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Invariants of Stable Maps between Closed Orientable Surfaces

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Abstract: In this paper, we will consider the problem of constructing stable maps between two closed orientable surfaces M and N with a given branch set of curves immersed on N. We will study, from a global point of view, the behavior of its families in different isotopies classes on the space of smooth maps. The main goal is to obtain different relationships between invariants. We will provide a new proof of Quine's Theorem.

Keywords: cusps; graphs; degree; Euler characteristic; stable maps

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1. Introduction

Stable maps between surfaces can only have fold curves with isolated cusp points on them (H. Whitney [1,2]). The image of the singular set consists of a finite number of immersed curves whose self-intersections are all transverse and disjoint of possible cusp points. Recently the interest of maps between surfaces has been increasing. Various authors (see [3–6]) have presented important results related to invariants for the classification of these maps.

The global description of a stable map between closed surfaces requires a study of the topological type of its regular set in the domain and the isotopy type of the image of its singular set in the range surface. To investigate the global classification of stable maps between surfaces, graphs of stable maps were introduced in [7] to provide a combinatorial description of the topology of the singular set. This graph, with weights on its vertices, describes the position of the singular and regular sets on the surface. In [8,9], we studied the properties of such graphs and we see that any bipartite graph with non-negatively weighted vertices is the graph of a stable map from a closed orientable surface to the plane, to the sphere and closed orientable surface. For a map into the sphere, the degree of a fold map can be obtained from of the information codified on its graph [8].

The aim of this paper is to study the relationships between the Euler characteristic number, cusps and degree of stable maps between closed orientable surfaces. We define new global invariants for stable maps that give us a necessary condition to associate for a given graph to some fold map (maps without cusps) between two closed and orientable surfaces. For this purpose, a detailed study about the behavior of the singular and regular sets of families of stable maps in different homotopy classes and cups degree was developed. We study in detail the list of codimension one transitions in the space $C^\infty(M,N)$ introduced in [6,10]. To properly understand the action of such transitions, we represent them as the result of a factorization given by the composition of a convenient immersion of the surface in 3-space followed by a convenient projection over N.

The main result of this paper is Theorem 3, which is equivalent to the result obtained by Quine (see [11]) for singularities of maps between closed and oriented 2-manifolds. In this paper, we present a different proof of Quine's theorem as a consequence of the results proved in Sections 2 and 3.

If N is a closed orientable surface, according to Eliashberg (see [12]), for each map of a homotopy class, there exists a stable map with irreducible contour. In [5], the authors determine which of these curves are minimal contours of a stable map from a closed orientable surface to the sphere. Here, we will see that these curves too can be minimal contour for some families of stable maps between closed orientable surfaces.

2. Stable Maps between Surfaces

Let $C^\infty(M, N)$ be the set of all smooth maps $f : M \longrightarrow N$, where M and N are closed orientable surfaces and the genus $g(N) = n$.

We will say that two smooth maps f and g in $C^\infty(M, N)$ are topologically equivalent, when diffeomorphisms (orientation-preserving) l and k exist such that $g \circ l = k \circ f$. When all maps sufficiently close to f (in the Whitney C^∞ topology) are equivalent to f, then we will say that f is a stable map.

If f is a stable map, the singular set Σf is a finite collection of closed regular simple curves on M made of fold points with possible isolated cusp points. Moreover, the image of Σf, known as the apparent contour or branch set of f, $\mathcal{B}f$, is a collection of closed curves on N with normal crossings and isolated singularities corresponding to the cusp points of f (see Figure 1). A stable map without cusp points is called a *fold map*.

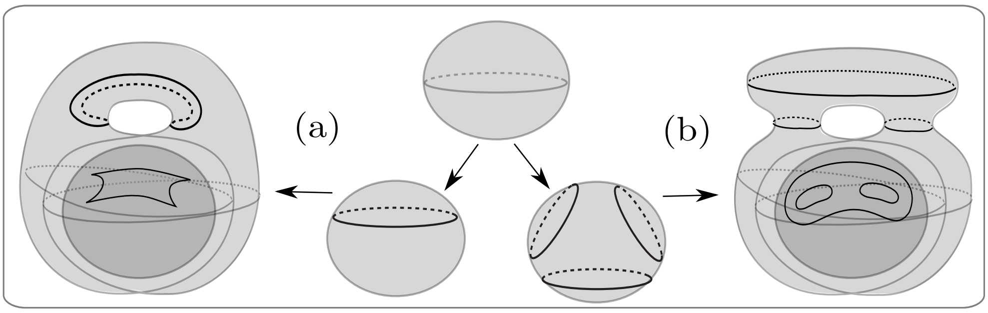

Figure 1. Example of stable maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ with dregree 2: (**a**) unique singular curve with four cusp points; (**b**) fold map with three singular curves.

