that $X_T \leq X_0 + (\Pi \cdot \mathbf{S})_t$ for an admissible strategy Π_t. Then from the super-replicability principle (see Karoui and Quenez (1995)) the problem (4.1) can be restated as to find

$$F(X_0) = \sup_{X_T\in\mathcal{X}} E[u(X_T)] \tag{4.2}$$

subject to the static budget constraint (3.6), where $\mathbf{Q}$ is a probability measure absolutely continuous w.r.t. $\mathbf{P}$. Once the optimal wealth $\widehat{X_T}$ is obtained, the optimal portfolio strategy $\widehat{\pi}$ can be determined.

4.1. Utility Function.

The investor is assumed to have a power utility function

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad \gamma > 0, \gamma \neq 1, x > 0 \tag{4.3}$$

This function maps to the reals, $u : \mathbb{R} \to \mathbb{R}$, and complies with the standard assumptions of being continuous, increasing, differentiable, strictly concave, and that the marginal utility tends to zero as the wealth tends to infinity

$$\lim_{x\to\infty} u'(x) = 0$$

The power utility function belongs to HARA's (hyperbolic absolute risk aversion) category. Its risk tolerance is an affine function of the wealth, and it satisfies the *Inada condition*, i.e.

$$u'(0) = \lim_{x \to 0_+} u'(x) = \infty \tag{4.4}$$

It also exhibits the iso-elasticity property

$$u(kx) = f(k)u(x) + g(k) \quad k > 0 \tag{4.5}$$

which implies that the optimal allocation weights are the same regardless the level of wealth. If in addition to the above-mentioned conditions, a utility function satisfies the following property

$$AE_{+\infty}(u) \doteq \limsup_{x \to \infty} \frac{xu'(x)}{u(x)} < 1$$

it is said to have the *reasonable asymptotic elasticity* property. For the power utility function we have $u'(x) = x^{-\gamma}$, hence the asymptotic elasticity is $AE_{+\infty}(u) = 1 - \gamma < 1$. This property is crucial to be able to find the optimizers for wealth and portfolio weights[3].

4.2. **Primal and dual problem.** The primal problem consists in finding the optimal terminal wealth $\widehat{X_T}$, in (4.2), that maximizes the expected utility for an initial wealth X_0, subject to the static budget constraint (3.6). To achieve this goal the following lagrangian functional is considered

$$L(X_T, \eta, \mathbf{Q}) = E[u(X_T)] - \eta(E^{\mathbf{Q}}[\bar{X}_T] - X_0) \tag{4.6}$$

[3]See Schachermayer (2003).

where $\eta > 0$ is the lagrangian multiplier. Let us define

$$\begin{aligned}
\Psi(\eta, \mathbf{Q}) &= \sup_{X_T \in \mathcal{X}} \{L(X_T, \eta, \mathbf{Q})\} \\
&= \sup_{X_T \in \mathcal{X}} \{E[u(X_T)] - \eta(E^{\mathbf{Q}}[\bar{X}_T] - X_0)\} \qquad (4.7) \\
&= \sup_{X_T \in \mathcal{X}} \left\{ E\left[u(X_T) - \eta \frac{d\mathbf{Q}}{d\mathbf{P}} \bar{X}_T \right] \right\} + \eta X_0
\end{aligned}$$

The last expression can be recognized as the convex conjugate (Legendre - Fenchel transform) $\tilde{u}(x)$ of the utility function $u(x)$. Recall that $\tilde{u}(x)$ is defined as

$$\widetilde{u}(p) = \sup_x \{u(x) - px\}, p > 0 \qquad (4.8)$$

The supremum is reached when $u'(\hat{x}) = p$, that is $\hat{x} = I(p)$, where $I(p)$ is the inverse of the utility function derivative $I(p) \doteq (u')^{-1}(p)$, thus $\tilde{u}(p) = u(I(p)) - pI(p)$. From the definition of the convex conjugate, we arrive at the inequality $u(x) \leq \widetilde{u}(p) + px$, that intuitively suggests us that we should get back to the original function $u(x)$ minimizing the right hand side with respect to variable p. In fact, if we define the inverse Legendre - Fenchel transform as

$$\widetilde{\widetilde{u}}(x) = \inf_p \{\widetilde{u}(p) + px\} \qquad (4.9)$$

then $u = \widetilde{\widetilde{u}}$ for a closed proper convex function u. Returning to the lagrangian functional, equation (4.7), and employing the convex conjugate we get

$$\Psi(\eta, \mathbf{Q}) = E\left[\tilde{u}\left(\eta \beta_T \frac{d\mathbf{Q}}{d\mathbf{P}} \right) \right] + \eta X_0 \qquad (4.10)$$

This expression allows us to define the dual problem as the minimization of the functional $\Psi(\eta, \mathbf{Q})$, where the EMM $\mathbf{Q}$ runs through the set $\mathcal{Q}$ of martingale probability measures $\mathbf{Q}$ absolutely continuous w.r.t $\mathbf{P}$. The relation with the primal problem is clarified defining the dual value function[4] $G(\eta)$

$$G(\eta) \doteq \inf_{\xi_T \in \mathcal{D}} \left\{ E\left[\tilde{u}\left(\eta \beta_T \frac{d\mathbf{Q}}{d\mathbf{P}} \right) \right] \right\} \qquad (4.11)$$

[4]In what remains of the section we present the optimal solution for the primal and dual problems. For proofs, refer to Kramkov and Schachermayer (1999).

where the space $\mathcal{Q}$ has been enlarged by defining $\mathcal{D}$ as the space of $\mathbf{P}-$ a.s. finite non-negative $\mathcal{F}_T-$measurable random variables ξ_T such that

$$\xi_T \leq \lim_{n\to\infty} \frac{d\mathbf{Q}_n}{d\mathbf{P}}$$

where the limit is in the sense of almost sure convergence. That has been made in order to assure that we can reach the saddle point of the Lagrangian. The minimum Radon - Nikodym derivative ξ_T (and hence the EMM $\widehat{\mathbf{Q}}$) exists and is unique. Hence, the dual problem can be restated as

$$\inf_{\eta>0,\xi_T\in\mathcal{D}} \{\Psi(\eta,\mathbf{Q})\} = \inf_{\eta>0}\{G(\eta)+\eta X_0\} \tag{4.12}$$

which is the inverse Legendre - Fenchel transform of $G(\eta)$. The functions $F(X_0)$ and G are conjugated and hence $G'(\widehat{\eta}) = -X_0$, or $\widehat{\eta} = F'(X_0)$. On the other hand, the lagrangian (4.6), as a concave function of X_T, has a unique maximum given by

$$\widehat{X_T} = I\left(\widehat{\eta}\beta_T \left.\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}\right|_{\mathcal{F}_T}\right) \tag{4.13}$$

At the optimum $F(X_0) = G(\widehat{\eta}) + \widehat{\eta}X_0$, and therefore the function $F(X_0)$ equals $L(\widehat{X_T}, \widehat{\eta}, \widehat{\mathbf{Q}})$. The point $(\widehat{X_T}, \widehat{\eta})$ is a saddle point, and the constraint (3.6) is transformed into an equality.

Summing up, for the market model defined in Section 2, and power utility function, Section 4.1, the primal problem solution and the dual problem solution are given. The resulting solutions are equivalent i.e., strong duality holds.

4.3. **Optimal Q.** First we fix $\eta > 0$ in the dual problem (4.12), and therefore we minimize over the set of EMM. Recall that a specific functional form for the EMM equation (3.1), that depends on variables $\{\theta, \varphi\}$, was chosen. Hence the optimization is carried out over this parameter space rather than over a function space. In what follows we change the notation accordingly. Note that the underlying dynamics for the EMM are the stochastic differential equation (3.2) and the stochastic volatility process (2.3).

If we regard $(\bar{\xi}, V)$ as a *controlled jump diffusion*, and (θ, φ) as the *control process*, belonging to $\mathcal{U}$, the *set of admissible controls*, then the dual problem can be restated as a *stochastic control problem* to find the *dual value function* $\Phi(\bar{\xi}, V, t)$ and an *optimal control* $(\widehat{\theta}, \widehat{\varphi}) \in \mathcal{U}$ such that

$$\mathcal{F}(\bar{\xi}, V, t) = \inf_{(\theta,\varphi)\in\mathcal{U}} \mathcal{J}^{(\theta,\varphi)}(\bar{\xi}, V, t) = \inf_{(\theta,\varphi)\in\mathcal{U}} E\left[\mathcal{G}\left(\bar{\xi}_T\right) \middle| \mathcal{F}_t \right] \tag{4.14}$$

Here $\mathcal{J} = \mathcal{J}^{(\theta,\varphi)}(\bar{\xi}, V, t)$ is the *performance criterion* and $\mathcal{G}$ is the *terminal cost function.* In this setting the value function is found by means of the Hamilton-Jacobi-Bellman (HJB) for optimal control of jump diffusions (see Øksendal and Sulem (2005)). The solution, function $\Phi(\bar{\xi}, V, t)$, satisfies the following relation

$$\inf_{(\theta,\varphi)\in\mathcal{U}} \left\{ \mathcal{A}\Phi(\bar{\xi}, V) \right\} = \frac{\partial \Phi(\tau)}{\partial \tau} \tag{4.15}$$

where $\tau = T - t$, also

$$\lim_{t\to T^-} \Phi(\bar{\xi}_t, V_t, t) = \mathcal{G}(\bar{\xi}_T, V_T, T) \tag{4.16}$$

If $(\widehat{\theta}(\bar{\xi}_{t-}, V_{t-}), \widehat{\varphi}(\bar{\xi}_{t-}, V_{t-}))$ is the optimal control hence $\Phi(\bar{\xi}_t, V_t, t) = \mathcal{F}(\bar{\xi}, V, t)$. The generator $\mathcal{A}$ associated to the stochastic processes $\bar{\xi}, V$, is

$$\begin{aligned} \mathcal{A}\Phi = {} & -r\bar{\xi}\frac{\partial \Phi}{\partial \bar{\xi}} + \kappa(\vartheta - V)\frac{\partial \Phi}{\partial V} + \frac{|\theta|^2}{2}\bar{\xi}^2\frac{\partial^2 \Phi}{\partial \bar{\xi}^2} + \frac{\sigma^2}{2}V\frac{\partial^2 \Phi}{\partial V^2} \\ & - \varrho\theta\bar{\xi}\frac{\partial^2 \Phi}{\partial \bar{\xi}\partial V} + \lambda \int_{\mathbb{R}} \left(\Phi(\varphi(z)\bar{\xi}) - \Phi(\bar{\xi}) + (1 - \varphi(z))\bar{\xi}\frac{\partial \Phi}{\partial \bar{\xi}} \right) \psi(dz) \end{aligned} \tag{4.17}$$

here $|\theta|^2 = \theta_1^2 + \theta_2^2$, and $\varrho = \sigma\sqrt{V}(\rho, \bar{\rho})$. For functions that satisfies the iso-elastic property (4.5)

$$\begin{aligned} \inf_{(\theta,\varphi)\in\mathcal{U}} \left\{ E\left[\tilde{u}(\eta\bar{\xi}_T(\theta, \varphi)) \middle| \mathcal{F}_t \right] \right\} &= \inf_{(\theta,\varphi)\in\mathcal{U}} \left\{ E\left[f(\eta)\tilde{u}(\bar{\xi}_T(\theta, \varphi)) + g(\eta) \middle| \mathcal{F}_t \right] \right\} \\ &= \inf_{(\theta,\varphi)\in\mathcal{U}} \left\{ E\left[\tilde{u}(\bar{\xi}_T(\theta, \varphi)) \middle| \mathcal{F}_t \right] \right\} \end{aligned} \tag{4.18}$$

that provides us with the terminal cost function $\mathcal{G}(\bar{\xi}_T) = \tilde{u}(\bar{\xi}_T)$.

4.4. **Portfolio Allocation.** Although we have found expressions for the optimum terminal wealth $\widehat{X}_T$ and also for the optimum EMM $\widehat{\mathbf{Q}}$, the portfolio weight is still missing. However, taking into account that the process resulting from the product of the time $t \in [0, T]$ discounted wealth and $\widehat{\mathbf{Q}}$ is a martingale, equation (3.7), the following relation holds

$$\beta_t \widehat{X}_t \left.\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}\right|_{\mathcal{F}_t} = E_t\left[\beta_T \widehat{X}_T \left.\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}\right|_{\mathcal{F}_T}\right] \tag{4.19}$$

or equivalently $\widehat{X}_t = e^{-r(T-t)} E^{\widehat{\mathbf{Q}}}\left[\widehat{X}_T \middle| \mathcal{F}_t\right]$. That means that today's portfolio value can be found as the present value of the future portfolio's expected value in a risk neutral world. As usual $\tau = T - t$ is the time remaining to portfolio maturity. The expression (4.19) allows us to find the time t wealth dynamics. Its associated stochastic differential equation should match that of equation (3.7), and then the portfolio weights can be found comparing terms.

5. Solution for Power Utility

In this section we address the portfolio allocation problem for the power utility function. First, the dual problem is solved and explicit expressions for the market price of risk and portfolio weights are given. Then, the lagrange multiplier and the optimal wealth are found.

5.1. **Power Utility.** We turn to the explicit calculation of the optimal control, details of which are given in appendix A. The parameters are obtained from the optimization in equation (4.15), and as a result we obtain the following expressions for the market price of risk, the market price of volatility risk and the market price of jump risk

$$\widehat{\theta}_1 = -b\rho\sigma\sqrt{V} + \gamma\widehat{\alpha}\sqrt{V}, \qquad \widehat{\theta}_2 = -b\bar{\rho}\sigma\sqrt{V}, \qquad \text{and} \qquad \widehat{\varphi} = (1 + \widehat{\alpha}\phi)^{-1/\gamma}$$

With these optimal controls the EMM, equation (3.1), is fully determined. Also, the optimal portfolio rule is retrieved,

$$\widehat{\alpha} = \frac{1}{\gamma}\frac{\mu - r}{V} + \frac{1}{\gamma}b\rho\sigma + \frac{1}{\gamma}\frac{\lambda}{V}E[\phi(1+\widehat{\alpha}\phi)^{-\gamma}] \tag{5.1}$$

where $b(\tau)$ is given in appendix - equation (A.9). An equivalent expression for the market price of risk, coming from restriction (3.3), is

$$\widehat{\theta}_1 = \frac{\mu - r}{\sqrt{V}} + \frac{E^z[\phi(z)\widehat{\varphi}(z)\lambda]}{\sqrt{V}} \tag{5.2}$$

Besides that, the market price of volatility risk can be written as

$$\widehat{\theta}_2 = \frac{\bar{\rho}}{\rho}(\widehat{\theta}_1 - \gamma\widehat{\alpha}\sqrt{V}) \tag{5.3}$$

On the other hand, the optimal lagrangian multiplier is

$$\widehat{\eta} = \left(\frac{X_0\beta_T^{-\tilde{\gamma}}}{E[\widehat{\xi_T^{\tilde{\gamma}}}]}\right)^{\frac{1}{\tilde{\gamma}-1}} \tag{5.4}$$

and the optimal wealth at time $t \in [0, T]$ is found to be (see appendix - equations (A.8) and (A.9))

$$\widehat{X_t} = \frac{X_0 B_t}{E[\widehat{\xi_T^{\tilde{\gamma}}}]}\widehat{\xi}_t^{\tilde{\gamma}-1}e^{(1-\tilde{\gamma})(a_2(\tau)+b(\tau)V)}$$

The portfolio value at t is the future value of the initial investment, X_0B_t, times a power of the EMM and a stochastic volatility factor.

5.2. **Analysis of the Market Price of Risk.** The first component on the right-hand side of the equation (5.2) is the well known Brownian motion market price of risk (MPR onwards). It measures the expected excess return over the riskfree rate, the risk premium, of an investment per volatility point. The second term is the jump contribution. Roughly speaking, it is the expected value of the product of the jump's amplitude change [5] times the event's frequency[6]. It measures the expected change in the return per volatility point of an asset attributable to a series of shocks of random

[5] i.e. the jump component part of dS_t/S_{t-}.

[6] $\widehat{\varphi}(z)\lambda$ is the jump intensity in the risk neutral world.

amplitude. Yet another way to view the second term is grouping $\phi(z)\widehat{\varphi}(z)$. Hence, we are in a risky world where the jumps occur with an intensity λ, but the jump's amplitude change is amplified by a factor of $\widehat{\varphi}(z)$. In both cases $\widehat{\varphi}(z)$ is determined by the investors risk aversion γ and depends on the market jump amplitude ϕ.

The market price of volatility risk (MPVR), equation (5.3), has two components that define its behavior. For an investor with near to zero risk aversion the MPVR, $\widehat{\theta}_2$, is approximately proportional to the MPR, $\widehat{\theta}_1$. The coefficient of proportionality is one when the coefficient of correlation is $\rho = 1/\sqrt{2} \approx 70.7\%$. On the other hand, the MPVR vanishes if the innovations in the returns are perfectly correlated with the instantaneous volatility, i.e. for a coefficient of correlation $\rho = \pm 1$. However, in a market with zero correlation $\rho = 0$ the portfolio rule becomes

$$\widehat{\alpha} = \frac{1}{\gamma}\frac{\widehat{\theta}_1}{\sqrt{V}}$$

In this case the MPVR attains the value of $-b\sigma\sqrt{V}$. As a result, the second term of the portfolio rule on the right-hand side of equation (5.1) disappears and the volatility risk enters via the instantaneous volatility. In that sense, the MPVR measures to what extent the volatility risk impacts the allocation weight.

6. Model Estimation

To estimate the model we first obtain the expression of the CCF for the market model return process Y_t, and then proceed to estimate its parameters. Once we have the results the portfolio weights are calculated.

6.1. **Model Parameters.** In order to estimate the model we use a discretization of the ECF. The parameters are found in such a way that the Euclidean distance of the ECF to the characteristic function is minimized. This is equivalent to the one step GMM with the identity as initial weighting matrix. An application of this procedure can be found in Chacko and Viceira (2003). They perform the optimization with a discretization along a finite set of integers. By contrast, in our work we restrict the

GMM in such a way that the first four sampling moments must match the theoretical moments.

The CF is unique in the sense that if there is another CF associated to the same probability function, both are equal. The CCF is the Fourier transform of the conditional density function. Its argument ζ belongs to the continuous dual (the frequency space) and is mapped into the complex plane. The CF can be found although the inverse transform, i.e. the density function, does not necessarily exist. For exponential affine stochastic processes that satisfy some regularity conditions the CCF have a specific exponential form (see Duffie, Pan, and Singleton (2000)).

Once we have found the CCF, the CF for the return process Y can be obtained. By definition of CF we know that $E[\exp(i\zeta Y_n) - \Phi(\zeta, \Theta)] = 0$, where $\Phi(\zeta, \Theta)$ is the CF, and Θ is the set of parameters that defines it. Hence, for $h(Y, \zeta, \Theta) = \exp(i\zeta Y) - \Phi(\zeta, \Theta)$, we can construct a finite set of moment conditions, by evaluating $h(Y, \zeta, \Theta)$ at a specified set of discrete frequencies $\zeta_1, \ldots, \zeta_k$, as $E[h(Y, \zeta, \Theta)] = 0$. Let us define

$$g(Y_n, \Theta) = \begin{bmatrix} \mathbf{Re}(h(Y_n, \zeta_1, \Theta)) \\ \vdots \\ \mathbf{Re}(h(Y_n, \zeta_k, \Theta)) \\ \mathbf{Im}(h(Y_n, \zeta_1, \Theta)) \\ \vdots \\ \mathbf{Im}(h(Y_n, \zeta_k, \Theta)) \end{bmatrix}$$

where $\mathbf{Re}(\cdot)$ is the real part function and $\mathbf{Im}(\cdot)$ is the imaginary part function. This forms a $2k$ set of moment conditions. The sampling counterpart is

$$G(Y, \Theta) = \frac{1}{N} \sum_{n=1}^{N} g(Y_n, \Theta)$$

where N is the data size. Thus, the one step GMM estimator is

$$\min_{\Theta} \quad G(Y, \Theta)' G(Y, \Theta)$$

To implement the procedure we use the first four theoretical moments of the process. For a process x_t, these can be retrieved by means of the formula

$$E_t\left[(x_{t+\tau}-E[x_t])^n\right]=\frac{1}{i^n}\frac{d^n}{d\zeta^n}\Phi(\zeta,\tau,\Theta,x_t-E[x_t])\bigg|_{\zeta=0} \tag{6.1}$$

The cumulant function would ease the calculation. Expressions for the mean, standard deviation, skewness and excess kurtosis of the return process are derived in appendix B.