2.1. Graphs of Stable Maps

In [7], a graph with weights on its vertices associated to stable maps f between two surfaces M and N was introduced, where the edges correspond to the singular curves, the vertices correspond to the regular regions (i.e., the connected components) and the weight of a vertex v correspond to the genus of the regular region corresponding to v (see Figures 2 and 3). If M is a closed orientable surface, a edge that connects the vertices u and v corresponds to a singular curve that lies in the border of the two regular regions: a positive region (f preserves the orientation) and a negative region (f reverse the orientation), because the graph is bipartite.

The number of edges of a stable map f, denoted by $E(f)$, corresponds to the number of connected components of Σf; the number of vertices V, denoted by $V(f)$, is the number ofregular regions of F. The number of total weights on its vertices will be denoted by W_f. We denote by $\mathcal{G}(V, E, W)$ or $\mathcal{G}$ the graph of a stable map between closed orientable surfaces, which is a topological invariant.

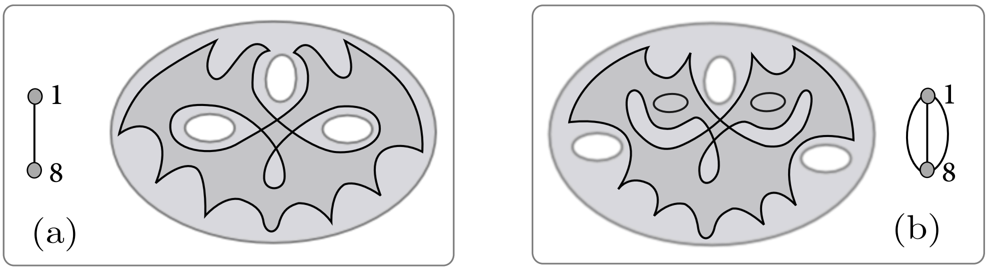

Figure 2. Example of apparent contours of stable maps to the 9-torus and its weighted graph: (**a**) corresponds to a graph type $\mathcal{G}(2,1,9)$. The singular curve has eight cusp points and four double points; (**b**) corresponds to a graph type $\mathcal{G}(2,3,9)$ with three singular curves.

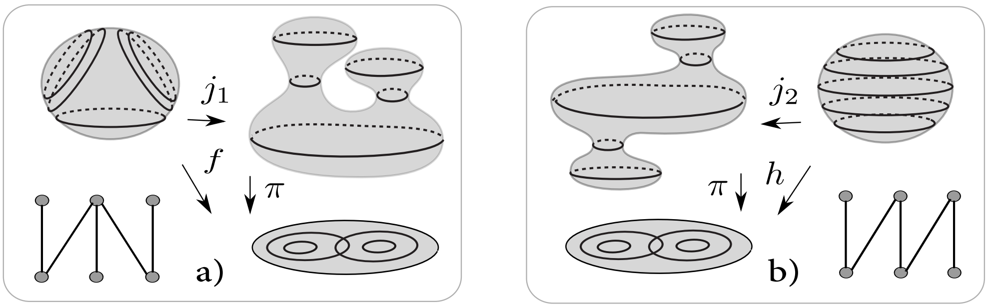

Figure 3. Examples of fold that are images of different maps with zero degree (see [13]) and equal apparent contours: (**a**) are two fold maps from $\mathbb{S}^2$ into the plane; (**b**) corresponds with two fold maps from torus into the plane.