6.2. **Conditional Characteristic Function.** The CCF for the process Y_t is defined as

$$\Phi(\zeta,\tau,\Theta,Y_t)=E_t\left[e^{i\zeta Y_{t+\tau}}\right] \tag{6.2}$$

where ζ is the argument of the CCF and Θ the set of parameters that determines it. Commonly, the derivation of the CCF is made using the Feynman−Kac Theorem. This asserts that the CCF, equation (6.2), is the solution to the associated partial differential−difference equation (PDDE) that results from the process' infinitesimal generator. As in section 4.3, it is tantamount to view this as a stochastic control problem, with the solution Φ satisfying equation (4.14). However, in the present case the underlying dynamics are those from the market model, Section 2, i.e.,

$$dS=S_-(\mu dt+\sqrt{V}dW_1+\phi dN)$$
$$dV=\kappa(\vartheta-V)dt+\varrho dW$$

It is convenient, both theoretically and numerically, to transform the asset price stochastic differential equation (SDE) by applying the logarithm function to S, and thus defining the variable $Y=Y_t\doteq\ln S$. Using Ito's lemma, the dynamics of Y is found to be

$$dY=d\ln S=\left(\mu-\frac{V}{2}\right)dt+\sqrt{V}dW_1+JdN \tag{6.3}$$

The resulting CCF depends on the stochastic volatility V, an unobservable variable. To be able to estimate the parameters, it is common to remove the dependence on V calculating the expected value (see Chacko and Viceira (2003)). The expectation w.r.t. the volatility V involves the knowledge of the probability density function of V or equivalently of its UCF. These functions are found in the next subsection.

6.2.1. *Volatility Unconditional Characteristic Function.* Consider the stochastic differential equation:

$$dV = \kappa(\vartheta - V)dt + \sigma\sqrt{V}dW_2$$

its characteristic function $\Phi^V(\zeta, \tau, \Theta^V, V)$, with vector of parameters Θ^V, solves the following partial differential equation (PDE)

$$\Phi^V_\tau = \kappa(\vartheta - V)\Phi^V_V + \frac{\sigma^2}{2}V\Phi^V_{V^2}$$

with boundary condition $\Phi^V(\zeta, 0, \Theta^V, V) = e^{i\zeta V}$. This PDE satisfies the conditions in Duffie, Pan, and Singleton (2000), thus the solution is of the exponential affine form $\Phi^V(\zeta, \tau, \Theta^V, V) = e^{a(\zeta,\tau)V + b(\zeta,\tau)}$, the parameter space being $\Theta^V = \{a, b\}$. The boundary condition implies that $a(\zeta, 0) = i\zeta$ and $b(\zeta, 0) = 0$. Inserting the partials of the function into the PDE, and separating variables we end up with the following two ordinary differential equations (ODE's)

$$a_\tau = -\kappa a + \frac{\sigma^2}{2}a^2$$

$$b_\tau = \kappa \upsilon a$$

Both equations have solution, see appendix - equation (C.1) for $a(\zeta, \tau)$ and appendix - equation (B.1) for $b(\zeta, \tau)$,

$$a(\zeta, \tau) = \frac{2\kappa i\zeta e^{-\kappa\tau}}{2\kappa - i\zeta\sigma^2(1 - e^{-\kappa\tau})}$$

$$b(\zeta, \tau) = \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa - i\zeta\sigma^2(1 - e^{-\kappa\tau})}\right)$$

Hence the CCF for the volatility is

$$\begin{aligned}\Phi^V(\zeta,\tau,\Theta^V,V) &= \exp\{a(\zeta,\tau)V+b(\zeta,\tau)\}\\ &= \exp\left\{\frac{2\kappa i\zeta e^{-\kappa\tau}}{2\kappa-i\zeta\sigma^2(1-e^{-\kappa\tau})}V+\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa-i\zeta\sigma^2(1-e^{-\kappa\tau})}\right)\right\}\end{aligned}$$

From this expression, the UCF is found as the limiting function of the CCF Φ^V when τ goes to infinity $\Phi^V(\zeta,\Theta^V)=\lim_{\tau\to\infty}\Phi^V(\zeta,\tau,\Theta^V,V)$,

$$\Phi^V(\zeta,\Theta^V)=\lim_{\tau\to\infty}\exp\left\{\frac{2\kappa i\zeta e^{-\kappa\tau}}{2\kappa-i\zeta\sigma^2(1-e^{-\kappa\tau})}V+\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa-i\zeta\sigma^2(1-e^{-\kappa\tau})}\right)\right\}$$

Therefore

$$\Phi^V(\zeta,\Theta^V)=\exp\left\{\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa-i\zeta\sigma^2}\right)\right\} \tag{6.4}$$

The associated probability density function is the gamma distribution

$$\psi^V(v)=\frac{\frac{2\kappa}{\sigma^2}^{\frac{2\kappa\vartheta}{\sigma^2}}v^{\frac{2\kappa\vartheta}{\sigma^2}-1}e^{-\frac{2\kappa v}{\sigma^2}}}{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)} \tag{6.5}$$

where $\Gamma(\cdot)$ is the gamma function.

6.2.2. *Market Model CCF.* From the return dynamics, equation (6.3), the market model becomes

$$\begin{aligned}dY &= \left(\mu-\frac{V}{2}\right)dt+\sqrt{V}dW_1+JdN\\ dV &= \kappa(\vartheta-V)dt+\varrho dW\end{aligned}$$

The associated CCF Φ_t, satisfies the partial differential-difference equation

$$\begin{aligned}\Phi_\tau = &\left(\mu-\frac{V}{2}\right)\Phi_Y+\kappa(\vartheta-V)\Phi_V\\ &+\frac{V}{2}(\Phi_{YY}+\sigma^2\Phi_{VV}+2\rho\sigma\Phi_{VY})+\lambda E[\Phi(Y+J)-\Phi(Y)]\end{aligned} \tag{6.6}$$

Again, this partial differential equation satisfies the conditions in Duffie, Pan, and Singleton (2000), hence the solution is of the exponential affine form $\Phi(\zeta,\tau,\Theta,Y)=$

$\exp\{c(\zeta,\tau)Y + a(\zeta,\tau)V + b(\zeta,\tau)\}$, with boundary condition $\Phi(\zeta, 0, \Theta, Y) = e^{i\zeta Y}$, implying $c(\zeta, 0) = i\zeta$, $a(\zeta, 0) = 0$ and $b(\zeta, 0) = 0$. The computation of the CCF is carried out in appendix B. And the solution, appendix - equation (B.2), is

$$\begin{aligned}(6.7)\quad & \Phi(\zeta,\tau,\Theta,\ln S) = \\ & \exp\left\{i\zeta \ln S + i\zeta\mu\tau + \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa(r_2 - r_1)}{(2\kappa - a(\zeta,\tau)\sigma^2)(r_2 e^{r_1\tau} - r_1 e^{r_2\tau})}\right)\right. \\ & \left. \qquad + \lambda\tau \exp\left(i\zeta\mu_J - \frac{\upsilon^2}{2}\zeta^2\right) - \lambda\tau\right\}\end{aligned}$$

where

$$\begin{aligned} a(\zeta,\tau) &= -\frac{2}{\sigma^2}\frac{r_1 r_2 e^{r_1\tau} - r_1 r_2 e^{r_2\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} \\ r_{1,2} &= \frac{1}{2}\left(i\rho\sigma\zeta - \kappa \pm \sqrt{(i\rho\sigma\zeta - \kappa)^2 - i\sigma^2\zeta(i\zeta - 1)}\right)\end{aligned}$$

The CF for the continuously compounded returns, $r_{t+\tau} = \ln(S_{t+\tau}/S_t)$, is easily obtained from the market model CCF as follows

$$\begin{aligned} \Phi(\zeta,\tau,\Theta,r) &= E_t[e^{i\zeta r_{t+\tau}}] \\ &= E_t[e^{i\zeta\ln(S_{t+\tau}) - i\zeta\ln S_t}] \\ &= E_t[e^{i\zeta\ln(S_{t+\tau})}]E_t[e^{-i\zeta\ln S_t}] \qquad (6.8) \\ &= E_t[e^{i\zeta\ln(S_{t+\tau})}]e^{-i\zeta\ln S_t} \\ &= \Phi(\zeta,\tau,\Theta,\ln S)e^{-i\zeta\ln S_t}\end{aligned}$$

In view of equation (6.7), the CCF for the continuously compounded returns neither depends on past values of $r = r_t$ nor on past volatility, hence it is the return's CF.

7. Numerical Results

As mentioned in Section 6.1, we use the one step restricted GMM methodology on the ECF to estimate the model's parameters for the data series. The dataset comprises the prices from January 1982 to October 2010[7] of the Standard and Poor's

[7] We chose this period of time to make a comparison with previous studies, see Penagos and Rubio (2011).

Composite Index, and two series of the ten portfolios formed on book to market from Kenneth French's web page (see French (2012)). The first series, low book to market, is referred to as Growth, and the second series, high book to market, is referred to as Value. The timeframe spanned by the data includes the crash of October 1987, the Gulf War I in August 1990, the Mexican crisis in December 1994, the Asian crisis of July 1997, the Russian crisis of August 1998, and the bursting of the dot.com bubble. The riskless rate of interest is the simple average of the three month Treasury Bill rates for the same period as above, and its value is 4.7% in annualized terms.

7.1. **Model Estimation.** Table 1 reports the descriptive statistics for the input data, parameters are given in daily basis, in percentage form. From the data we observe that the Value mean is 0.14%, twice the Growth mean of 0.07% and almost five times the S&P500 mean which is 0.03%. However, the volatility is higher for Growth series than for Value series, in agreement with the well known value premium. The S&P500 volatility is similar to that of Growth series.

Skewness and excess kurtosis are higher, in absolute terms, for Value series and lower for S&P500 series, skewness being negative. This reflects a distribution with heavier tails for Value portfolios with a higher tendency of having large negative returns. All series are leptokurtic with values well above 3, that of a normal distribution. The largest excess kurtosis is for the Value's which doubles the S&P500's excess kurtosis and is well above the Growth's excess kurtosis. These findings are suggested in figure 1. The graphics for density distributions exhibit high peaks, which is an indication of high kurtosis, and the non straight QQ plot is a symptom of skewness.

For comparison purposes, the graphics of the ECF against the normal distribution CF are plotted in figure 2. In each case the parameters used in the normal distribution are the mean and standard deviation of the data series under study. On the left-hand side the absolute value of the characteristic functions is plotted and in the right-hand side the phase diagram is shown. Although the modulo appears to be similar in all the three cases, the phase seems to be closer to the normal distribution phase for the Value series. On the contrary, the S&P500 displays a more irregular behavior.

Initially, an *unconstrained* optimization is performed using wide intervals for the parameter's values. Afterwards, the procedure is repeated but the long term variance is *constrained* to fit the values obtained in a Garch modeling of the series. The results of the latter are in Table 2, as shown, the variance is larger for Growth series than for Value series, with S&P500's variance in the middle but closer to the former, in agreement with descriptive statistics.

The parameters' estimates[8] are reported in Table 3, they are the result of applying the one step restricted GMM methodology to the data (refer to section 6.1). The results show that for the unconstrained case the Growth series' return is approximately half of Value series' return. The magnitude of the volatility σ is approximately half the value of the associated long run variance ϑ. Also note that the volatility σ for Growth series is twice the value of the other series volatility. The parameter σ is thought to explain the kurtosis, however, the pattern is broken for Value series, which has the highest kurtosis with the smallest σ. The same behavior is followed by the long run variance ϑ, by the jump volatility ν and by the mean reversion κ. The persistence of the series is larger for Value series[9] suggesting that a shock to the volatility fades away more slowly for Value investors than for Growth ones.

The jump amplitude is commonly used to explain the skewness. Table 3 reports a negative jump mean for all the series implying negative skewness. For the unconstrained estimates the Value series jump mean is larger in magnitude than the Growth jump mean, the S&P500 jump mean being the lowest. A similar pattern is found in the skewness where the Value series is the most left skewed. On the other hand, the jump mean becomes more positive as long as the jump frequency increases, thus jumps often happen for S&P500 at a rate of one per week. This occurs once each 20 days for Growth series, and less frequently, once each 1.3 months,

[8]We have set $\tau = 1$, hence the resulting parameters have the same frequency of the input data.
[9]Higher values of κ implies lower persistence.

for Value. Notice that regarding the jump mean magnitude, i.e., its absolute value, smaller shocks happen more frequently that larger shocks.

In Panel A of Table 4 the theoretical values of the first four moments are reported. These are the result of the restrictions imposed on the GMM estimator. As we can see, they are quite accurate, indicating that the model parameter's estimates adjust very well to the data on first moments restrictions. Panel B reports the unconditional moments of the return series split into the jump part and the stochastic volatility part. It is clear that skewness and kurtosis depend almost exclusively on the jump contribution. The skewness is mainly due to jump mean, which also determines the sign, and to Poisson intensity. The kurtosis strongly depend on jump volatility, which is bigger than jump mean, and on the Poisson intensity as well (see appendix - equation (B.3)). On the other hand, the theoretical mean is composed by the diffusion mean μ compensated by the volatility stochastic mean, term $\vartheta/2$, plus the jump mean $\lambda\mu_J$ (see appendix - equation (B.3)). However, the dominant term is the diffusion mean, that is because the long term variance is comparatively low in magnitude. And in addition, the jump impact in the overall mean is low because the jump mean is reduced by the jump intensity. The variance is made up of a jump component and a stochastic volatility component in the same order of magnitude. However, the jump term is more important for the S&P500 series. The volatility seems to depend on model volatility parameters: jump volatility, long term volatility and volatility of stochastic volatility, and also on mean reversion κ.

In Panel C of Table 4 we make use of appendix - equation (B.3) to disentangle the jump component of the series' variance. Taking a closer look we notice that the jump contribution is the product of the Poisson intensity λ and the term $(\mu_J^2 + \upsilon^2)$. The latter is similar for Growth and Value series, because jump volatility υ also is similar for both series and jump mean μ_J is much lower than υ. However, the jump contribution to the variance of the Value series is less than a half the contribution to the variance of Growth series, approximately 45.39%. This number is explained by the ratio of the jump intensity of Value series over Growth series of approximately 49.34%.

On the other hand, the stochastic volatility contribution to variance was separated into the product of two factors, the first is the long run volatility ϑ and the second is an expression which is function of the volatility σ, mean reversion κ and the correlation ρ (see appendix - equation (B.3)). The last expression has a magnitude of about 1, this is because κ is much larger than the volatility σ, and only one term of the expression is not neglectable, therefore the whole expression is approximately equal to the long run variance ϑ. The ratio of the unconstrained long run variance between Growth and Value is roughly 1.97, which is approximately the same ratio encountered between the stochastic volatility contribution of Growth series over Value series (see Panel B of Table 4). In conclusion the variance of the Growth series is larger than the variance of the Value series mainly since the lower frequency of occurrence of unexpected large events in Value series lessens its impact on its own variance and also because the largest long term volatility of the Growth series compared with that of Value series.

For the for the constrained estimates we have that the Growth series' return is approximately half of Value series' return. The long run volatility is similar, as expected, to the series' standard deviation. Their values are 0.980%, 1.10% and 0.70% for the S&P500, Growth and Value respectively. The constrained series estimates follow a similar behavior than those of the unconstraint series for the parameters related with the diffusion part of the asset dynamics, and for the stochastic volatility parameters, except for the persistence which is lower for the Value series compared to Growth's persistence. The parameters related with the jump part also exhibit a quite different pattern. The jump mean and Poisson intensity are the lowest for S&P500 and largest for Value. However, the skewness is the lowest for Value series. Recall that the function errors, Table 3, are the highest for constrained estimates. This poor adjustment points to a less reliable parameter's estimates.

7.2. **Portfolio Parameters Estimates.** We assume that a risk averse investor wishes to allocate certain proportion α of money in the risky asset and the remaining $1-\alpha$ in the riskless bond. His goal is to obtain the best result, maximum expected utility wealth, according to the changing investment opportunities, i.e., stochastic volatility in the presence of unexpected large events. In what follows, the optimal portfolio is calculated for each of the series S&P500, Growth and Value. That is conducted for the unconstrained and constrained case with constant risk aversion coefficients ranging from 2 to 10. Additionally, a sensitivity analysis is carried out to asses the impact of the interest riskless rate.

Equation (5.1) shows that instantaneous volatility is a key component for the portfolio weight, therefore two approaches are carried out in order to estimate its value: first, the average instantaneous volatility is calculated from an approximation of the stochastic processes driving the asset prices and stochastic volatility (see Chesney and Elliot (1995)), and second, taking into account that parameters are found as the result of an optimization process for a 28 year series, the long run volatility parameter from the stochastic volatility model, equation (2.3), is used as a proxy.

For comparison purposes, we calculate the portfolio allocation for an investor who does not recognize neither jumps events nor stochastic volatility behavior. These results are reported in Table 5. The main feature of the allocation is that the weights diminish as risk aversion increases. The participation in Value series is larger than in Growth series, being the lower for S&P500 series. Note also that while an investor's participation in the S&P500 is a fraction of his wealth, this investor is willing to borrow at the market interest rate to invest in Growth and Value portfolios.

Table 6 shows the proportions of the total wealth invested in the risky asset when investor acknowledge jumps and stochastic volatility. The results are obtained using the instantaneous volatility estimates for the three series. Comparing these results with those of Table 5, we notice that the allocation is higher for investors that disregard the presence of jumps and stochastic volatility. Motivated by the work of

Merton (1969), we calculate a myopic demand as the risk premium divided by the instantaneous volatility times the risk aversion. Then, we define the intertemporal hedging demand as the difference between the total portfolio weight and the myopic contribution. As expected, the participation in the risky asset decreases with an increasing risk aversion for both the myopic and the intertemporal hedging components. Figure 3 illustrates the behavior of the myopic demand of the risky portfolio weight against the intertemporal hedging demand (IHD from now onwards). The change in the invested proportion on the myopic component is of 80% for a risk aversion ranging from 2 to 10. Although it is lower, in absolute terms, for S&P500, and larger for Value.

Recall that an investor allocates part of his wealth in risky assets looking for to perform better at the end of the period. This proportion is chosen accordingly to the single period optimal value, the myopic component. However, in a multiperiod environment, intertemporal hedging demands arise when returns are correlated with changing investment opportunities and when there is a nonzero frequency of jump occurrences. Investors wish to hedge themselves from changing investment opportunities deviating their initial myopic portfolio. In the present setting, for a risk averse investor with $\gamma > 1$, the asset trend is to do worse, mainly because the negative jump amplitude mean μ_J, which, as stated in the previous section, accounts for the negative skewness of the model. Therefore, the myopic component is reduced by a negative IHD.

The ratio of the IHD over the myopic demand is plotted in figure 4. As expectations of asset's performance worsen, investors hedge against adverse changes in investment opportunities. A negative IHD is equivalent to holding a short position on the asset and as a result the myopic participation is reduced. It is remarkable that this ratio is fairly constant for the S&P500 series, with a value of 61.42%, but in general it lessens, in magnitude, as γ increases, due to a faster reduction in the IHD component compared to the myopic one. Indeed, the change in the proportion invested in the IHD from a risk aversion of 2 to 10 is about 80.00% for S&P500,

80.04% for Growth, and 80.70% for Value, which explains the aforementioned behavior. In the long run, the IHD diminishes in a proportion similar to the myopic component, i.e. with $1/\gamma$, the difference in the three series is an anomaly in the IHD at the few first values of risk aversion that disappears at a fast pace for Value series. Hence each series tend to a constant ratio, suggesting an asymptotic risk premium associated to stochastic volatility and jumps. In addition, the IHD - myopic ratio also suggest that S&P500 series performs the worst and Value the better.

An approximate analysis of sensitivities is conducted (the results are compiled in Table 7), with the riskless rate as the perturbed variable, for a reduced set of risk aversion coefficients. In all cases the allocation on the risky asset lessens for an increasingly riskless rate of return. Investors favor holding the riskless asset more than the risky one for higher rates. S&P500 portfolios are the most sensitive to changes in interest rates and Value portfolios the least. Indeed, one percentage point change in interest rates represents approximately 10% change in portfolio for S&P500 and Growth, and 20% for Value. However, compared with the total share, it is a small part of the allocation, about 3%, whereas for the S&P500 the difference is about 10% of the total share. The absolute deviations are lesser for risk aversion 6 and 10 but the sensitivity increases with interest rates. When facing rates increments, investors give up relatively larger proportions of the risky asset as they become more conservative. This is particularly true for S&P500 portfolios.

For portfolios calculated with the long run variance, Tables 8 and 9, the behavior is similar to that described previously. It should be mentioned that as volatilities are smaller compared to those of instantaneous estimates the proportion invested in the risky asset becomes larger, but sensitivities maintain around similar values.

Table 10 is constructed in order to gauge the accuracy of the common belief that investors should allocate 60% of their wealth in risky assets. This is yet another way to stress the idea that S&P500 portfolios are less appealing than Growth and Value portfolios, being the latter the preferable investment. Indeed, in order to hold only 60% of Value portfolio the agent's risk aversion must be as high as 21.36 for

Value in contrast with 1.05 of a S&P500 portfolio investment. Recall that risky asset allocation diminishes as risk aversion increases.