We denote by M_f^+ (resp. M_f^-) the union of all the positive (resp. negative) regions of f, including their boundaries (curves of the singular set Σf). The number of connected component of M_f^+ (resp. M_f^-) will be denoted by $V^+(f)$ (resp. $V^-(f)$) and the total genus of M_f^+ (resp. M_f^-) will be denoted by $W^+(f)$ (resp. $W^-(f)$). We will omit f in the notations above when it does not cause misunderstanding. Of course, $\mathcal{G}$ has E edges, V^+ (resp. V^-) vertices with sings "+" (resp. "−") and W^+ (resp. W^-) is the total weight of the vertices V^+ (resp. V^-). Then, $V = V^+ + V^-$ and $W(f) = W^+ + W^-$.

Definition 1. *The difference between the number of positive and negative vertices will be denoted by $\theta_V = V^+ - V^-$ and the difference between the total positive weight and total negative weight will be denoted by $\theta_W = W^+ - W^-$. We will denote by $\beta_1(\mathcal{G}) + 1 - V + E$ the number of cycles of the graph.*

The next results are consequences of others that were proved in [9].

Theorem 1. *Let $\mathcal{G}(V,E,W)$ be a bipartite graph and let M be a closed orientable surfaces. We can assume that:*

1. *If $\mathcal{G}(V,E,W)$ is the graph of a stable map $f : M \longrightarrow N$, then the genus of M is given by $g(M) = \beta_1(\mathcal{G}) + W$.*
2. *If genus $g(M) = \beta_1(\mathcal{G}) + W$, then $\mathcal{G}(V,E,W)$ is the graph of some stable map of M about some closed orientable surface N, where the maximum (absolute value) for the degree of f is given by $(m-1)/(n-1)$.*

2.2. Relation between Degree and Euler Characteristic Number

In [3], Demoto studied isotopy classes of stable maps from the sphere to the sphere, in which case the branch set is a connected and closed curve which may have cusps and/or self-intersections. Demoto proved that the number of cusps of f is at least $2d$ where $deg(f) \geq 2$ and the branch set has no self-intersections. In [5], the authors extended these results to stable maps from orientable surfaces to the sphere.

Definition 2. *Let $f : M \longrightarrow N$ be a stable map. We denote by*

1. $\chi(M_f^{\pm})$ *the Euler characteristic of* $M_f^{\pm}$*,*
2. $\theta_\chi(f) = \chi(M_f^+) - \chi(M_f^-)$ *the the difference of the* $\chi(M_f^+)$ *and* $\chi(M_f^-)$*.*

For a stable map between two closed orientable surfaces, there are some maps with the singular set empty. These maps are known as covering maps. Of course, the graph of the covering maps $\rho : M \longrightarrow N$ has only one vertex with $W = g(M)$.

Remark 1. *For a stable map between between two closed orientable surfaces M and N, with degree d and without singular points (covering map):*

1. *the Euler characteristic of M is $\chi(M) = d(f)\chi(N)$,*
2. *the genus of M is given by $g(M) = d(f)(n-1)+1$, where $n = g(N)$.*

Indeed, we can decompose M in $d(f)$ regions M_i, with genus $n-2$ and four boundary components, covering the surface N only once (see Figure 4), with the identification of the pairs of bordering. Then $\chi(M) = \sum_{i=1}^{d} \chi(M_i) = d(f)[2-2g(M_i)-4] = d(f)[2-2(n-2)-4] = d(f)(2-2n) = d(f)\chi(N)$.

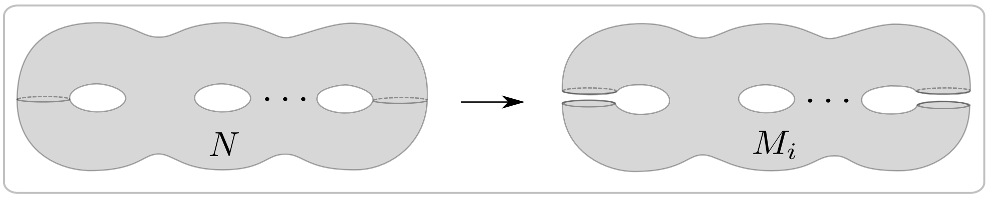

Figure 4. Example of a decomposition of M.