8. Conclusions

In this paper, the allocation problem of an investor maximizing the expected utility on terminal wealth was solved by relying on martingale and duality methods. The market model takes into account the occurrence of unexpected large events in a market with a changing opportunity investment set, i.e. stochastic volatility. As a result, the expression for the portfolio allocation rule is found. Furthermore, the market price of risk, the market price of volatility risk and the market price of jump risk are retrieved. The portfolio weight for the risky asset is inversely proportional to risk aversion and reduces to the standard myopic rule when the correlation between the asset prices and volatility is zero, and the frequency of extreme event vanishes. The market price of risk was found to be composed by the Brownian market price of risk plus a jump contribution. The market price of volatility risk is approximately proportional to the market price of risk for a very low risk adverse investor. If the innovations in the returns are perfectly correlated with the instantaneous volatility the market price of volatility risk is zero.

The model was tested with three market series, the Standard and Poor's Composite (S&P500), the low Book to Market portfolio (Growth series) and the high Book to Market portfolio (Value series) of Fama and French. The numerical results show that skewness and excess kurtosis depend almost exclusively on the jump contribution. The skewness is mainly due to jump mean, which also determines the sign, and to Poisson intensity. The kurtosis strongly depends on jump volatility and also on the Poisson intensity. The theoretical mean is dominated by the diffusion mean μ which is compensated by a half of the long run variance $\vartheta/2$. It also includes a Poisson term, however, its contribution is minimal. The variance is made up of a jump component and a stochastic volatility component in the same order of magnitude, however, the jump term is more important for the S&P500 series. The volatility depends on model volatility parameters: jump volatility, long term volatility and volatility of stochastic volatility, and also on the mean reversion κ. A closer look at Growth and Value series reveals that the largest variance of Growth series, compared with that of Value series, is mainly due to the low frequency of occurrence of unexpected large events in Value series, that lessens its impact in its own variance, and also because of the larger long term volatility of the Growth series.

The parameter estimates are used to calculate the portfolio allocation. The weights are found to be low compared with those of a standard diffusion model. This is because the investor perceives more risk coming from jumps and stochastic volatility. We calculate a myopic demand as the ratio of the risk premium over the instantaneous volatility times the risk aversion, it proves to be greater than the total portfolio weight. The discrepancy could be explained if we define the intertemporal hedging demand as the difference between the total portfolio weight and the myopic contribution. Given that the expectations of asset performance worsen, the intertemporal hedging demand is negative thus reducing the participation of the myopic component. This is mainly due to the negative jump mean. The ratio IHD - myopic demand is calculated, it is fairly constant for S&P500 and decreases for Growth and Value. They seems to converge, suggesting an asymptotic risk premium associated to stochastic volatility and jumps.

Finally, a portfolio composed of a 60% on risky assets was established as a benchmark to compare investments in the three series. The results suggest that investors would prefer investing in Value portfolios. A sensitivity analysis reveals that S&P500 allocation is the most sensitive to interest rate changes.

There are at least two topics for future research. Firstly, the analysis of the impact on the moments by varying the parameters of the market model was performed in numerical fashion in this instance, however, a sensitivity analysis is necessary to draw general conclusions. Secondly, the term in the portfolio rule involving the jump component can be further exploited to determine both its impact in the allocation and its role in the market price of jump risk. It would also enhance the numerical calculations.

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Appendix A. Solution to HJB

The HJB equation (4.15) conveys a partial differential-difference equation (PDDE) for the value function and an optimization problem w.r.t. the set of variables $\{\theta,\varphi\}$. The PDDE comes from the generator equation (4.17), when evaluated at the optimum point $\{\widehat{\theta},\widehat{\varphi}\}$ it takes the following form

$$\frac{\partial\Phi}{\partial\tau} = -r\widehat{\xi}\frac{\partial\Phi}{\partial\widehat{\xi}} + \kappa(\vartheta - V)\frac{\partial\Phi}{\partial V} + \frac{|\widehat{\theta}|^2}{2}\widehat{\xi}^2\frac{\partial^2\Phi}{\partial\widehat{\xi}^2} + \frac{\sigma^2}{2}V\frac{\partial^2\Phi}{\partial V^2}$$
$$- \varrho\theta\widehat{\xi}\frac{\partial^2\Phi}{\partial\widehat{\xi}\partial V} + \lambda E^z\left[\Phi(\widehat{\varphi}(z)\widehat{\xi}) - \Phi(\widehat{\xi}) + (1-\widehat{\varphi}(z))\widehat{\xi}\frac{\partial\Phi}{\partial\widehat{\xi}}\right]$$

where $\widehat{\xi}$ is the optimum of $\bar{\xi}$ and $\widehat{\xi}$ is the optimum of ξ respectively. From equations (4.18) and (4.16), the boundary condition reads

$$\lim_{t\to T^-}\Phi(\widehat{\xi}_t, V_t, t) = \widetilde{u}(\widehat{\xi}_T) \tag{A.1}$$

and the infimum is calculated from

$$\inf_{(\theta,\varphi)\in\mathcal{U}}\left\{\frac{|\theta|^2}{2}\bar{\xi}^2\frac{\partial^2\Phi}{\partial\bar{\xi}^2} - \varrho\theta\bar{\xi}\frac{\partial^2\Phi}{\partial\bar{\xi}\partial V} + \lambda E^z\left[\Phi(\varphi(z)\bar{\xi}) - \Phi(\bar{\xi}) + (1-\varphi(z))\bar{\xi}\frac{\partial\Phi}{\partial\bar{\xi}}\right]\right\}$$

Subject to restriction (3.3),

$$\mu - r - \theta_1\sqrt{V} + \lambda E^z[\phi(z)\varphi(z)] = 0$$

This assures that $\overline{S}$ is a $\mathcal{F}_t-$martingale. The HJB differential equation is solved by the method of variation of parameters and variable separation. In order to find a candidate we first calculate the Legendre-Fenchel transform (LFT), equation (4.8) of the modified power utility

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}H(V,\tau)$$

For this function $I(p) = (pH^{-1})^{-1/\gamma}$, hence the LFT is

$$\widetilde{u}(p) = -\frac{p^{\bar{\gamma}}}{\bar{\gamma}}H^{1/\gamma} \qquad \bar{\gamma} = \frac{\gamma-1}{\gamma}$$

Notice that $\bar{\gamma} - 1 = -1/\gamma$, and

$$\widetilde{u}(kp) = -\frac{(kp)^{\bar{\gamma}}}{\bar{\gamma}} = k^{\bar{\gamma}}\widetilde{u}(p)$$

With $f(k) = k^{\bar{\gamma}} > 0$ for $k > 0$, and $g(k) = 0$, therefore $\widetilde{u}(p)$ is also iso-elastic. Motivated by the LFT, we guess that the dual value function has the form

$$\Phi(\bar{\xi}, V, \tau) = -\frac{\bar{\xi}^{\bar{\gamma}}}{\bar{\gamma}} e^{(1-\bar{\gamma})(a(\tau)+b(\tau)V)} \tag{A.2}$$

Notice that $\Phi(\bar{\xi}, V, \tau) < 0$ for $\bar{\xi} > 0$. Its partials are

$$\frac{\partial \Phi}{\partial \bar{\xi}} = \frac{\bar{\gamma}}{\bar{\xi}}\Phi \qquad \frac{\partial \Phi}{\partial V} = (1-\bar{\gamma})b\Phi \qquad \frac{\partial \Phi}{\partial \tau} = (1-\bar{\gamma})\left(a_\tau + b_\tau V\right)\Phi$$
$$\frac{\partial^2 \Phi}{\partial \bar{\xi}^2} = \frac{\bar{\gamma}(\bar{\gamma}-1)}{\bar{\xi}^2}\Phi \qquad \frac{\partial^2 \Phi}{\partial V^2} = (1-\bar{\gamma})^2 b^2 \Phi \qquad \frac{\partial \Phi}{\partial \bar{\xi}\partial V} = \frac{\bar{\gamma}(1-\bar{\gamma})}{\bar{\xi}} b\Phi$$

Inserting into the HJB

$$\begin{aligned}(1-\bar{\gamma})(a_\tau + b_\tau V)\Phi \\ = \inf_{(\theta,\varphi)\in\mathcal{U}} \Big\{ -\bar{\gamma}r + (1-\bar{\gamma})b\kappa(\vartheta - V)\Phi + \frac{\bar{\gamma}(\bar{\gamma}-1)}{2}|\theta|^2\Phi \\ +(1-\bar{\gamma})\frac{\sigma^2}{2}b^2 V\Phi - \bar{\gamma}(1-\bar{\gamma})b\varrho\theta\Phi + \lambda E^z[\varphi^{\bar{\gamma}}(z) - 1 + \bar{\gamma}(1-\varphi(z))]\Phi \Big\}\end{aligned} \tag{A.3}$$

First we obtain the control process and then the result is inserted into the PDDE and it is subsequently solved.

A.1. **Optimal θ, φ.** The optimization problem for θ, φ is

$$\sup_{(\theta,\varphi)\in\mathcal{U}} \left\{ \frac{\bar{\gamma}(\bar{\gamma}-1)}{2}|\theta|^2 + \bar{\gamma}(\bar{\gamma}-1)b\varrho\theta + \lambda E^z\left[\varphi^{\bar{\gamma}} - \bar{\gamma}\varphi\right] \right\} \tag{A.4}$$

To find the optimum we construct the lagrangian

$$\begin{aligned}\mathcal{L}(\theta,\varphi,\chi) = \frac{\bar{\gamma}(\bar{\gamma}-1)}{2}|\theta|^2 + \bar{\gamma}(\bar{\gamma}-1)b\varrho\theta \\ + \lambda E^z[\varphi^{\bar{\gamma}}(z) - \bar{\gamma}\varphi(z)] - \chi(\mu - r - \theta_1\sqrt{V} + \lambda E^z[\varphi(z)\phi(z)])\end{aligned} \tag{A.5}$$

where χ is the lagrangian multiplier. To find the set of optimal values we match to zero the lagrangian partials w.r.t. each variable. After reordering we arrive at the following expressions

$$\begin{aligned}\widehat{\theta}_1 &= -b\rho\sigma\sqrt{V} - \frac{\widehat{\chi}}{\bar{\gamma}(\bar{\gamma}-1)}\sqrt{V} \\ \widehat{\theta}_2 &= -b\bar{\rho}\sigma\sqrt{V} \\ E[\widehat{\varphi}^{\bar{\gamma}-1}] &= 1 + \frac{\widehat{\chi}}{\bar{\gamma}}\bar{\phi} \\ 0 &= \mu - r - \widehat{\theta}_1\sqrt{V} + \lambda E[\widehat{\varphi}\phi]\end{aligned} \tag{A.6}$$

The variable $\widehat{\chi}$ is still missing. To reach to an explicit expression we use equation (5.4), (4.19), and relation (4.13), hence

$$\widehat{X}_t = X_0 B_t \frac{E_t^{\widehat{\mathbf{Q}}}[\widehat{\xi}_T^{\bar{\gamma}-1}]}{E[\widehat{\xi}_T^{\bar{\gamma}}]} \tag{A.7}$$

from this and the dual value function, equation (A.2), evaluated at the optimum we get the optimal wealth at time $t \in [0, T]$ as

$$\widehat{\xi}(t)\widehat{X}(\widehat{\xi}, V, t) = \frac{X_0 B_t}{E[\beta_T^{\bar{\gamma}}\widehat{\xi}_T^{\bar{\gamma}}]}\beta_t^{\bar{\gamma}}\widehat{\xi}^{\bar{\gamma}}(t)e^{(1-\bar{\gamma})(a(\tau)+b(\tau)V)} \tag{A.8}$$

Applying Ito's lemma we obtain

$$\begin{aligned}d\widehat{\xi}\widehat{X} = \left(\mathcal{A}_{\widehat{\xi}\widehat{X}} - \frac{\partial}{\partial\tau}\right)(\widehat{\xi}\widehat{X}) - \widehat{\xi}(\widehat{\xi}\widehat{X})_{\widehat{\xi}}\widehat{\theta}' dW_t \\ + (\widehat{\xi}\widehat{X})_V \varrho dW_t + \int_{\mathbb{R}}((\widehat{\xi}\widehat{X})(\widehat{\varphi}(z)\widehat{\xi}_{-}) - (\widehat{\xi}\widehat{X})(\widehat{\xi}_{-}))M(dt, dz)\end{aligned}$$

where $\mathcal{A}_{\widehat{\xi}\widehat{X}}$ is the infinitesimal generator for $\widehat{\xi}\widehat{X}$, thus

$$d\widehat{\xi}_t\widehat{X}_t = -(\bar{\gamma}\widehat{\theta}' + (\bar{\gamma}-1)b\varrho)\widehat{\xi}_t\widehat{X}_t dW_t + \int_{\mathbb{R}}(\widehat{\varphi}^{\bar{\gamma}}(z) - 1)\widehat{\xi}_{t^-}\widehat{X}_{t^-}M(dt, dz)$$

because $\widehat{\xi}_t\widehat{X}_t$ is a martingale. Comparing terms with equation (3.7) we found

$$\begin{aligned}\widehat{\theta}_1 &= -b\rho\sigma\sqrt{V} - \frac{\widehat{\alpha}}{\bar{\gamma}-1}\sqrt{V} \\ \widehat{\theta}_2 &= -b\bar{\rho}\sigma\sqrt{V}\end{aligned}$$

$$\widehat{\varphi} = (1+\widehat{\alpha}\phi)^{\frac{1}{\bar{\gamma}-1}}$$

In order to match these expressions with those of (A.6) we require that $\widehat{\alpha} = \widehat{\chi}/\bar{\gamma}$. We also found

$$\widehat{\alpha} = \frac{1}{\gamma}\frac{\mu - r}{V} + \frac{1}{\gamma}b\rho\sigma + \frac{1}{\gamma}\frac{\lambda}{V}E[\phi(1+\widehat{\alpha}\phi)^{-\gamma}]$$

which is the expression for the optimal portfolio weight.

A.2. **Solution of the PDDE.** At the optimal point $(\widehat{\theta}, \widehat{\varphi})$ the partial differential-difference equation (A.3) becomes

$$\begin{aligned} a_\tau + b_\tau V = \frac{\bar{\gamma}}{\bar{\gamma}-1}r + b\kappa(\vartheta - V) - \frac{\bar{\gamma}}{2}|\widehat{\theta}|^2 \\ + b^2(1-\bar{\gamma})\frac{\sigma^2}{2}V - b\bar{\gamma}\varrho\widehat{\theta} + \frac{\lambda}{1-\bar{\gamma}}E^z[\widehat{\varphi}^{\bar{\gamma}}(z) - 1 + \bar{\gamma}(1-\widehat{\varphi}(z))] \end{aligned}$$

From equations (A.1) and (A.2) the boundary conditions are $a(0) = 0$ and $b(0) = 0$. Notice that

$$\begin{aligned} \varrho\widehat{\theta} &= -b\sigma^2 V - \rho\sigma V\frac{\widehat{\alpha}}{(\bar{\gamma}-1)} \\ |\widehat{\theta}|^2 &= b^2\sigma^2 V + 2b\rho\sigma V\frac{\widehat{\alpha}}{\bar{\gamma}-1} + V\frac{\widehat{\alpha}^2}{(\bar{\gamma}-1)^2} \end{aligned}$$

Inserting the solutions into the PDDE, we end up with a system of two ordinary differential equations of the Ricatti type

$$\begin{aligned} b_\tau &= b^2\frac{\sigma^2}{2} + b\left(2\rho\sigma\frac{\bar{\gamma}\widehat{\alpha}}{1-\bar{\gamma}} - \kappa\right) - \frac{\bar{\gamma}\widehat{\alpha}^2}{2(\bar{\gamma}-1)^2} \\ a_\tau &= \frac{\bar{\gamma}}{\bar{\gamma}-1}r + b\kappa\vartheta + \frac{\lambda}{1-\bar{\gamma}}E[\widehat{\varphi}^{\bar{\gamma}} - 1 + \bar{\gamma}(1-\widehat{\varphi})] \end{aligned}$$

The solutions (see equations (C.4) and (C.5)) are

$$\begin{aligned} a(\tau) &= \frac{\bar{\gamma}}{\bar{\gamma}-1}r\tau + a_2(\tau) \\ a_2(\tau) &= \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{r_2 - r_1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) + \frac{\lambda\tau}{(1-\bar{\gamma})}E[\widehat{\varphi}^{\bar{\gamma}} - 1 + \bar{\gamma}(1-\widehat{\varphi})] \qquad \text{(A.9)} \\ b(\tau) &= -\frac{2}{\sigma^2}\frac{r_1 r_2 e^{r_1\tau} - r_1 r_2 e^{r_2\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} \end{aligned}$$

where

$$r_{1,2} = \frac{1}{2}\left(2(\gamma-1)\rho\sigma\widehat{\alpha} - \kappa \pm \sqrt{(2(\gamma-1)\rho\sigma\widehat{\alpha} - \kappa)^2 + \gamma(\gamma-1)\sigma^2\widehat{\alpha}^2}\right) \tag{A.10}$$

Appendix B. Estimation

This appendix is devoted to the construction of the conditional characteristic function from the ODE's that arises in the estimation procedure.

B.1. **Conditional Characteristic function.** Consider the ODE's

$$\begin{aligned} b_\tau &= \kappa\vartheta a(\zeta,\tau) \\ b &= \kappa\vartheta \int a(\zeta,\tau)d\tau + C \\ &= 2\kappa^2\vartheta i\zeta \int \frac{e^{-\kappa\tau}d\tau}{2\kappa - i\zeta\sigma^2(1-e^{-\kappa\tau})} + C \\ &= 2\kappa^2\vartheta i\zeta \int \frac{d\tau}{(2\kappa - i\zeta\sigma^2)e^{\kappa\tau} + i\zeta\sigma^2} + C \end{aligned}$$

The integral, see Gradshteyn and Ryzhik (2007), is

$$\int \frac{d\tau}{(2\kappa - i\zeta\sigma^2)e^{\kappa\tau} + i\zeta\sigma^2} = \frac{1}{\kappa i\zeta\sigma^2} \ln\left(\frac{e^{\kappa\tau}}{(2\kappa - i\zeta\sigma^2)e^{\kappa\tau} + i\zeta\sigma^2}\right)$$

Therefore

$$b(\zeta,\tau) = \frac{2\kappa\vartheta}{\sigma^2} \ln\left(\frac{e^{\kappa\tau}}{(2\kappa - i\zeta\sigma^2)e^{\kappa\tau} + i\zeta\sigma^2}\right) + C$$

applying the boundary condition $b(\zeta,0)=0$, then $C = (2\kappa\vartheta/\sigma^2)\ln 2\kappa$, and

$$b(\zeta,\tau) = \frac{2}{\sigma^2}\kappa\vartheta \ln\left(\frac{2\kappa}{2\kappa - i\zeta\sigma^2(1-e^{-\kappa\tau})}\right) \tag{B.1}$$

On the other hand, to solve the differential equation (6.6), we insert the partials of the CCF solution $\Phi(\zeta,\tau,\Theta,Y) = \exp\{c(\zeta,\tau)Y + a(\zeta,\tau)V + b(\zeta,\tau)\}$ into the PDDE. As a result we obtain

$$c_\tau Y + a_\tau V + b_\tau = \left(\mu - \frac{V}{2}\right)c + \kappa(\vartheta - V)a$$

$$+\frac{V}{2}(c^2+\sigma^2a^2+2\rho\sigma ac)+\lambda E[e^{cJ}-1]$$

separating variables

$$c_\tau = 0$$
$$a_\tau = \frac{c^2}{2}-\frac{c}{2}+(\rho\sigma c-\kappa)a+\frac{\sigma^2}{2}a^2$$
$$b_\tau = \mu c+\kappa\vartheta a+\lambda E[e^{cJ}-1]$$

From the first ODE $c(\zeta,\tau)$ is a constant on τ. Applying the boundary condition we have $c(\zeta)=i\zeta$. The second ODE it a Riccati type whose solution, equation (C.4), is

$$a(\zeta,\tau)=-\frac{2}{\sigma^2}\frac{r_1r_2e^{r_1\tau}-r_1r_2e^{r_2\tau}}{r_2e^{r_1\tau}-r_1e^{r_2\tau}}$$

with

$$r_{1,2}=\frac{1}{2}\left(i\rho\sigma\zeta-\kappa\pm\sqrt{(i\rho\sigma\zeta-\kappa)^2-i\sigma^2\zeta(i\zeta-1)}\right)$$

For the second ODE we have

$$\dot{b}=\mu c+\kappa\vartheta a+\lambda E[e^{cJ}-1]$$
$$b=\mu c\tau+\lambda E[e^{cJ}-1]\tau+\kappa\vartheta\int a(\zeta,\tau)d\tau+C$$

From (C.5), and applying boundary condition

$$b=\mu c\tau+\lambda E[e^{cJ}-1]\tau+\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{r_2-r_1}{r_2e^{r_1\tau}-r_1e^{r_2\tau}}\right)$$

The expected value is calculated over an exponential function which in turn depends on the normal random variable $J\sim N(\mu_J,\nu^2)$. This expression is the characteristic function of a normal random variable

$$E[e^{cJ}]=\exp\left(i\zeta\mu_J-\frac{\upsilon^2}{2}\zeta^2\right)$$

Summing up, the CCF for the market model is:

$$
\begin{aligned}
\Phi(\zeta,\tau,\Theta,\ln S,V) &= \exp\{c(\zeta,\tau)\ln S + a(\zeta,\tau)V + b(\zeta,\tau)\} \\
&= \exp\left\{ i\zeta\ln S + i\zeta\mu\tau - \frac{2}{\sigma^2}\frac{r_1r_2e^{r_1\tau} - r_1r_2e^{r_2\tau}}{r_2e^{r_1\tau} - r_1e^{r_2\tau}}V \right. \\
&\left. +\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{r_2-r_1}{r_2e^{r_1\tau} - r_1e^{r_2\tau}}\right) + \lambda\exp\left(i\zeta\mu_J - \frac{\upsilon^2}{2}\zeta^2\right)\tau - \lambda\tau \right\}
\end{aligned}
$$

To deal with the unobservable volatility parameter we take the expected value of the market model CCF w.r.t V, $\Phi(\zeta,\tau,\Theta,\ln S) = E^V[\Phi(\zeta,\tau,\Theta,\ln S,V)]$. This can be made using the unconditional probability density function, equation (6.5). Alternatively, if we realize that the expected value of the exponentiated volatility is its characteristic function, that has previously been calculated in equation (6.4), we simply need to insert it into the previous equation,

$$
\begin{aligned}
\Phi(\zeta,\tau,\Theta,\ln S) &= \exp\left\{ i\zeta\ln S + i\zeta\mu\tau + \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa}{2\kappa - a(\zeta,\tau)\sigma^2}\right)\right. \\
&\left. +\frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{r_2-r_1}{r_2e^{r_1\tau} - r_1e^{r_2\tau}}\right) + \lambda\exp\left(i\zeta\mu_J - \frac{\upsilon^2}{2}\zeta^2\right)\tau - \lambda\tau \right\}
\end{aligned}
$$

Rearranging terms we end up with

$$
\begin{aligned}
\text{(B.2)}\quad &\Phi(\zeta,\tau,\Theta,\ln S) = \\
&\exp\left\{ i\zeta\ln S + i\zeta\mu\tau + \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa(r_2-r_1)}{(2\kappa - a(\zeta,\tau)\sigma^2)(r_2e^{r_1\tau} - r_1e^{r_2\tau})}\right)\right. \\
&\left. +\lambda\tau\exp\left(i\zeta\mu_J - \frac{\upsilon^2}{2}\zeta^2\right) - \lambda\tau \right\}
\end{aligned}
$$

B.2. **Moments.** The return's unconditional moments are recovered with the aid of the cumulants, that is, the derivatives of the cumulant function $\ln\Phi$, being Φ the return's CF, equation (6.8),

$$
\begin{aligned}
&\ln\Phi(\zeta,\tau,\Theta,r) = \\
&\quad i\zeta\mu\tau + \frac{2\kappa\vartheta}{\sigma^2}\ln\left(\frac{2\kappa(r_2-r_1)}{(2\kappa - a(\zeta,\tau)\sigma^2)(r_2e^{r_1\tau} - r_1e^{r_2\tau})}\right)
\end{aligned}
$$

$$+ \lambda\tau \exp\left(i\zeta\mu_J - \frac{\upsilon^2}{2}\zeta^2\right) - \lambda\tau$$

Notice that

$$\begin{aligned}(2\kappa - a(\zeta,\tau)\sigma^2)(r_2 e^{r_1\tau} - r_1 e^{r_2\tau}) &= 2[\kappa r_2 e^{r_1\tau} - \kappa r_1 e^{r_2\tau} + r_1 r_2 e^{r_1\tau} - r_1 r_2 e^{r_2\tau}] \\ &= 2[r_2 e^{r_1\tau}(\kappa + r_1) - r_1 e^{r_2\tau}(\kappa + r_2)]\end{aligned}$$

If we call $A(\zeta) = e^{r_1\tau}(\kappa + r_1) - r_1 e^{r_2\tau}(\kappa + r_2)$ and $B(\zeta) = r_2 - r_1$ the expression

$$f(\zeta) = \ln\left(\frac{2\kappa(r_2 - r_1)}{(2\kappa - a(\zeta,\tau)\sigma^2)(r_2 e^{r_1\tau} - r_1 e^{r_2\tau})}\right)$$

can be written as

$$\begin{aligned}f(\zeta) &= \ln\left(\frac{\kappa B(\zeta)}{A(\zeta)}\right) \\ &= \ln(\kappa) + \ln(B(\zeta)) - \ln(A(\zeta))\end{aligned}$$

The first derivative is

$$f^{(1)}(\zeta) = \frac{B^{(1)}(\zeta)}{B(\zeta)} - \frac{A^{(1)}(\zeta)}{A(\zeta)}$$

The higher order derivatives are found making use of the Leibniz theorem

$$f^{(n)}(\zeta) = \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} \left[B^{(i+1)}(B^{-1})^{(n-1-i)} - A^{(i+1)}(A^{-1})^{(n-1-i)}\right], \qquad n > 1$$

Recall that cumulants are defined as

$$K_n = \frac{1}{i^n}\frac{d}{d\zeta}\ln\Phi(\zeta)\bigg|_{\zeta=0}$$

Therefore the first four cumulants are given by

(B.3)

$$K_1 = \left[\mu - \frac{\vartheta}{2} + \lambda\mu_J\right]\tau$$

$$K_2 = (\mu_J^2 + \upsilon^2)\lambda\tau + \frac{\vartheta}{4\kappa^3}((\sigma^2 - 4\kappa\rho\sigma + 4\kappa^2)\kappa\tau - (\sigma^2 - 4\kappa\rho\sigma)(1 - e^{-\kappa\tau})) - \frac{\vartheta}{4\kappa}(\sigma^2 - 4\kappa\rho\sigma)(1 - e^{-\kappa\tau})$$

$$K_3 = (3\mu_J\upsilon^2 + \mu_J^3)\lambda\tau - \frac{3\sigma\vartheta}{8\kappa^5}(\sigma - 2\kappa\rho)[(\sigma^2 - 4\kappa\rho\sigma + 4\kappa^2)\kappa\tau - 2(\sigma^2 - 4\kappa\rho\sigma + 2\kappa^2) + ((\sigma^2 - 4\kappa\rho\sigma)\kappa\tau + 2(\sigma^2 - 4\kappa\rho\sigma + 2\kappa^2))e^{-\kappa\tau}]$$

$$K_4 = (3\upsilon^4 + 6\mu_J^2\upsilon^2 + \mu_J^4)\lambda\tau + \frac{3\sigma^2\vartheta}{32\kappa^7}\{[(4\kappa^2 - 4\rho\kappa\sigma + \sigma^2)^2 + 4\sigma(4\kappa\rho - \sigma)(2\kappa\rho - \sigma)^2 e^{-\kappa\tau}]\kappa^2\tau^2 - [4(4\kappa^2 - 4\rho\kappa\sigma + \sigma^2)(4\kappa^2\rho^2 - 2\kappa^2 - 4\kappa\rho\sigma + \sigma^2) + 2Ae^{-\kappa\tau}]\kappa\tau - [B + 2Ce^{-\kappa\tau}]\}$$

where

$$A = 64\kappa^4\rho^2 - 128\kappa^3\rho^3\sigma - 80\kappa^3\rho\sigma + 176\kappa^2\rho^2\sigma^2 + 20\kappa^2\sigma^2 - 72\kappa\rho\sigma^3 + 9\sigma^4$$

$$B = 32\kappa^4 + 26\sigma^4 + 384\kappa^4\rho^2 + 64\kappa^4\rho^4 - 208\kappa\rho\sigma^3 + 16\kappa^2\sigma^2(33\rho^2 + 7) - 448\kappa^3\rho\sigma(\rho^2 + 1)$$

$$C = 128\kappa^4\rho^2 + 16\kappa^4 - 160\kappa^3\rho^3\sigma - 160\kappa^3\rho\sigma + 216\kappa^2\rho^2\sigma^2 + 40\kappa^2\sigma^2 - 88\kappa\rho\sigma^3 + 11\sigma^4$$

The moments can be retrieved from the cumulants in the usual form, namely

(B.4)

$$\text{Mean} = K_1$$

$$\text{Standard Deviation} = \sqrt{K_2}$$

$$\text{Skewness} = \frac{K_3}{K_2^{3/2}}$$

$$\text{Excess Kurtosis} = \frac{K_4}{K_2^2}$$

Appendix C. Some Results in Ordinary Differential Equations

Explicit calculations for ODE's that arise in the paper are carried out here. Related results are also included (refer to Gradshteyn and Ryzhik (2007) for details).

C.1. **Bernoulli ODE.** Consider the differential equation

$$y' = ay + by^2$$

where the function y depends on the independent variable x and a and b are constants. Using the variable change $w = 1/y$, with derivative $w' = -y'/y^2$ the differential equation becomes

$$w' = -aw - b$$

The integrating factor is

$$M(x) = \exp\left(\int a dx\right) = e^{ax}$$

Multiplying both sides and integrating

$$\int (w'e^{ax} + ae^{ax}w)dx = -b\int e^{ax}dx$$

then

$$e^{ax}w = \int (we^{ax})'dx = -b\int e^{ax}dx = -\frac{b}{a}e^{ax} + C$$

hence

$$y = \frac{a}{-b + aCe^{-ax}}$$

applying boundary condition $y(x_0) = y_0$ we get

$$y_0 = \frac{a}{-b + aCe^{-ax_0}}$$

and after some algebra

$$C = \frac{a + by_0}{ay_0}e^{ax_0}$$

Therefore the solution is

$$y = \frac{a y_0 e^{a(x-x_0)}}{a + b y_0 (1 - e^{a(x-x_0)})} \tag{C.1}$$

For an ODE with non constant coefficients

$$y' = a(x)y + b(x)y^2$$

$$y(x) = \exp\left\{-\int a(x)dx\right\}\left[C - \int b(x)\exp\left\{\int a(x)dx\right\}dx\right] \tag{C.2}$$

for a constant of integration C.

C.2. **Riccati ODE.** The following ODE

$$a_\tau = q_0 + q_1 a + q_2 a^2$$

is of Riccati type. To solve it we first consider the self-adjoint ODE

$$u_{\tau^2} - R u_\tau + S u = 0$$

where $R = q_1$ and $S = q_0 q_2$. The solution to this equation is of type $u(\tau) = e^{r\tau}$ with derivatives $u_\tau(\tau) = re^{r\tau}$ and $u_{\tau\tau}(\tau) = r^2 e^{r\tau}$. Inserting these expressions into the ODE, it is transformed into the algebraic equation $r^2 - Rr + S = 0$. Its roots are given by

$$r_{1,2} = \frac{R \pm \sqrt{R^2 - 4S}}{2} \tag{C.3}$$

the general solution is $u(\tau) = c_1 e^{r_1\tau} + c_2 e^{r_2\tau}$ with derivative $u_\tau(\tau) = c_1 r_1 e^{r_1\tau} + c_2 r_2 e^{r_2\tau}$, then a solution to the Riccati ODE is:

$$a(\tau) = -\frac{u_\tau}{q_0 u} = -\frac{2}{\sigma^2}\frac{c_1 r_1 e^{r_1\tau} + c_2 r_2 e^{r_2\tau}}{c_1 e^{r_1\tau} + c_2 e^{r_2\tau}}$$

Applying the boundary condition $a(0) = 0$

$$0 = -\frac{2}{\sigma^2}\frac{c_1 r_1 + c_2 r_2}{c_1 + c_2}$$

then $c_1 r_1 + c_2 r_2 = 0$, so $c_1 = r_2$ and $c_2 = -r_1$, hence

$$a(\tau) = -\frac{2}{\sigma^2}\frac{r_1 r_2 e^{r_1\tau} - r_1 r_2 e^{r_2\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} \tag{C.4}$$

C.3. **Ricatti's solution Integral.** We need the integral of the equation (C.4), that is

$$I(\tau) = \int a(\tau) d\tau$$

If we let $r_2 - r_1 = -\sqrt{R^2 - 4S} = \Lambda$, we first calculate the integral

$$\begin{aligned}
\int \frac{e^{r_1\tau} - e^{r_2\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} d\tau &= \int \frac{e^{r_1\tau} d\tau}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} - \int \frac{e^{r_2\tau} d\tau}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}} \\
&= \int \frac{d\tau}{r_2 - r_1 e^{\Lambda\tau}} - \int \frac{d\tau}{r_2 e^{-\Lambda\tau} - r_1} \\
&= \frac{1}{r_2\Lambda} \ln\left(\frac{e^{\Lambda\tau}}{r_2 - r_1 e^{\Lambda\tau}}\right) - \frac{1}{r_1\Lambda} \ln\left(\frac{e^{-\Lambda\tau}}{r_2 e^{-\Lambda\tau} - r_1}\right) \\
&= \frac{1}{r_2\Lambda}\left[\ln\left(\frac{e^{r_2\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) + r_2\tau\right] - \frac{1}{r_1\Lambda}\left[\ln\left(\frac{e^{r_1\tau}}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) + r_1\tau\right] \\
&= \frac{1}{r_2\Lambda} \ln\left(\frac{1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) - \frac{1}{r_1\Lambda} \ln\left(\frac{1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) \\
&= \left(\frac{1}{r_2} - \frac{1}{r_1}\right)\frac{1}{\Lambda} \ln\left(\frac{1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) \\
&= -\frac{1}{r_1 r_2} \ln\left(\frac{1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right)
\end{aligned}$$

The integral in the second line can be found in Gradshteyn and Ryzhik (2007), therefore

$$\int a(\tau) d\tau = \frac{2}{\sigma^2} \ln\left(\frac{1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) + C$$

Applying the boundary condition $I(0) = 0$

$$\begin{aligned}
0 &= \frac{2}{\sigma^2} \ln\left(\frac{1}{r_2 - r_1}\right) + C \\
C &= \frac{2}{\sigma^2} \ln(r_2 - r_1)
\end{aligned}$$

then

$$I(\tau) = \frac{2}{\sigma^2} \ln\left(\frac{r_2 - r_1}{r_2 e^{r_1\tau} - r_1 e^{r_2\tau}}\right) \tag{C.5}$$

Table 1

Descriptive Statistics

This table reports the descriptive statistics of the daily returns for the Standard and Poor's Composite Index series along with the Growth and Value portfolios constructed by Fama and French from their ten Book-to-Market portfolios. The data comprises the returns from January 1982 to October 2010. The statistics are given in daily basis and percentage format, except for the skewness and excess kurtosis.

Statistic	S&P500	Growth	Value
Min	-9.47	-9.7	-7.71
Mean	0.0294	0.0677	0.1382
Median	0.0369	0.0900	0.1297
Max	10.25	8.89	7.41
Volatility	0.9925	1.0364	0.7213
Skewness	-0.1451	-0.2351	-0.4616
Excess Kurtosis	9.64	10.23	18.22

Table 2

Garch Estimates

This table reports the results of a Garch(1,1) estimates: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1}$ for the Standard and Poors 500 series along with the Growth and Value portfolios constructed by Fama and French from their ten Book-to-Market portfolios. The data comprises the returns from January 1982 to October 2010, the long term variance is reported at the bottom of the table.

Parameter	S&P500	Growth	Value
α_0	0.000065	0.000142	0.000054
α_1	5.237735	12.248057	10.195925
β_1	94.114545	86.619471	88.698658
$\frac{\alpha_0}{1-\alpha_1-\beta_1}$	0.009980	0.012579	0.004885

Table 3

Parameter Estimates for the Return and Stochastic Volatility Processes

This table reports the estimates of the parameters for the stochastic volatility - jump diffusion log-returns model for the S&P500, Growth and Value series.

$$d \ln S = \left(\mu - \frac{V}{2}\right) dt + \sqrt{V} dW_1 + J dN$$
$$dV = \kappa(\vartheta - V)dt + \varrho dW$$

where $W = (W_1, W_2)'$ is a $\mathbb{R}^2$ standard Brownian motion, $\varrho = \sigma\sqrt{V}(\rho, \bar{\rho})$ is the correlation vector, N is a Poisson Process and $J \sim Normal(\mu_J, \nu^2)$ is the percentage change conditioned on the occurrence of a jump. The parameters are given in daily basis in percentage form.

Series		μ	σ	κ	ϑ	ρ	μ_J	ν	λ	ε
S&P500	Unc	0.0408	0.0017	1.49	0.0035	-0.8561	-0.0745	2.2173	0.1287	3.77e-14
	Cons	0.0380	0.0028	1.88	0.0096	-0.3683	-3.5644	14.6230	0.0001	3.53e-07
Growth	Unc	0.0808	0.0031	1.58	0.0062	-0.4254	-0.1931	2.9450	0.0519	2.46e-15
	Cons	0.0741	0.0110	0.64	0.0121	-0.6787	-2.7246	12.3100	0.0002	1.02e-06
Value	Unc	0.1470	0.0016	0.61	0.0032	-0.4976	-0.2829	2.8166	0.0256	2.28e-15
	Cons	0.1418	0.0024	0.97	0.0049	-0.4978	-1.8930	6.9413	0.0006	4.49e-12

Table 4

Contribution to Moments

This table reports the theoretical moments, appendix - equation (B.4), breaking down into jump and stochastic volatility contributions. Panel A of the table reports the theoretical moments: mean, standard deviation, skewness and excess kurtosis, calculated for the unconstrained parameters in the instantaneous volatility case. The function error is also given. Panel B reports the theoretical moments separated into jump and stochastic volatility (SV) components, the diffusion contribution to the mean is included in the jump part. Panel C disentangles the two main jump components of the variance.

Panel A. Theoretical Moments

Series	Mean	Volatility	Skewness	Kurtosis	ε
S&P500	0.0294	0.9925	-0.1451	9.64	9.72e-10
Growth	0.0677	1.0364	-0.2351	10.23	6.39e-08
Value	0.1382	0.7213	-0.4616	18.22	4.52e-08

Panel B. Contribution of Jump and SV to the Theoretical Moments

Series	Mean		Variance		Skewness		Kurtosis	
	Jump	SV	Jump	SV	Jump	SV	Jump	SV
S&P500	3.12e-04	-1.76e-05	6.33e-05	3.52e-05	-0.1446	-5.19e-04	9.63	-1.29e-05
Growth	7.08e-04	-3.11e-05	4.52e-05	6.22e-05	-0.2344	-6.99e-04	10.23	-6.88e-06
Value	1.40e-03	-1.58e-05	2.05e-05	3.15e-05	-0.4608	-8.14e-04	18.21	-2.12e-04

Panel C. Disentangled Variance's Jump Component

Series	λ	Factor
Growth	0.0519	8.7e-04
Value	0.0256	8.0e-04

Table 5

Portfolio Results for a Diffusion Model

The table reports the portfolio weights for an investor who is not taking into account the possible occurrence of jumps and considers a constant volatility, which is the Merton's portfolio problem. Weights are calculated for the S&P500, the Fama and French's Growth portfolio, and the Fama and French's Value portfolio. The interest rate is 4.7% annualized, and the risk aversion coefficient ranges from 2 to 10.

γ	S&P500	Growth	Value
2	0.5464	2.2824	11.4846
3	0.3643	1.5216	7.6564
4	0.2732	1.1412	5.7423
5	0.2186	0.9130	4.5938
6	0.1821	0.7608	3.8282
7	0.1561	0.6521	3.2813
8	0.1366	0.5706	2.8711
9	0.1214	0.5072	2.5521
10	0.1093	0.4565	2.2969

Table 6

Portfolio Results Using Instantaneous Volatility Estimates

Panel A of this table reports the portfolio weights for an investor who invests in the Standard and Poors 500, and a riskless asset with an annualized return rate of 4.7%. The investor maximizes the power expected utility of terminal wealth with constant relative risk aversion which varies from 2 to 10. He is taking into account occurrences of unexpected jumps and assumes that volatility is stochastic. The column labeled α_1 corresponds to the weight associated with the unconstrained SVJ parameter estimates, α_{myo_1} is the associated portfolio myopic component and ε_1 is the function error, the parameters subscripted with 2 are for the constrained SVJ estimates. Volatility is chosen to be the average instantaneous volatility estimate which is 0.013529024 (in percentage) for the S&P500. Panel B reports the results of the Fama and French's Growth portfolio with volatility 0.014328558, and Panel C does the same for Fama and French's Value portfolio, the volatility is 0.00722463.