Proposition 1. *If $\mathcal{G}(V,E,W)$ is the graph of a stable map f between two closed orientable surfaces, then $\theta_\chi(f) = 2(\theta_V - \theta_W)$.*

Proof. The Euler characteristic of $M_i^{\pm}$ is $\chi(M_i^{\pm}) = 2 - 2w_i^{\pm} - E_i^{\pm}$, where $w_i^{\pm}$ and $E_i^{\pm}$ denotes (respectively) the genus and the number of boundary components for each surface $M_i^{\pm}$. Thus, $\chi(M^{\pm}) = \sum_{i=1}^{V^{\pm}} \chi(M_i^{\pm}) = 2V^{\pm} - 2W^{\pm} - E$. Then, it follows from Definition 2, $\theta_\chi(f) = \chi(M_i^+) - \chi(M_i^-) = 2(\theta_V - \theta_W)$. $\square$

Proposition 2. *If f is a fold map with degree d between two closed orientable surfaces M and N, then $\theta_\chi(f) = d(f)\chi(N)$.*

Proof. Let $f : M \longrightarrow N$ be a fold map with $d(f) \geq 0$ and $\Sigma f \neq \varnothing$. We denote by N_0 the complement of the apparent contour $\mathcal{B}f$ in N and by $M_0^{\pm}$ the union of the inverse images of N_0 contained in $M^{\pm}$. Note that, for each region R_i^- in M_0^-, there is a copy R_i^+ in M_0^+, then $\chi(R^-) = \chi(R^+)$. Denoted by $M_r^+ = M^+ \setminus \chi(\bigcup_i R_i^+)$, the complement of $\bigcup_i R_i^+$ in M^+. Then, $\chi(M^-) = \chi(\bigcup_i R_i^-) = \chi(\bigcup_i R_i^+)$ and $\chi(M^+) = \chi(M_r^+)) + \chi(\bigcup_i R_i^+))$. Thus, $\chi(M^+) - \chi(M^-) = \chi(M_r^+)$. Since M_r^+ forms a cover for N with degree d, by Remark 1, we have $\chi(M_r^+) = d(h)\chi(N)$. By replacing the last equality in the previous equality, we have $\chi(M^+) - \chi(M^-) = d(h)\chi(N)$. Then, the result follows from Definition 2. $\square$

Corollary 1. *If $\mathcal{G}(V,E,W)$ is a graph of a fold map $f : M \longrightarrow N$ with degree d, then $\theta_V - \theta_W = d(f)(1-n)$.*

Proof. All fold maps are stable maps. Then, by Propositions 1 and 2, we have equality $d(f)\chi(N) = 2(\theta_V - \theta_W)$, which proves the proof. $\square$

For $N = \mathbb{S}^2$, we have $d = \theta_V - \theta_W$ (see [7]).

3. The Effects of Codimension One on the Transitions

Our objective in this section is to study codimension one transitions and their effects on the topology of the regular set and the singular set from a global point of view.

Codimension One Transitions

A codimension one transition corresponds to a generic isotopy from a given stable map to another one lying in a different path component of the subset $\mathcal{E}$ of stable maps in $C^\infty(M, N)$. In other words, this means a path that is transversal to all the strata of the complement Δ, known as the discriminant set, of $\mathcal{E}$ in $C^\infty(M, N)$.

In [6], Ohmoto-Aicardi presents a detailed list of the codimension one transitions from the local viewpoint. Some of these transitions, regardless of the orientation of the arcs of curves, are illustrated in Figure 5, where we denote by B the beaks transition, by L the lips transition and by S the swallowtail transition. These three transitions change the number of cusps for ± 2. The transitions B and L also change the number of singular components. The other transitions that do not alter the number of cusps and the number of singular components are tangency between two folds, denoted by T^i, $i = 1, 2, 3$, tangency between one fold and a cusp, denoted by TC^i, $i = 1, 2$, where i refers to the number of pairs of preimage inside the new region bounded by the two arcs of curves. Both transitions change the number of double points. We will omit here the triple point because it does not affect the invariants (cusp, double points, number of singular components and number of regular components).