Panel A. Standard and Poors Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	0.31523	0.69824	0.81714	0.71395	7.52e-09	3.01e-09
3	0.21016	0.46559	0.54476	0.47597	1.78e-09	6.46e-09
4	0.15762	0.34920	0.40857	0.35698	1.03e-09	1.19e-08
5	0.12609	0.27936	0.32686	0.28558	2.70e-09	2.26e-08
6	0.10508	0.23287	0.27238	0.23798	3.81e-09	2.70e-08
7	0.09007	0.19960	0.23347	0.20399	4.60e-09	2.21e-08
8	0.07881	0.17465	0.20428	0.17849	5.18e-09	1.85e-08
9	0.07005	0.15524	0.18159	0.15866	5.64e-09	1.57e-08
10	0.06305	0.13972	0.16343	0.14279	6.01e-09	1.34e-08

Panel B. Fama and French Growth Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	1.37844	1.85516	2.16935	1.93478	4.10e-09	1.85e-09
3	0.91946	1.23905	1.44623	1.28985	1.15e-08	1.13e-08
4	0.68976	0.93008	1.08467	0.96739	5.37e-09	1.45e-08
5	0.55188	0.74440	0.86774	0.77391	9.07e-09	1.42e-08
6	0.45995	0.62050	0.72312	0.64493	3.23e-09	4.76e-09
7	0.39429	0.53195	0.61981	0.55279	3.05e-08	5.67e-09
8	0.34501	0.46554	0.54234	0.48369	3.65e-09	1.30e-08
9	0.30667	0.41381	0.48208	0.42995	1.71e-08	4.83e-08
10	0.27601	0.37250	0.43387	0.38696	3.36e-08	2.06e-09

Panel C. Fama and French Value Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	6.30781	7.17544	8.88137	8.52080	1.71e-09	1.38e-09
3	4.23240	4.99744	5.92091	5.68053	9.98e-09	9.21e-09
4	3.18359	3.80336	4.44068	4.26040	5.85e-09	4.85e-09
5	2.55107	3.06462	3.55255	3.40832	1.73e-08	1.35e-09
6	2.12818	2.56468	2.96046	2.84027	1.60e-08	1.47e-08
7	1.82554	2.20447	2.53753	2.43451	5.68e-09	8.69e-09
8	1.59826	1.93275	2.22034	2.13020	1.33e-08	3.64e-09
9	1.42126	1.72056	1.97364	1.89351	4.79e-09	9.33e-09
10	1.27953	1.55028	1.77627	1.70416	3.08e-08	1.37e-08

Table 7

Portfolio Sensitivities for Instantaneous Volatilities

This table reports the portfolio weights and function errors calculated with the average instantaneous volatility for riskless rates of return of 3%, 4%, 5% and 6%, and for relative risk aversion coefficients 2, 6 and 10 respectively. Panel A reports the results for the Standard and Poors 500 series, its instantaneous volatility is 0.013529024, Panel B if for Fama and French Growth portfolio which has instantaneous volatility of 0.014328558 and Panel C is for Fama and French Value portfolio with instantaneous volatility of 0.00722463. Myopic weights are also given.

Panel A. Standard and Poors Portfolio Weights with Instantaneous Volatility

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	0.48486	0.94415	1.06646	0.96327	2.13e-10	6.51e-09
6	3	0.16164	0.31490	0.35549	0.32109	4.31e-09	1.03e-08
10	3	0.09699	0.18894	0.21329	0.19265	9.90e-09	3.88e-08
2	4	0.38512	0.79950	0.91980	0.81661	1.57e-08	7.39e-09
6	4	0.12841	0.26665	0.30660	0.27220	3.70e-08	5.39e-09
10	4	0.07704	0.16004	0.18396	0.16332	3.36e-08	5.41e-08
2	5	0.28529	0.65477	0.77314	0.66996	7.85e-09	4.96e-09
6	5	0.09512	0.21835	0.25771	0.22332	2.62e-08	8.87e-09
10	5	0.05707	0.13105	0.15463	0.13399	2.44e-08	4.90e-08
2	6	0.18541	0.50993	0.62649	0.52330	6.14e-09	1.03e-09
6	6	0.06180	0.17004	0.20883	0.17443	9.79e-09	2.02e-08
10	6	0.03711	0.10203	0.12530	0.10466	5.38e-08	1.37e-08

Panel B. Fama and French Growth Portfolio Weights with Instantaneous Volatility

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	1.55580	2.07914	2.40475	2.17018	4.53e-09	1.20e-08
6	3	0.51923	0.69608	0.80158	0.72339	5.55e-09	1.71e-09
10	3	0.31162	0.41795	0.48095	0.43404	5.32e-09	3.73e-08
2	4	1.45149	1.94754	2.26628	2.03171	4.77e-09	7.53e-09
6	4	0.48436	0.65162	0.75543	0.67724	1.27e-08	2.11e-08
10	4	0.29065	0.39122	0.45326	0.40634	5.89e-08	1.26e-09
2	5	1.34710	1.81548	2.12780	1.89324	1.34e-09	2.74e-09
6	5	0.44949	0.60711	0.70927	0.63108	1.35e-08	2.17e-08
10	5	0.26976	0.36446	0.42556	0.37865	4.22e-08	1.40e-08
2	6	1.24263	1.68300	1.98933	1.75476	5.09e-09	8.45e-09
6	6	0.41461	0.56252	0.66311	0.58492	3.71e-08	3.99e-08
10	6	0.24877	0.33769	0.39787	0.35095	4.16e-08	3.14e-09

Panel C. Fama and French Value Portfolio Weights with Instantaneous Volatility

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	6.64002	7.44180	9.34825	8.98767	3.10e-09	5.90e-09
6	3	2.24298	2.69274	3.11608	2.99589	7.58e-10	1.80e-08
10	3	1.34886	1.62934	1.86965	1.79753	3.05e-08	3.43e-08
2	4	6.44491	7.28787	9.07361	8.71304	4.70e-09	6.39e-09
6	4	2.17551	2.61768	3.02454	2.90435	8.97e-09	4.55e-09
10	4	1.30812	1.58301	1.81472	1.74261	1.38e-10	3.39e-08
2	5	6.24887	7.12610	8.79898	8.43841	8.21e-09	3.89e-09
6	5	2.10792	2.54187	2.93299	2.81280	6.47e-09	1.21e-08
10	5	1.26735	1.53620	1.75980	1.68768	3.13e-08	1.06e-09
2	6	6.05196	6.95657	8.52435	8.16377	7.09e-09	7.60e-09
6	6	2.04012	2.46532	2.84145	2.72126	3.48e-09	1.40e-08
10	6	1.22639	1.48908	1.70487	1.63275	3.48e-08	2.01e-09

Table 8

Portfolio Results Using Long Term Volatility Estimates

Panel A of this table reports the portfolio weights for an investor who invests in the Standard and Poors 500, and a riskless asset with an annualized return rate of 4.7%. Volatility is chosen to be the long run estimate in the unconstraint and constraint cases, those are 0.0035182 and 0.0095963 respectively (in percentage). Panel B reports the results of the Fama and French's Growth portfolio with volatilities 0.0062229 and 0.012057, and Panel C does the same for Fama and French's Value portfolio, the volatilities are 0.0031513 and 0.004884 respectively

Panel A. Standard and Poors Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	0.63472	0.97885	3.14226	1.00654	2.90e-09	4.47e-09
3	0.42330	0.65291	2.09484	0.67103	1.40e-08	9.52e-09
4	0.31747	0.48975	1.57113	0.50327	7.72e-10	5.40e-09
5	0.25398	0.39186	1.25690	0.40262	7.03e-09	1.81e-09
6	0.21170	0.32655	1.04742	0.33551	1.73e-08	1.79e-08
7	0.18146	0.27998	0.89779	0.28758	1.37e-08	2.77e-08
8	0.15877	0.24499	0.78557	0.25164	1.10e-08	1.93e-08
9	0.14113	0.21777	0.69828	0.22368	8.87e-09	1.29e-08
10	0.12702	0.19599	0.62845	0.20131	7.19e-09	7.77e-09

Panel B. Fama and French Growth Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	2.39599	2.18410	4.99503	2.29930	6.38e-09	1.15e-08
3	1.60066	1.46075	3.33002	1.53286	1.93e-09	4.28e-10
4	1.20170	1.09700	2.49752	1.14965	9.92e-09	1.41e-08
5	0.96190	0.87823	1.99801	0.91972	3.58e-09	1.96e-08
6	0.80190	0.73226	1.66501	0.76643	1.76e-08	2.42e-08
7	0.68751	0.62782	1.42715	0.65694	5.51e-09	2.34e-09
8	0.60171	0.54949	1.24876	0.57482	2.92e-08	1.87e-08
9	0.53494	0.48852	1.11001	0.51095	3.42e-08	2.03e-08
10	0.48145	0.43975	0.99901	0.45986	2.88e-08	4.28e-08

Panel C. Fama and French Value Portfolio Estimates

γ	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	10.17925	15.46256	20.36131	12.60434	1.93e-09	4.77e-04
3	6.97065	6.30971	13.57421	8.40289	1.18e-09	1.63e-08
4	5.29217	4.94855	10.18066	6.30217	4.73e-09	2.95e-09
5	4.26322	4.04893	8.14452	5.04174	9.51e-09	6.37e-09
6	3.56858	3.41941	6.78710	4.20145	2.62e-09	1.32e-08
7	3.06831	2.95679	5.81752	3.60124	7.79e-09	2.66e-09
8	2.69096	2.60324	5.09033	3.15109	2.60e-09	5.28e-09
9	2.39618	2.32468	4.52474	2.80096	3.04e-09	1.28e-08
10	2.15957	2.09962	4.07226	2.52087	3.59e-09	1.16e-09

Table 9

Portfolio Sensitivities for Long Term Volatility

The following panels reports the sensitivity analysis for the portfolio weights, the riskless rate is varied from 3 percent (annualized) to 6 percent for the selected group of relative risk aversion coefficients, 2, 6 and 10. Panel A reports the results and the function errors for Standard and Poors 500 series, Panel B for Fama and French Growth portfolio and Panel C for Fama and French Value portfolio. Myopic weights are also given.

Panel A. Standard and Poors Portfolio Sensitivity to Rate of Return

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	0.97552	1.32230	4.10099	1.35804	9.42e-10	3.34e-09
6	3	0.32546	0.44145	1.36700	0.45268	2.30e-08	1.30e-08
10	3	0.19532	0.26491	0.82020	0.27161	4.38e-08	3.95e-08
2	4	0.77516	1.12046	3.53703	1.15128	9.30e-11	3.97e-09
6	4	0.25854	0.37389	1.17901	0.38376	7.79e-09	1.67e-08
10	4	0.15513	0.22437	0.70741	0.23026	8.18e-09	2.68e-08
2	5	0.57451	0.91810	2.97307	0.94452	2.61e-10	6.36e-09
6	5	0.19157	0.30632	0.99102	0.31484	3.55e-09	2.50e-08
10	5	0.11494	0.18379	0.59461	0.18890	1.17e-08	1.57e-09
2	6	0.37362	0.71528	2.40911	0.73776	3.45e-09	2.02e-09
6	6	0.12454	0.23855	0.80304	0.24592	1.29e-08	4.81e-09
10	6	0.07472	0.14317	0.48182	0.14755	1.60e-08	1.71e-08

Panel B. Fama and French Growth Portfolio Sensitivity to Rate of Return

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	2.69965	2.44451	5.53707	2.57905	4.33e-09	3.83e-09
6	3	0.90426	0.82092	1.84569	0.85968	1.15e-09	2.48e-08
10	3	0.54304	0.49309	1.10741	0.51581	1.16e-08	1.77e-08
2	4	2.52124	2.29173	5.21822	2.41449	1.28e-09	3.86e-09
6	4	0.84407	0.76881	1.73941	0.80483	4.00e-09	4.23e-10
10	4	0.50685	0.46174	1.04364	0.48290	5.89e-09	2.62e-08
2	5	2.34214	2.13792	4.89938	2.24993	9.26e-09	4.10e-09
6	5	0.78375	0.71653	1.63313	0.74998	1.36e-08	5.49e-09
10	5	0.47059	0.43027	0.97988	0.44999	5.96e-09	7.39e-09
2	6	2.16253	1.98319	4.58054	2.08536	4.45e-09	4.31e-09
6	6	0.72334	0.66413	1.52685	0.69512	9.88e-10	4.09e-09
10	6	0.43425	0.39876	0.91611	0.41707	3.27e-08	1.59e-09

Panel C. Fama and French Value Portfolio Sensitivity to Rate of Return

γ	$r(\%)$	α_1	α_2	α_{myp_1}	α_{myp_2}	ε_1	ε_2
2	3	10.62826	15.46258	21.43167	13.29497	9.66e-10	4.10e-04
6	3	3.74295	3.55862	7.14389	4.43166	1.25e-08	8.85e-11
10	3	2.26680	2.19163	4.28633	2.65899	2.28e-08	1.19e-08
2	4	10.36581	15.46258	20.80205	12.88872	2.81e-09	4.49e-04
6	4	3.64079	3.47760	6.93402	4.29624	1.50e-08	1.17e-08
10	4	2.20389	2.13791	4.16041	2.57774	2.22e-08	1.38e-09
2	5	10.09859	15.46254	20.17243	12.48247	1.23e-09	4.89e-04
6	5	3.53746	3.39422	6.72414	4.16082	9.62e-09	2.24e-08
10	5	2.14045	2.08302	4.03449	2.49649	1.80e-08	1.38e-08
2	6	9.82652	15.46257	19.54280	12.07622	1.90e-10	5.29e-04
6	6	3.43322	3.30840	6.51427	4.02541	2.90e-09	1.76e-08
10	6	2.07646	2.02705	3.90856	2.41524	1.23e-08	1.20e-08

Table 10

60% Portfolio

This table reports the implied coefficient of risk aversion of an investor that wishes to allocate 60% of his wealth in the risky asset. Calculations are made for S&P500, Growth and Value series, for Instantaneous Volatility (IV) and Long Term Volatility (LTV) estimates.

Volatility		S&P500		Growth		Value	
		γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
IV	γ	1.05	2.33	4.60	6.21	21.36	25.99
	ε	2.67e-09	1.03e-09	1.13e-09	2.86e-10	1.48e-09	5.83e-10
LTV	γ	2.12	3.26	8.02	7.33	36.31	35.71
	ε	9.73e-10	1.49e-09	1.07e-09	1.99e-09	3.74e-10	6.27e-10

Figure 1

Density Functions and QQ Plots

The figure shows the empirical probability densities for each series contrasted against the normal densities calculated using the sample mean and volatility. Also the QQ plots are drawn to asses the divergence from normality.

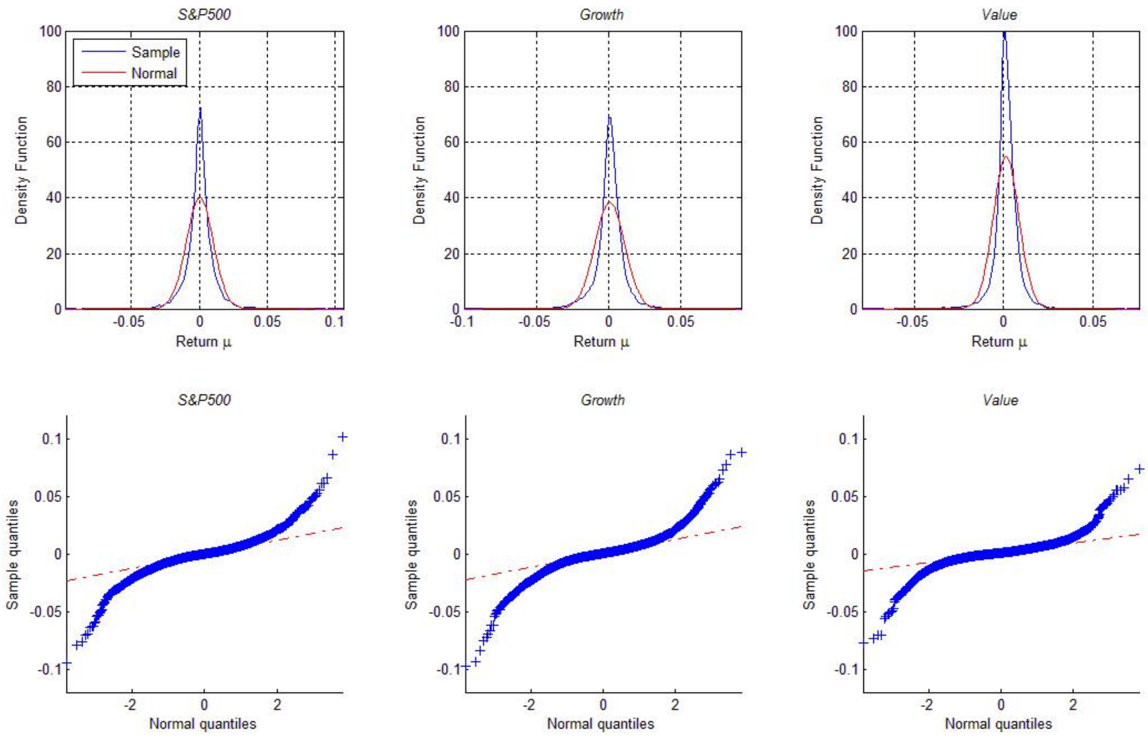


Figure 2

Characteristic Functions

This exhibit shows a comparison between the ECF and the normal distribution CF for each series. In the left-hand side appears the absolute value plotted in the same scale for the three series. In the right-hand side the phase diagram is shown.

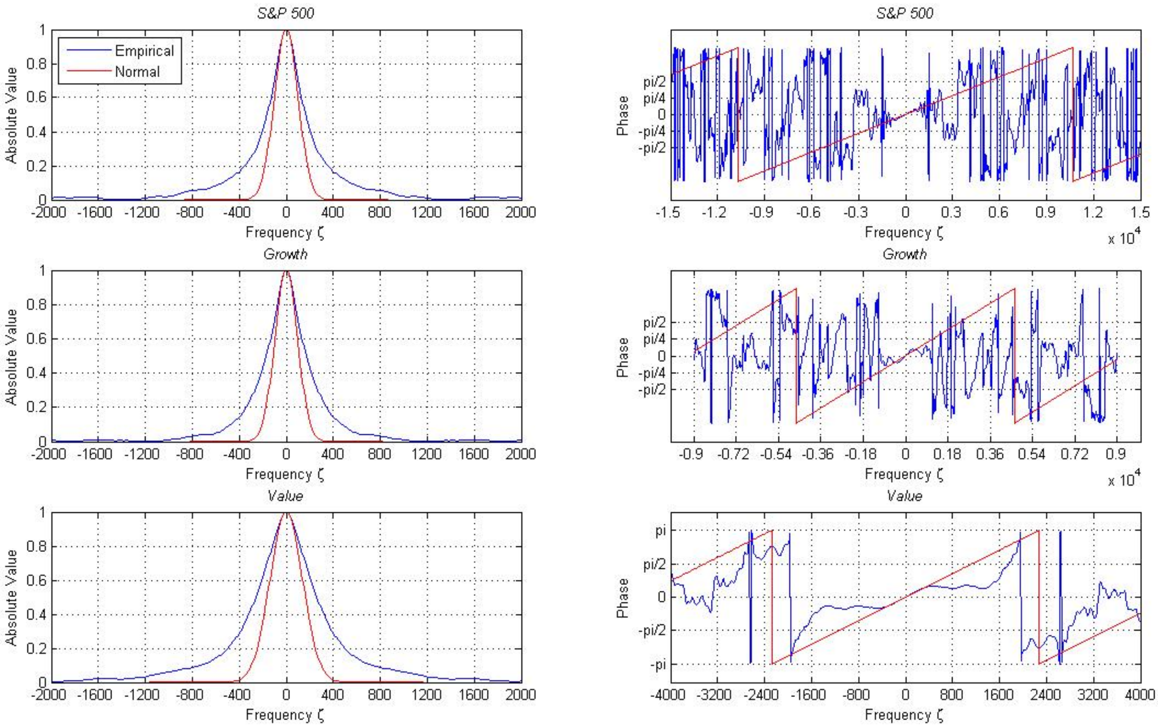


Figure 3

Portfolio Components

This figure shows the myopic demand and intertemporal hedging demand for the portfolios formed with the S&P500, Growth and Value series, the abscissa is the coefficient of risk aversion that ranges from 2 to 10. The exhibit displays the results for instantaneous volatility (IV) and long term volatility (LTV).

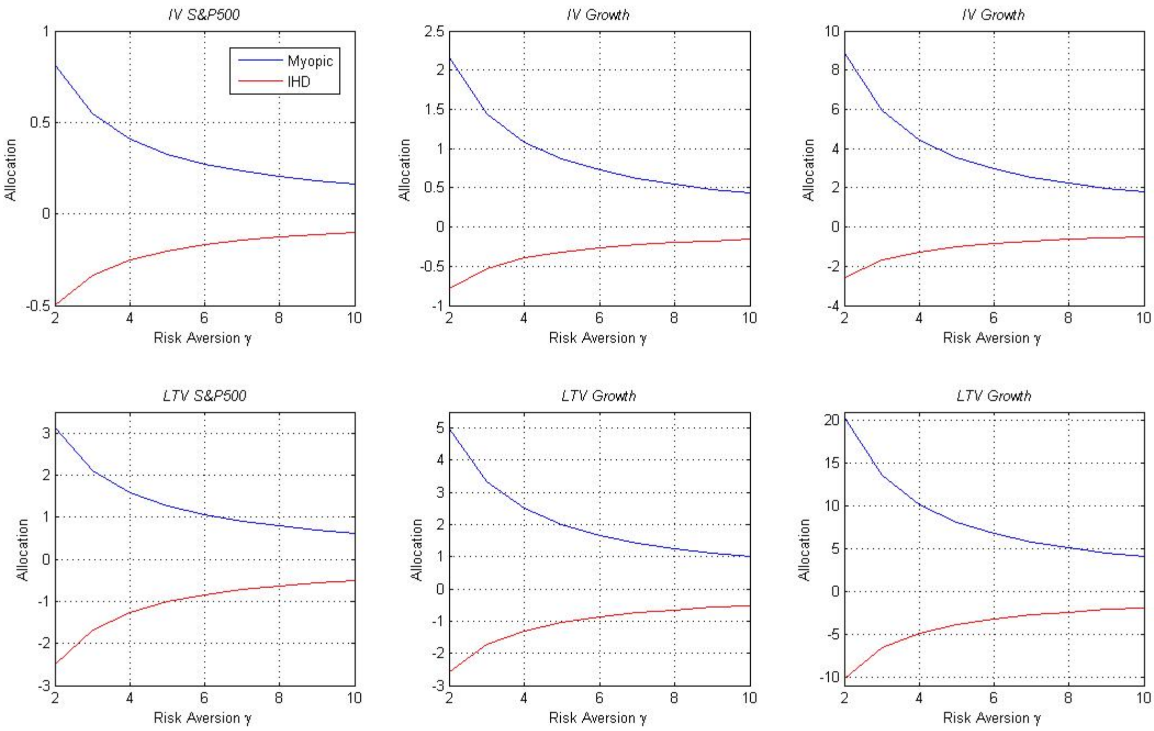
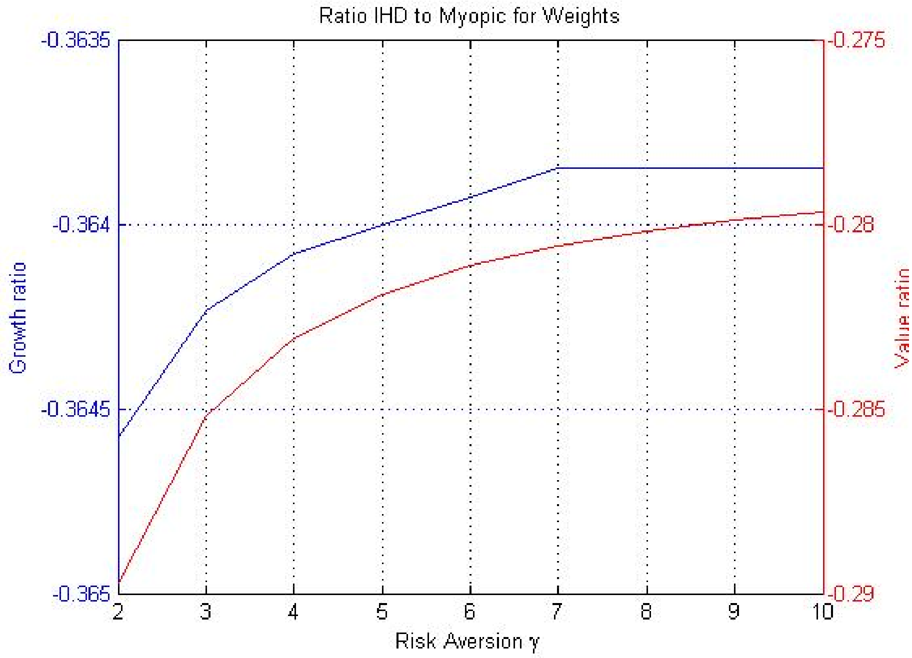


Figure 4

Portfolio Components

This figure shows the portfolio weight's ratio of the intertemporal hedging demand to the myopic demand for the unconstrained model parameters calculated with the instantaneous volatility (IV). The abscissa is the coefficient of risk aversion that ranges from 2 to 10.



TIME VARYING RELATIVE RISK AVERSION AND FACTOR PRICING MODELS WITH HABIT PERSISTENCE

Abstract. The consumption based, external habit model of Campbell and Cochrane is estimated for the period of January 1960 to December 2010. The priced returns are from the 25 portfolios formed on size and book to market by Fama and French. Moreover, the estimation is carried out for a six and twelve months lagged versions of the consumption time series. The curvature coefficient has plausible values in spite of the high relative risk aversion. The time dependent and countercyclical relative risk aversion and the stochastic discount factor are rendered. These series are plotted using as a framework the crisis periods. Two factor pricing models are derived and estimated, where the risk factors defined are consumption growth and risk aversion growth. The results of the market price of risk are consistent with the theory for twelve months lagged consumption.

1. Introduction

In an early study Mehra and Prescott (1985) introduced the equity premium puzzle. They consider an economy with a representative agent whose preferences are modeled with a time separable power utility function. They use a variation of the pure exchange model of Lucas (1978), in which is assumed that the growth endowment rate follows a Markov process. It turns out that the values for the subjective discount factor and relative risk aversion that result from calibration are not in concordance with the observed equity premium and the real interest rate. This contradiction arises from the restriction imposed by the Euler equation through the Hansen and Jagannathan (1991) bound, which provides the limits for the returns that can be priced by a stochastic discount factor (SDF hereafter). Conversely, the bound characterizes the possible SDF's that can price a given set of returns.

The market data for the postwar U.S. shows a high equity premium implying either a high relative risk aversion or a higher consumption growth rate volatility

JEL classification: C22, C23, G10, G12
Key words: Asset pricing, habit persistence in preferences, equity premium puzzle, generalized method of moments, Fama MacBeth, Ultimate Consumption Risk

than the actually observed. A high relative risk aversion has not economic sense by itself given the implications in consumption decisions. Moreover, accepting this value results in a high level and high volatile risk free rate, which is not consistent with the empirical evidence, the so called risk free rate puzzle.

Among others proposals, the models based on internal and external habit preferences have been developed to deal with the equity premium puzzle. The approach is to let the current utility to depend not only on the present consumption but also on the past history. In this context, authors as Constantinides (1990) address the problem defining a subsistence level of consumption as the exponentially weighted sum of past consumption and then subtracted it from consumption. Likewise, Sundaresan (1989) use the internal habit approach in defining a consumption standard.

High equity premium can also be explained by using external habit models, based on aggregated consumption. Some researchers as Abel (1998) explains the equity premium through the use of a ratio approach. This model has the drawback that does not accounts for interest rate's volatility and does not allows changing risk aversion. On the other hand, Campbell and Cochrane (1999) propose a model driven by an independent and identically distributed consumption growth and an external habit. They define a surplus process which is the ratio of consumption in excess of habit over consumption. The dynamics is defined in such a way that the risk free interest rate is constant. In this specification, habit depends on aggregate consumption i.e. an external habit. The model explains the time varying counter cyclical risk premium which is replicated by a time varying risk aversion. This results in a relative risk aversion that can be high despite the low curvature parameter for the power utility function. Moreover, the price-dividend ratio as a function of surplus inherits the persistence of the latter, and is able of forecasting long horizon returns.

The variation in the expected returns has been reported among others by Fama and French (1988) who investigate the relation between time varying expected returns and prices. Afterwards, Fama and French (1989) study how expected returns are related to economic conditions. Additionaly, Campbell and Shiller (1988) show that the ratio of real earnings to stock price is a predictor of stock's returns, and that the returns are too volatile if considering the news about future dividends. Campbell (1991) determines that expected stock returns changes in a persistent way. Moreover, Lettau and Ludvigson (2001) investigate the predictability of stock returns and excess returns over a Treasury bill rate by means of logarithmic aggregate consumption - wealth ratio.

From the works of Samuelson (1969), Merton (1969) and Merton (1971), is well known that under constant investment opportunities, and specifically when returns are IID, the myopic portfolio remains optimal for multi-period settings. Otherwise, a new term, the inter-temporal hedging demand, comes up modifying investor's portfolio composition. In this context the characterization of varying excess return and relative risk aversion becomes important. Our purpose in this paper is to estimate the consumption based, external habit model of Campbell and Cochrane (1999) using a series of consumption data and two lags of six months and twelve month respectively, in order to asses the model parameters validity according to the empirical evidence. We aim to set the basis of study for the time varying relative risk aversion as a useful tool for portfolio allocation.

We first use the iterated GMM methodology to estimate the subjective discount factor and the curvature parameter. That allows to determine the stochastic discount factor and also derive two linear factor pricing models. These involve consumption growth and risk aversion growth as risk factors. We render the surplus and risk aversion series and regress the excess return agains the battery of regressors, i.e., the covariance of excess return with the risk factors. For this we employ the Fama - MacBeth procedure, obtaining the market price of risk to each risk factor.

The paper is organized as follows. The theoretical framework is introduced in Section 2. In Section 3 the parameter estimation methodologies are explained. The results are presented in Section 4. Finally, the conclusions are drawn in Section 5.

2. Model

2.1. **External Habit Model.** We assume that agent's preferences, at time t, are expressed by a power utility function with curvature parameter γ, that accounts for the aversion to intertemporal substitution. The investor's goal is to maximize the expected utility

$$E\left[\sum_{t=0}^{\infty}\rho^t\frac{(C_t-X_t)^{1-\gamma}-1}{1-\gamma}\right] \tag{2.1}$$

where ρ is the subjective discount factor, C_t is consumption, and X_t is the *habit level*. Following Cochrane (2005) we define the *surplus* as:

$$S_t=\frac{C_t-X_t}{C_t} \tag{2.2}$$

and specify the logarithmic consumption growth as a random walk

$$\Delta c_{t+1}=\bar{c}+u_{t+1} \tag{2.3}$$

where $u_{t+1}\sim N(0,\sigma_c^2)$. Additionally, it is assumed that the logarithmic surplus follows the dynamics

$$s_{t+1}=(1-\phi)\bar{s}+\phi s_t+\lambda(s_t)u_{t+1} \tag{2.4}$$

where $\bar{s}=\ln\bar{S}$, and

$$\bar{S}=\sigma_c\sqrt{\frac{\gamma}{1-\phi}}$$

Finally, a sensitivity factor $\lambda_t=\lambda(s_t)$ is defined such that

$$\lambda_t=\lambda(s_t)\doteq\begin{cases}\bar{S}^{-1}\sqrt{1-2(s_t-\bar{s})}-1 & s_t<=s_{max}\\ 0 & s_t>s_{max}\end{cases} \tag{2.5}$$

where $s_{max}=\bar{s}+(1-\overline{S}^2)/2$. Note that $\lambda(s_t)=\partial s_t/\partial c_t$, and from the definition of surplus, equation (2.2), we get $\lambda(s_t)+1=S_t^{-1}$.

The power utility on consumption - habit level difference, $u(C_t-X_t)=u(S_tC_t)$, is time non-separable, allowing past time consumption to affect current utility. It is also assumed that the agent's habit level depends on other's current and past aggregate consumption, that is, an *external* habit. In this context C_t does not affects the future habit level X_t, it is an exogenous state variable. As a consequence, from equation (2.1), the first order condition for optimal consumption reads

$$E_t\left[\rho\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{it+1}\right]=1 \tag{2.6}$$

where R_{it+1} is the gross return of asset i at time $t+1$. The first term inside the expectation is the growth in marginal utility, i.e., the stochastic discount factor (SDF). It is defined between dates t and $t+1$, and is denoted as $M_{t+1}=M_{t,t+1}$,

$$\begin{aligned} M_{t+1} &= \rho\frac{u'(S_{t+1}C_{t+1})}{u'(S_tC_t)} \\ &= \rho\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \rho m_{t+1}^{-\gamma} \end{aligned} \tag{2.7}$$

where the prime denotes the derivative with respect to C_t, and

$$m_{t+1}=\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}$$

Using equation (2.6) we found the expected returns to be given by

$$E_t[R_{it+1}]=\frac{1}{E_t[M_{t+1}]}-\frac{cov_t(M_{t+1},R_{t+1})}{E_t[M_{t+1}]} \tag{2.8}$$

For the excess return R^e_{t+1} the pricing equation reads $E_t\left[M_{t+1}R^e_{t+1}\right]=0$, and therefore

$$E_t[R^e_{it+1}]=-\frac{cov_t(M_{t+1},R^e_{t+1})}{E_t[M_{t+1}]} \tag{2.9}$$

The term $R^s_{ft}=1/E_t[M_{t+1}]$ is defined as the shadow price of the riskless rate or zero beta rate from time t to $t+1$. The shadow riskfree rate assumes the role of market risk free rate if a riskfree asset is no traded in the market. As such, the agents are indifferent to borrowing or lending at that rate. However, in equilibrium if a market riskfree rate is available and is higher than the shadow riskfree rate, an agent would invest more in the riskfree asset thus reducing its consumption.

2.2. **Linear Factor Models.** From equation (2.6), and taking into account that the process C_t and S_t are lognormal[1], the following expression for the expected return is obtained[2]

$$E_t[r_{it+1}] = -\ln\rho + \gamma\bar{c} - \gamma(1-\phi)(s_t - \bar{s}) \\ - \frac{\gamma^2}{2}(\lambda(s_t)+1)^2\sigma_c^2 - \frac{\sigma_{it}^2}{2} + \gamma(\lambda(s_t)+1)\mathrm{cov}_t(r_{it+1}, \Delta c_{t+1})$$

Subtracting the associated equation for the risk free rate we get

$$E_t[r_{it+1}] - r_{ft+1} = -\frac{\sigma_{it}^2}{2} + \gamma(\lambda(s_t)+1)\mathrm{cov}_t(r_{it+1}, \Delta c_{t+1})$$

which can be written as

$$E_t[r_{it+1}^e] + \frac{\sigma_{it}^2}{2} = \gamma_t\sigma_{cit} \tag{2.10}$$

where r_{it+1}^e is the excess return of the i^{th} asset over the risk free rate, and $\sigma_{cit} \doteq \mathrm{cov}_t(r_{it+1}, \Delta c_{t+1})$. Note that the term $\sigma_{it}^2/2$ is the adjustment to the Jensen's inequality. Also, $\gamma_t = \gamma/S_t = \gamma(\lambda(s_t)+1)$ is the relative risk aversion. Equation (2.10) can be restated as:

$$\begin{aligned} E_t[r_{it+1}^e] + \frac{\sigma_{it}^2}{2} &= \gamma\frac{\sigma_{cit}}{S_t} \\ E_t[R_{it+1}^e] &\approx \gamma\frac{\sigma_{cit}}{S_t} \end{aligned} \tag{2.11}$$

Hence, if an asset return covaries positively with consumption growth it delivers low payoffs when consumption is low. This asset does not smooth consumption over time; thus, it is riskier and therefore demands a higher return. In this specification the excess return moves counter cyclically with the surplus. This is because in contractions the surplus is low, due to a low consumption, implying a high relative risk aversion and vice versa. In case of a realization of a low surplus (low consumption relative to habit) a higher return is demanded.

[1]We also assume that asset returns and consumption are jointly lognormal

[2]Lowercase indicates that the logarithm has been taken to the variable.

Yet another specification of the previous factor model can be found if we use the definition of relative risk aversion, which for convenience we rename as $RA_t = \gamma_t = \gamma/S_t$. If we insert that into the stochastic discount factor, equation (2.7), we arrive to

$$M_{t+1} = \rho\left(\frac{C_{t+1}}{C_t}\frac{RA_t}{RA_{t+1}}\right)^{-\gamma}$$

To get an expression for the expected excess return we first expand the stochastic discount factor in Taylor series and maintain the first two terms, obtaining

$$\begin{aligned} M_{t+1} &= \exp\left(\ln\rho - \gamma\Delta c_{t+1} + \gamma\Delta ra_{t+1}\right) \\ &\approx 1 + \ln\rho - \gamma\Delta c_{t+1} + \gamma\Delta ra_{t+1} \end{aligned}$$

where ra_t is the logarithm of RA_t. Inserting this into equation (2.9) we get

$$E_t[R^e_{t+1}] = \lambda cov_t(\Delta c_{t+1}, R^e_{t+1}) - \lambda cov_t(\Delta ra_{t+1}, R^e_{t+1}) \tag{2.12}$$

Likewise to equation (2.11), for some positive constant λ, an asset with high positive correlation with consumption growth is considered risky and demands a higher excess return. Regarding the second term, recall that in recessions the consumption growth is low and the risk aversion growth is high. Thus, fixing the consumption growth, if the excess return covaries negatively with risk aversion growth, the asset delivers low payoffs when risk aversion is high and a higher return is demanded. To the best of our knowledge, this specification of the model has not been tested before.

3. Model Estimation

We estimate the model through the use of the stochastic discount factor. To this end, we apply the GMM where the moment conditions are drawn from Euler equation (2.6). Once we have the parameter estimates, we run a cross sectional estimation using the Fama and MacBeth's (1973) methodology on equation (2.11), equation (2.12), and on two auxiliary regression equations.

3.1. **Generalized Method of Moments (GMM).** The fundamental pricing equation (2.6) can be restated as $E_t[M_{t+1}R_{it+1} - 1] = 0$ for all assets $i \in \{1, \ldots, N\}$, which includes the risk free rate series. This vector of N conditional expectations can be recognized as a set of moment conditions. The $N-$dimensional vector of non linear functions f depends on consumption and surplus, and it is defined on the vector of true parameters $\theta_0 = (\rho_0, \gamma_0)'$. Each coordinate component of f is given by

$$f_i(C_t, S_t; \theta_0) = \rho_0 \left(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma_0} R_{it+1} - 1 \tag{3.1}$$

Moment conditions are noted $g(\theta_0) = E_t[f(C_t, S_t; \theta_0)]$. As the number of conditions, N, is greater than the number of parameters the system is over identified and we apply the GMM estimator.[3] We look for a vector of parameters θ that minimizes the distance of the sample moment conditions g_T to the zero vector. To overcome scaling effects we use a $N \times N$ symmetric and positive definite matrix that weights the moment conditions according to some criteria. As such, we have implicitly define a metric, and our goal is to find an optimal parameter vector θ that minimizes the quadratic form $Q_T(\theta) = g_T'(\theta) W_T g_T(\theta)$. In this setting the GMM estimator is

$$\widehat{\theta}_{GMM}(W_T) = \arg\min_{\theta} \{g_T'(\theta) W_T g_T(\theta)\}$$

If we assume that $E_T[f(C_t, S_t; \Theta)] \to E[f(C_t, S_t; \theta)]$ for $T \to \infty$ hence, under some regularity conditions, the GMM estimator is consistent, that is, $\widehat{\theta}_{GMM} \to \theta_0$ as $T \to \infty$. Further on, assuming that $\sqrt{T} g_T(\theta_0) \to N(0, S)$, where S is the asymptotic variance, therefore the asymptotic distribution of the GMM estimator is given by $\sqrt{T}(\widehat{\theta}_{GMM} - \theta_0) \to N(0, S)$, and the asymptotic variance is $V = (D'WD)^{-1} D'WSWD(D'WD)^{-1}$, where

$$D = E\left[\frac{\partial f(C_t, S_t; \theta)}{\partial \Theta'} \right]$$

[3]We use Hansen's notation $E_T = 1/T \sum_{t=1}^{T}$, see Hansen (1982)

The sample counterpart reads

$$D_T = \frac{\partial g_T(\theta)}{\partial \theta'} = E_T\left[\frac{\partial f(C_t, S_t; \theta)}{\partial \theta'}\right] \tag{3.2}$$

Selecting the weighting matrix to be $W_T = S_T^{-1}$ we obtain the efficient GMM estimator whose asymptotic matrix is $\widehat{V} = (D_T' W_T D_T)^{-1}$. As estimator of S we use the heteroskedasticity and autocorrelation consistent (HAC) variance estimator

$$S_T(\theta) = \Gamma_T(0) + \sum_{i=0}^{T-1} \omega_i(\Gamma_T(i,\theta) + \Gamma_T'(i,\theta))$$

where

$$\Gamma_T(i,\theta) = \frac{1}{T}\sum_{t=i+1}^{T} f(C_t, S_t; \theta) f'(C_{t-i}, S_{t-i};, \theta)$$

with the *Bartlett kernel*

$$\omega_i \doteq k\left(\frac{i}{B}\right) = \begin{cases} 1 - \frac{i}{B} & \text{for } \frac{i}{B} > 0 \\ 0 & \text{for } \frac{i}{B} \leq 0 \end{cases}$$

which is known as the $Newey - West$ *estimator*. To find the optimal parameter vector θ we use the iterated GMM estimator, it has the advantage that the estimates are invariant to the selected initial matrix.

To assess the validity of the model we use the test for overidentifying restrictions via the J statistic of Hansen. Given that $g_T(\theta_0) \overset{a}{\sim} N(0, T^{-1}S)$, for the optimal weights $W_T \to S^{-1}$ then $\widehat{\theta}_{GMM} \to \theta_0$ and

$$J = T g_T'(\widehat{\theta}_{GMM}) W_T g_T(\widehat{\theta}_{GMM}) = T Q_T(\theta) \to \chi^2(R - K)$$

The derivatives of the functions (3.1), needed to find the sample variance, are computed in the appendix.

3.2. **Fama MacBeth Methodology (FMM).** The general form that we use in our study is

$$E[R_i] = \lambda_0 + \lambda_1 cov(R_i, f_1) + \cdots + \lambda_j cov(R_i, f_j) + \cdots + \lambda_k cov(R_i, f_k)$$

where i runs across the set of asset returns, and k is the number of risk factors. f_j is the risk factor j, and the covariance measures the quantity of risk to which the asset i is exposed with respect to risk factor j. On the other hand, λ_j is the market price of risk associated to risk factor j. The intercept to the expected return axis, λ_0, is interpreted as the riskless rate of return. When a riskless asset is traded in the market the value of λ_0 is restricted to be the riskless rate of return. Otherwise, λ_0 is called the zero-beta rate or shadow rate and have to be estimated.

As a simple modification to this model we subtract the traded riskless rate of return to each asset's rate leading to the excess return. For that case λ_0, the shadow rate, is not longer present in the representation, and as a result

$$E[R_i^e] = \lambda_1 b_{1i} + \cdots + \lambda_j b_{ji} + \cdots + \lambda_k b_{ki}$$

where b_{ji} is the covariance between the excess return of asset i and the risk factor j, $b_{ji} = cov(R_i^e, f_j)$ for $j = 1 \ldots k$. The variable b_{ji} determines the evolution of the asset's excess return throughout the business cycle, whose dynamics is described by the associated risk factor.

We use Fama and MacBeth (1973) methodology to find the standard errors and tests statistics for the model as well as the pricing errors. To that end we first we chose a time window of size $1 < T_0 < T$. Then, for each $t = T_0 + 1, \ldots, T$ we run a cross sectional regression to the model[4]

$$E_{t,T_0}[R_{it}^e] = \lambda_{0t} + \lambda_{1t}\widehat{b}_{1it} + \cdots + \lambda_{jt}\widehat{b}_{jit} + \cdots + \lambda_{kt}\widehat{b}_{kit} + a_{it} \tag{3.3}$$

where a_{it} are the pricing errors. The covariances are estimated as $\widehat{\sigma}_{jit} = E_{t,T_0}[(R_{it}^e - \overline{R}_{it,T_0}^e)(f_{jt} - \overline{f}_{jt,T_0})]$, where the rolling averages are given by $\overline{R}_{it,T_0}^e = E_{t,T_0}[R_{it}^e]$ and $\overline{f}_{jt,T_0} = E_{t,T_0}[f_{jt}]$. Later on, the estimate of λ_j, for $j = 0, \ldots, k$; is calculated with

[4]We use the operator $E_{s,T_0} = 1/T_0 \sum_{t=s-T_0}^{s-1}$ which is a extended version of Hansen's notation for an arbitrary time interval $[s - T_0, s - 1]$.

the use of

$$\widehat{\lambda}_j = \frac{1}{T - T_0} \sum_{t=T_0+1}^{T} \widehat{\lambda}_{jt} \tag{3.4}$$

And the standard errors are given by

$$\widehat{\sigma}(\widehat{\lambda}_j) = \sqrt{\frac{1}{(T - T_0)^2} \sum_{t=T_0+1}^{T} (\widehat{\lambda}_{jt} - \widehat{\lambda}_j)^2} \tag{3.5}$$

Recall that the theory suggest that the intercept $\widehat{\lambda}_0$ should be zero.

We estimate two factor pricing equations whose risk factors were identified previously. From the two alternative models in equations (2.11) and (2.12) we define three regressors: the logarithmic excess return and consumption growth covariance over surplus, the covariance between excess returns and consumption growth, and the covariance between excess returns and risk aversion growth. Thus the first regression equation reads

$$R^e_{it} \approx \lambda_{0t} + \gamma_{1t} \frac{\sigma_{cit}}{S_t} + a_{it} \qquad i = 1, \ldots, N-1 \qquad \text{for each } t \tag{3.6}$$

And the second regression equation is

$$R^e_{it} = \lambda_{0t} + \lambda_{1t}(\sigma_{cit} - \sigma_{rait}) + a_{it} \qquad i = 1, \ldots, N-1 \qquad \text{for each } t \tag{3.7}$$

For comparison purposes we estimate two auxiliary equations, where the expected excess returns are driven by consumption growth and risk aversion growth respectively,

$$\begin{aligned} R^e_{it} &= \lambda_{0t} + \lambda_{1t}\sigma_{cit} + a_{it} \\ R^e_{it} &= \lambda_{0t} + \lambda_{1t}\sigma_{rait} + a_{it} \end{aligned} \qquad i = 1, \ldots, N-1 \qquad \text{for each } t \tag{3.8}$$

The first one is consumption based and the second focus on recessions.

3.3. **Consumption Risk.** As suggested at the outset, the differences in expected returns for the U.S. market are not fully explained merely with the use of consumption growth. So far, we have deal with this issue through the use of a time non-separable utility function that gives rise to a time varying relative risk aversion. The non separability is the result of evaluating the standard power utility function on the consumption - habit level difference.

We can also address this phenomena by changing the measure of risk by using the ultimate risk to consumption. That is the covariance of the asset's returns and the consumption growth over several periods of the return[5]. The key insight is the time delay in the effect of portfolio decisions to consumption.

An important feature of ultimate consumption risk is that the derived stochastic discount factor exhibits a business cycle behavior. Due to the serial correlation of the consumption, which is low during recessions, the stochastic discount factor achieves its maximum values before a crisis.

4. Results

The consumption data are the monthly personal consumption expenditures for nondurable goods and services for the period of January 1960 through December 2010. The data comes from the Bureau of Economic Analysis.[6] We use the 25 portfolios formed on book to market by Fama and French. These information, along with the risk free rate series, comes from French's (2012) web page. The price over dividend is from Shiller's (2012) home page. In addition to the original consumption series (c0), and in order to determine the effects of ultimate consumption risk, we use two lagged versions of it. Those series are a six months lagged one, referred as to c6, and a twelve months lagged version, referred as to c12.

Table 1 reports the basic descriptive statistics. The average riskfree rate is of 0.43% and the monthly value premium for the period is of 0.33%, while the volatility growth - value spread is of 3.11%, this difference is somewhat suggested by the maximum - minimum rates spreads. The descriptive statistics for the logarithmic consumption growth rate for each series are found with the help of equation (2.3). The results are presented in Panel A of the Table 2, all coefficients are estimated

[5] see Parker and Julliard (2005)

[6] Bureau of Economic Analysis, U.S. Department of Commerce, URL http://www.bea.gov/

with precision. As expected, the mean of consumption growth increases with lags. Additionally, the c6's standard deviation is 2.6 times as much as c0's standard deviation, and c12's standard deviation is 4.4 times as much as c0's standard deviation. We then expect that the more lagged series explains better the observed equity premium.

The persistence of the surplus ratio is obtained from the logarithmic price over dividend series which is modeled as an AR(1) process. The estimated parameters for the series associated with c0, c6, and c12 are given in Panel B of the Table 2. The estimates are statistically significant and the three of them match to the third significant figure with an estimated value of 0.995, this implies that the surplus is highly persistent.

Afterwards, we estimate the Campbell and Cochrane (1999) model (CC henceforth). To this end, we use the initial values[7] of ρ and γ to calculate the surplus ratio equation (2.4). Then, we estimate the CC model and obtain $\widehat{\rho}$ and $\widehat{\gamma}$. We start over the process recursively with the estimated values $\{\widehat{\rho}, \widehat{\gamma}\}$ as initial values, and continue until convergence. The final estimates of $\{\widehat{\rho}, \widehat{\gamma}\}$ are used to obtain the surplus ratio S_t, the stochastic discount factor M_t, and the relative risk aversion γ_t.

Table 3 shows that the estimates of the subjective discount factor, $\widehat{\rho}$, are obtained with precision, while the estimates of the curvature, $\widehat{\gamma}$, are obtained with little precision. In overall the over identifying restrictions test rejects the model at a significance level of 5% but are not rejected at a significance level of 1%. The forth column reports the mean square pricing error (MSPE), its value decreases consistently as long as the lag increases. However, the model estimated using genetic algorithms for c6 series exhibits the lowest MSPE.

We plot the price over dividend ratio, the surplus ratio, the relative risk aversion, and the stochastic discount factor for the simulated annealing estimates. To set a framework we highlight the crisis time periods with red strips. The time varying

[7]The genetic algorithms technique starts with a set of random values ranging from 0 to 1 for ρ and between 0 and 100 for γ. While simulated annealing starts with the mean of the inverse riskfree rate gross return as ρ and 3 for γ

relative risk aversion for c0 series is plotted in Panel A of Figure 1. This graphic clearly suggest the changing nature of relative risk aversion. Its value mainly ranges from 70 to 90, however it exhibits a steep rise that starts right before the 2008 financial crisis. This step up reach values as high as 110, despite the lower power utility parameter γ which is of about 19.40 to 21.59.[8] The figure shows a countercyclical behavior, in recessions the relative risk aversion becomes higher, when surplus is low. The magnitude of γ diminishes as we increase the lag of the ultimate consumption series, it decreases from a value of 19.40 to a value of 2.6 for GA estimates (similar behavior is seen for SA estimates).

Figure 2 simultaneously shows the price over dividend and surplus ratio for c0 series. In the habit model the price over dividend, which is low in recessions, is a linear function of the surplus which is the state variable. This is not the case for market data, although the patterns are similar. For c0 the two series appear to have the same local highs, while the trend is opposed after 1990. For c6 series, Figure 3, the highs and lows are exaggerated and sharp, and the series are situated away to each other, but after the crossing the two series becomes well correlated until the end of the last crisis indicated on the graph, where they switch directions again. Figure 4 shows that the surplus of the c12 series is the smoothest of the three, and although it exhibits a crossing in the 90's decade, where the series are partially opposed in direction at the outset, they seems to be more positive correlated.

The stochastic discount factor evolution for c0 series is plotted in Panel B of Figure 2. It appears to be very volatile attaining deep lows and high peaks that ranges from 0.5 to 2.5. Its volatility is about 25% (GA estimates) and 27% (SA estimates). It seems to follow a trend with highs at crisis periods but this feature is hidden because its highly fluctuating character. Panel B of Figure 3 shows the stochastic discount factor for c6 series, its volatility reduces to 16% for GA estimates and 18% for SA estimates. The figure shows a business cycle pattern with peaks at the outset of a crisis period and deep falls at the end of the crisis period. Its excursions are

[8]Genetic algorithms and simulated annealing estimates respectively, see Table 3.

less marked, ranging from 0.7 to 2.2. Lastly, Figure 4 shows the stochastic discount factor for c12, its value ranges from 0.8 to 2.2, and its volatility is approximately 18% for both GA and SA estimates. It is remarkable how its pattern becomes well defined with respect to that of c6 series, experiencing highs at the start of recessions and lows at the end, in concordance with theory.

Table 4 shows the results of the cross sectional regression for the model given in equation (3.6), for the contemporaneous and ultimate consumption risk respectively. All the coefficients are estimated with precision, including the intercept, $\widehat{\lambda}_0$, which is positive and equal for both GA and SA estimates. The market price of risk is similar for GA and SA, despite the difference in c6 series value. As we move from contemporaneous to ultimate consumption risk, by increasing the series lag, the price of risk increases from a value of approximately -10% to a positive value of 2.60% (GA estimates), and 2.67% (SA estimates) for c12 series. And, although the price of risk is estimated with less precision for ultimate consumption risk in c12 series, the sign becomes consistent with theory. The data suggest that ultimate consumption risk specification with time varying aversion seems to explain relatively well the cross section of average returns.

Table 5 reports the results for model in equation (3.7) and the sub models in (3.8), for the contemporaneous and ultimate consumption risk. As before the estimates of the intercept are positive and significant for all models. We first discuss the results from the sub-models in (3.8). Panel A reports the results for the model using consumption growth as a risk factor. As before all the coefficients are estimated with precision, however, only the price of risk of c12 has the required sign. Panel B reports the results for the model using risk aversion growth as a risk factor. The price is estimated with precision for c0 and c12 series, however, the sign is correct for c6 series in GA estimates and c12 series in both GA and SA estimates. Regarding the sub models' market price, Panel A and B of Table 5, the price of risk for c12 series has a value of 6.65 when using logarithmic consumption growth, and -23.70 for logarithmic risk aversion growth. Hence the excess returns seems to be more sensitive to risk aversion. The estimates for the complete model are given in Panel C. As before the coefficients are estimated with precision for c0 and c12 series but the sign agrees with the theory only for the latter.

Comparing the two estimated pricing models we notice that the market price is about 2.6 for logarithmic consumption growth with surplus weighting risk factor, and about 4.4 for logarithmic consumption growth - logarithmic risk aversion difference risk factor. On the other hand, Table 4 and Table 5 reports the statistic $\bar{R}^2$, which is the average of the R^2 for the cross section regressions. All of them are very similar, suggesting that the models explain about 25% of the cross-sectional variation of returns. However, in all the pricing models we obtained a very high and significant intercept suggesting the overall rejection of them.

5. Conclusions

The consumption model with external habit of Campbell and Cochrane (1999) accounts for time varying and countercyclical expected returns, as well as the high equity premium with a low and steady riskfree rate. This model has the feature of deliver a counter cyclical varying risk aversion, and allows predictability of asset returns. In this paper we estimate and test the model throughout the use of GMM methodology. Afterwards, we test two specifications of pricing models that includes surplus and risk aversion, under contemporaneous and ultimate consumption risk. To that purpose, we examine the 25 portfolios formed on size and book to market by Fama and French.

We first estimate the model using the stochastic discount factor and found the subjective discount factor and the curvature. The actual parameters are plausible, and in particular the curvature exhibit low values, in spite of the high relative risk aversion. The actual magnitude of the curvature parameter lessens and the stochastic discount factor becomes smoother as the lag of the ultimate consumptions series increases. The stochastic discount factor exhibits a business cycle, attaining maxima at the start of recessions and dropping at the end of them. With these parameters we calculate the surplus ratio and thus the relative risk aversion. The linear relationship between the price over dividend and the surplus ratio in the Campbell and Cochrane model is most closely followed under c12 series. The risk aversion, on the other hand, proved to be time varying and countercyclical.

Further, we estimate two factor pricing models derived from the habit model specification. The risk factor for the first model is the contemporaneous and ultimate consumption risk, and the regressor is its covariance with the lognormal excess return weighted by the inverse of the surplus. The risk factors for the second model are the contemporaneous and ultimate consumption risk minus the risk aversion growth. The sign of the estimated prices of risk are consistent with theory for the twelve months consumption lagged series. However, the estimated intercept is statistically significant indicating an overall rejection of the models. The data suggest that ultimate consumption risk specification with time varying aversion seems to explain relatively well the cross section of average returns. The excess returns seems to be more sensitive to risk aversion than to consumption growth.

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Appendix . Derivatives to the Stochastic Discount Factor

In order to implement the test of overidentifiying restrictions we need the derivatives (3.2) of the vector of functions (3.1),

$$\frac{\partial f(C_t, S_t; \theta)}{\partial \theta'}$$

To this end, we calculate the derivatives of the SDF $M_{t+1} = \rho m_{t+1}^{-\gamma}$, w.r.t. the parameter vector θ. For the subjective discount factor the resulting expression is $m_{t+1}^{-\gamma} = M_{t+1}/\rho$. Besides, the derivative of w.r.t. the risk aversion coefficient γ is

$$\begin{aligned}
\frac{\partial M_{t+1}}{\partial \gamma} &= \rho \frac{\partial m_{t+1}^{-\gamma}}{\partial \gamma} \\
&= \rho \frac{\partial e^{-\gamma \ln m_{t+1}}}{\partial \gamma} \\
&= \rho e^{-\gamma \ln m_{t+1}} \left(\ln m_{t+1} + \gamma \frac{m'_{t+1}}{m_{t+1}} \right) \\
&= M_{t+1} \left(\ln m_{t+1} + \gamma \frac{m'_{t+1}}{m_{t+1}} \right)
\end{aligned}$$

where the derivative of m_{t+1} is given by

$$\frac{\partial m_{t+1}}{\partial \gamma} = \frac{C_{t+1}}{C_t} \frac{\partial}{\partial \gamma} \left(\frac{S_{t+1}}{S_t} \right)$$

and

$$\frac{\partial}{\partial \gamma} \left(\frac{S_{t+1}}{S_t} \right) = \frac{\partial e^{\Delta s_{t+1}}}{\partial \gamma}$$

The derivative of s_{t+1} is $s'_{t+1} = (1-\phi)\bar{s}' + \phi s'_t + \lambda'_t(\Delta c_{t+1} - \bar{c})$. Recalling the definition of $\bar{S}$ and $\bar{s}$ we have $\bar{s}' = 1/2\gamma$ and

$$\begin{aligned}
\frac{\partial \bar{S}}{\partial \gamma} &= \frac{\sigma}{\sqrt{1-\phi}} \frac{\partial \sqrt{\gamma}}{\partial \gamma} \\
&= \frac{\sigma}{\sqrt{1-\phi}} \frac{1}{2\sqrt{\gamma}} = \frac{\bar{S}}{2\gamma}
\end{aligned}$$

Also

$$
\begin{aligned}
\frac{\partial \lambda_t}{\partial \gamma} &= \frac{\partial}{\partial \gamma}\left(\frac{1}{\bar{S}}\right)\sqrt{1-2(s_t-\bar{s})} + \frac{1}{\bar{S}}\frac{\partial}{\partial \gamma}\sqrt{1-2(s_t-\bar{s})} \\
&= -\left(\frac{\bar{S}'}{\bar{S}^2}\right)\sqrt{1-2(s_t-\bar{s})} - \frac{1}{\bar{S}}\frac{s_t'-\bar{s}'}{\sqrt{1-2(s_t-\bar{s})}} \\
&= -\left(\frac{\bar{S}'}{\bar{S}}\right)(\lambda_t+1) - \frac{1}{\bar{S}^2}\frac{s_t'-\bar{s}'}{(\lambda_t+1)}
\end{aligned}
$$

From the definition of S_t, and the previous relation for the derivative of $\overline{S}$ we get

$$
\begin{aligned}
\frac{\partial \lambda_t}{\partial \gamma} &= -\left(\frac{\bar{S}'}{\bar{S}}\right)(\lambda_t+1) - \frac{1}{\bar{S}^2}\frac{s_t'-\bar{s}'}{(\lambda_t+1)} \\
&= -\frac{\overline{S}}{2\gamma}\frac{1}{\overline{S}}\frac{1}{S_t} - \frac{S_t}{\overline{S}^2}(s_t'-\bar{s}') \\
&= -\frac{\gamma_t}{2\gamma^2} - \frac{S_t}{\overline{S}^2}(s_t'-\bar{s}')
\end{aligned}
$$

where we have used the definition of relative risk aversion $\gamma_t = \gamma/S_t$.

TABLE 1

Descriptive Statistics

This table reports the descriptive statistics for the riskless interest rate, the Value, and the Growth portfolios of Fama and French. Those are the extreme cases of the set of 25 constructed portfolios. The statistics are reported in percentage.

Statistic	Value	Growth	Riskless
Minimum	−19.13	−34.18	0.00
Mean	0.992	0.659	0.426
Maximum	17.5	39.68	1.35
Standard Deviation	4.98	8.09	0.24

TABLE 2

Consumption and P/D Processes Parameter Estimates

Panel A of this table reports the estimates of the random walk process for consume. Panel B shows the estimates of the AR(1) process for the price over dividend data. We form the time series pd0, pd6, and pd12 in order to match the consumption series c0, c6, and c12 time periods respectively. The standard errors are in parenthesis.

Panel A. Consumption process' parameter estimates

Series	$\bar{c}$	σ_c
c0	0.001740	0.003501
	(0.000141)	(0.000079)
c6	0.010505	0.009055
	(0.000380)	(0.000228)
c12	0.021176	0.015394
	(0.000669)	(0.000445)

Panel B. Price over dividend process' parameter estimates

Series	Constant	ϕ
pd0	3.669072	0.995657
	(0.353533)	(0.003022)
pd6	3.638722	0.9954188
	(0.342845)	(0.003054)
pd12	3.64788	0.995521
	(0.350530)	(0.003045)

Table 3

Model Parameter Estimates

This table reports the parameter estimates for the habit model. The results are given for the contemporaneous consumption data series (c0), and the ultimate consumptions series: the six month lagged series (c6), and the twelve months lagged version (c12). Columns two and three report the estimates of the subjective discount factor ρ and the curvature parameter γ, respectively, for the power utility function. The standard errors are reported in parenthesis below each parameter. Column four shows the mean square pricing error. The following columns report Hansen's test related parameters, the sample size, test statistic J, and $p-$value.

Panel A. Parameter estimates using the genetic algorithm routine

Series	$\widehat{\rho}$	$\widehat{\gamma}$	MSPE	T	J	$p-$value
c0	0.960	19.40	6.7146$e-$02	612	38.46	0.031
	(0.01)	(20.06)				
c6	1.000	3.13	2.4238$e-$02	606	37.84	0.036
	(0.17)	(17.00)				
c12	0.997	2.62	3.0833$e-$02	600	37.02	0.044
	(0.31)	(14.01)				

Panel B. Parameter estimates using the simulated annealing routine

Series	$\widehat{\rho}$	$\widehat{\gamma}$	MSPE	T	J	$p-$value
c0	0.959	21.59	7.6745$e-$02	612	38.35	0.032
	(0.01)	(20.89)				
c6	1.000	6.00	3.2253$e-$02	606	36.78	0.046
	(0.18)	(18.40)				
c12	1.000	2.75	3.1069$e-$02	600	37.01	0.044
	(0.31)	(14.03)				

Table 4

Linear Factor Model Calibration for Consumption Growth Risk Factor with Surplus Weighting

This table reports the estimates of the parameters $\{\widehat{\lambda}_0, \widehat{\gamma}_1\}$ associated with the linear model $R^e_{it} \approx \lambda_{0t} + \gamma_{1t}\sigma_{cit}/S_t + a_{it}$. Panel A reports the results using the genetic algorithm routine estimates and Panel B reports the results using the simulated annealing routine estimates.

Panel A. Parameter estimates using genetic algorithms.

Series	$\widehat{\lambda}_0$	$\widehat{\gamma}_1$	$\bar{R}^2$
c0	0.0103 (0.0003)	−9.74 (2.21)	25.67
c6	0.0123 (0.0003)	−2.05 (1.39)	25.73
c12	0.0109 (0.0002)	2.60 (1.36)	27.82

Panel A. Parameter estimates using simulated annealing.

Series	$\widehat{\lambda}_0$	$\widehat{\gamma}_1$	$\bar{R}^2$
c0	0.0103 (0.0003)	−10.22 (2.31)	25.67
c6	0.0123 (0.0003)	−3.17 (1.94)	25.73
c12	0.0109 (0.0002)	2.67 (1.40)	27.82

TABLE 5

Linear Factor Model Calibration for Consumption Growth and Risk Aversion Growth Risk Factors

This table reports the estimates of the parameters $\{\widehat{\lambda}_0, \widehat{\lambda}_1\}$ associated with the linear model $R^e_{it} = \lambda_{0t} + \lambda_{1t}(\sigma_{cit} - \sigma_{rait}) + a_{it}$. Each panel reports the results for genetic algorithm (GA) estimates and for simulated annealing (SA) estimates. Panel A shows the estimates when the regressor is the covariance between the excess return and logarithmic consumption growth. Panel B is for the covariance between the excess return and logarithmic risk aversion growth. In Panel C the explicative variable is the difference between the covariance of the excess return and and logarithmic consumption growth and the covariance between the risk aversion growth with the excess return.

Panel A. Parameter estimates using covariance between the excess return and logarithmic consumption growth as explicative variable.

Series	$\widehat{\lambda}_0$	$\widehat{\lambda}_1$	$\bar{R}^2$
c0	0.0064	−12.80	25.61
	(0.0003)	(8.21)	
c6	0.0085	−9.37	21.99
	(0.0003)	(4.66)	
c12	0.0073	6.65	24.68
	(0.0003)	(3.23)	

Panel B. Parameter estimates using the covariance between the excess return and logarithmic risk aversion growth as explicative variable.

GA	$\widehat{\lambda}_0$	$\widehat{\lambda}_1$	$\bar{R}^2$
c0	0.0066	6.61	25.07
	(0.0003)	(3.14)	
c6	0.0094	−3.46	24.34
	(0.0003)	(4.52)	
c12	0.0072	−23.70	25.60
	(0.0003)	(6.56)	

SA	$\widehat{\lambda}_0$	$\widehat{\lambda}_1$	$\bar{R}^2$
c0	0.0066	6.88	25.07
	(0.0003)	(3.32)	
c6	0.0092	1.16	24.34
	(0.0003)	(4.93)	
c12	0.0072	−23.54	25.60
	(0.0003)	(6.55)	

Panel C. Parameter estimates using as explicative variable the difference between the covariance of the excess return and logarithmic consumption growth and the covariance between the risk aversion growth and the excess return.

GA	$\widehat{\lambda}_0$	$\widehat{\lambda}_1$	$\bar{R}^2$
c0	0.0066	−4.37	25.21
	(0.0003)	(2.26)	
c6	0.0091	−0.39	23.63
	(0.0003)	(1.89)	
c12	0.0072	4.44	25.70
	(0.0003)	(1.68)	

SA	$\widehat{\lambda}_0$	$\widehat{\lambda}_1$	$\bar{R}^2$
c0	0.0066	−4.48	25.21
	(0.0003)	(2.35)	
c6	0.0089	−2.64	23.63
	(0.0003)	(2.13)	
c12	0.0072	4.49	25.70
	(0.0003)	(1.70)	

FIGURE 1

Risk Aversion for c0 Series

The figure shows the monthly risk aversion for the simulated annealing estimates of $c0$. The crisis periods are highlighted in red.

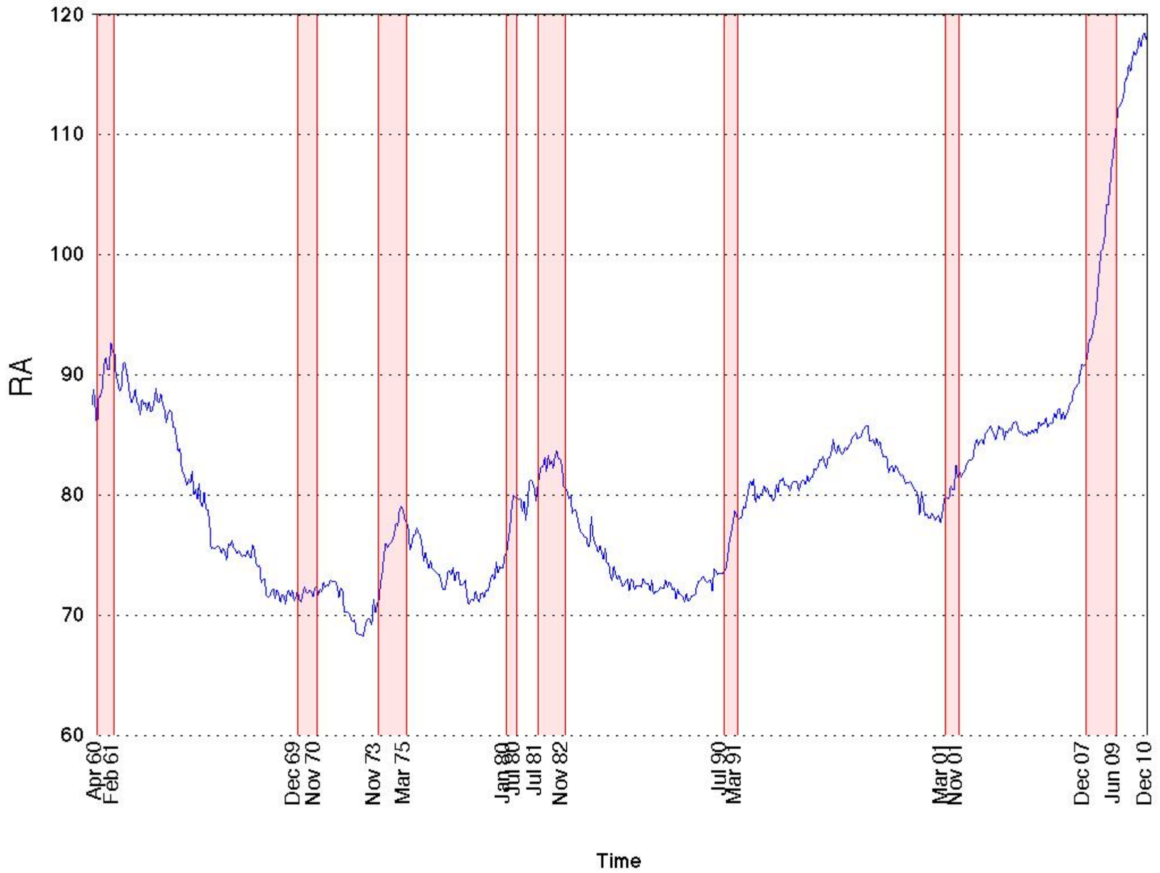

FIGURE 2

Surplus and Price/Dividend and Stochastic Discount Factor for c0 Series

The figures show the monthly surplus and monthly price over dividend and the stochastic discount factor for the simulated annealing estimates of $c0$. The crisis periods are highlighted in red.

Figure A. Surplus and Price/Dividend

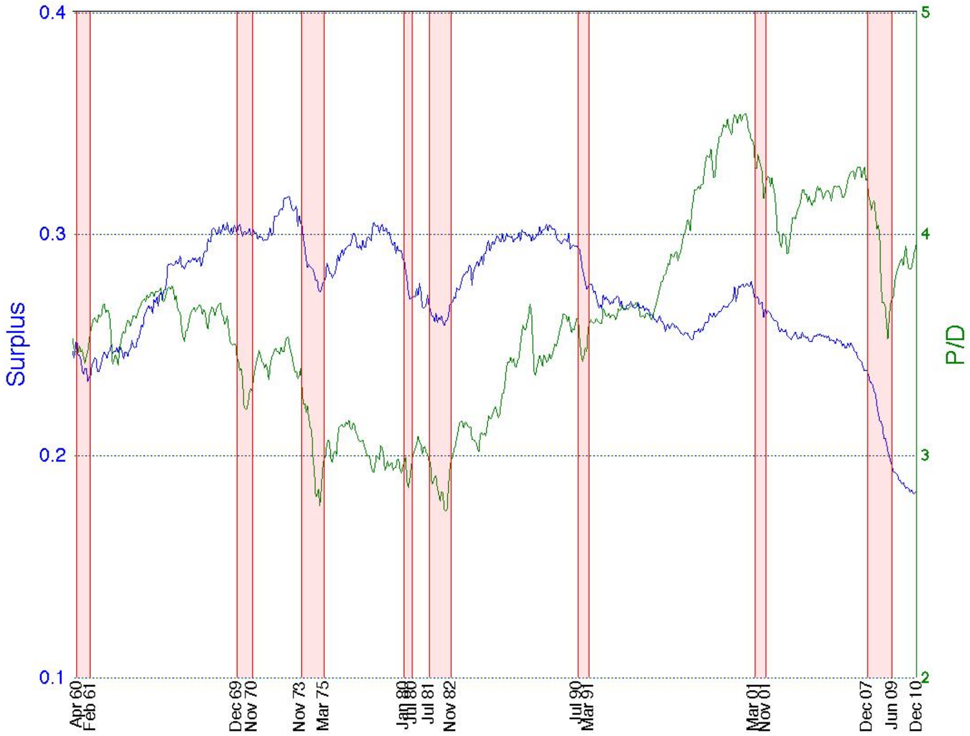

Figure B. Stochastic Discount Factor

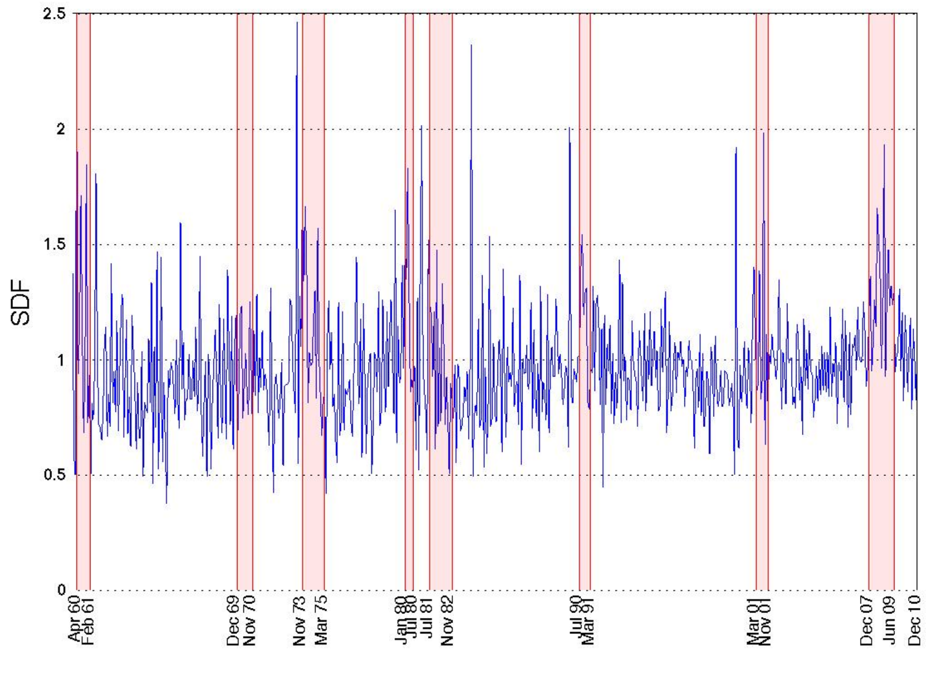

Figure 3

Surplus and Price/Dividend and Discount Factor for c6 Series

The figures show the monthly surplus and monthly price over dividend and the stochastic discount factor for the simulated annealing estimates of $c6$. The crisis periods are highlighted in red.

Figure A. Surplus and Price/Dividend

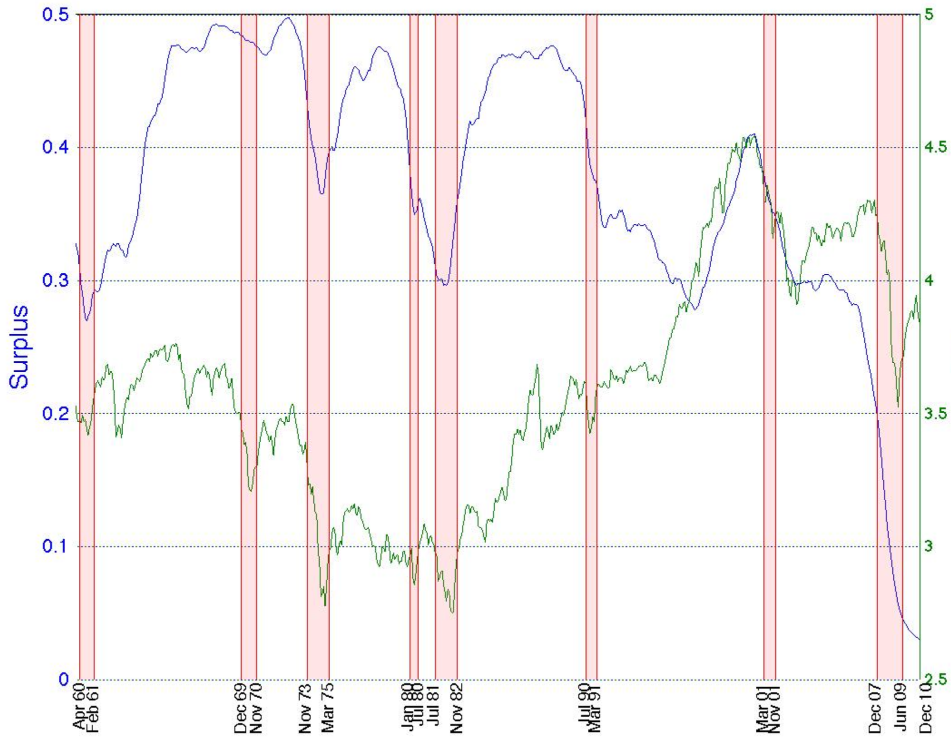

Figure B. Stochastic Discount Factor

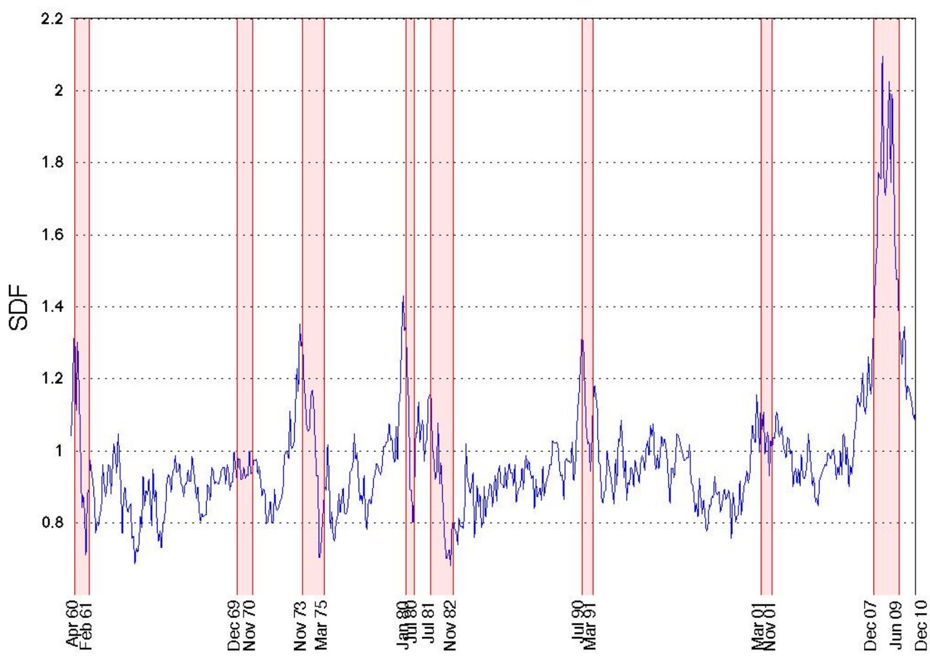

Figure 4

Surplus and Price/Dividend and Discount Factor for c12 Series

The figures show the monthly surplus and monthly price over dividend and the stochastic discount factor for the simulated annealing estimates of $c12$. The crisis periods are highlighted in red.

Figure A. Surplus and Price/Dividend

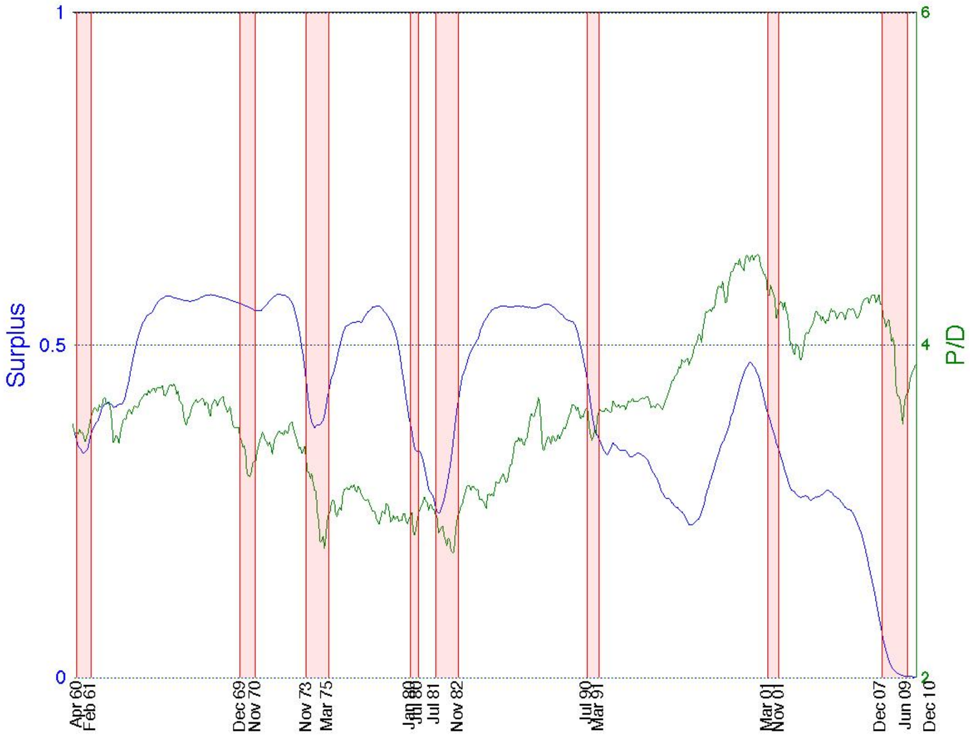

Figure B. Stochastic Discount Factor