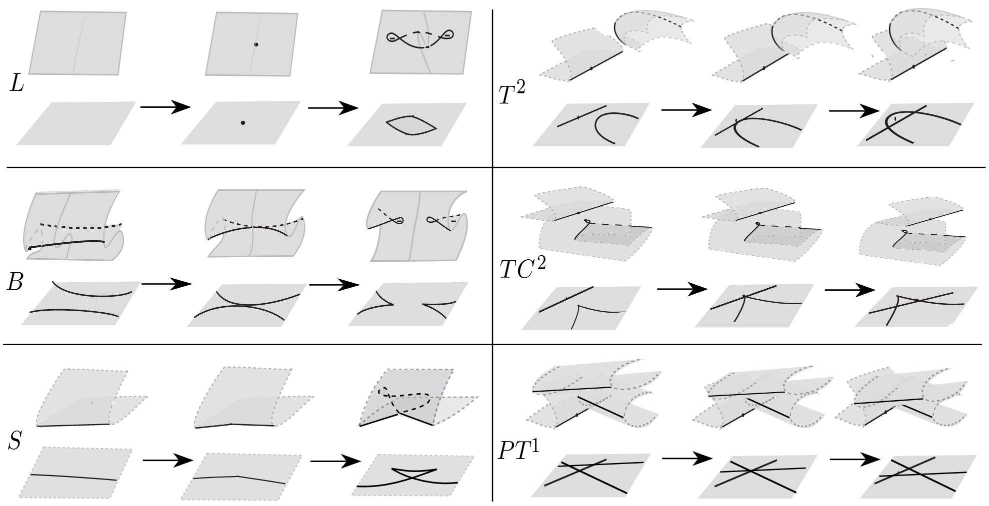

Figure 5. Examples of some codimension one transitions (see [6]).

If f and h are two stable maps in the same homotopy class in $C^\infty(M, N)$, then f and h can be joined by a generic homotopy, in the sense that only passes through codimension one transitions in $C^\infty(M, N)$. These transitions can change the numbers of cusps, double points, singular curves and regular regions. Clearly, the transitions do not alter the degree, for the new map remains in the same path component of $C^\infty(M, N)$. We are interested in the transitions that change the topology of the regular set and the singular set and the number of cusps, namely the beaks and lips transitions.

We call the sign (positive or negative) of a cusp point if the singular set in a neighborhood of any cusp point separates M into a positive M_x^+ and a negative M_x^+ immersed small region. The point x is said to be positive (resp., negative) if $f(M_x^-)$ (resp., $f(M_x^-)$) is connected in $N \setminus f(\Sigma f)$. In other words, x is a positive (resp., negative) cusp point if all the points of $f(M_x^-)$ (resp., $f(M_x^+)$) have the same number of preimages (exactly 3, in a small neighborhood of x, as shown in Figure 6). We denote by C^+ (C^-) the total number of positive (negative) cusp points.

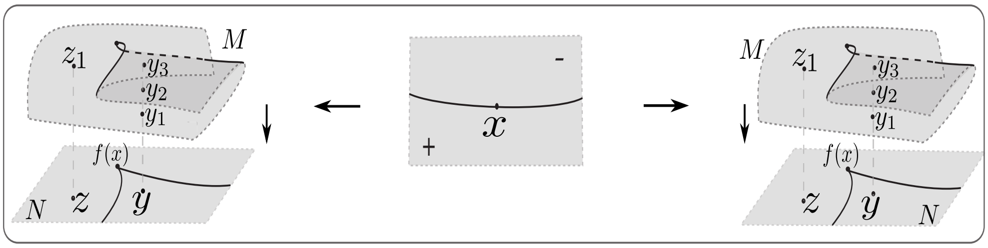

Figure 6. Negative and positive cusps.

The codimension one transitions that change the number of cusps and singular curves are the beaks and lips transitions:

(i) **Lips transitions:** We denote the lips transition (see Figure 6) by L^γ, where γ corresponds o the sign ("+" if positive and "$-$" if negati(ve) of the pair of cusps.

The transition $L^\pm$ occurs in a region $X \subset M^\pm$ and always increase the number singular components and the number of regular components of $M^\mp$. The effect of this transition $L^\pm$ on the graph of f corresponds to adding a new edge in E and a new vertex in $V^\mp$, corresponding to the initial region, now renamed $X_1^\mp$ (see Figure 7).

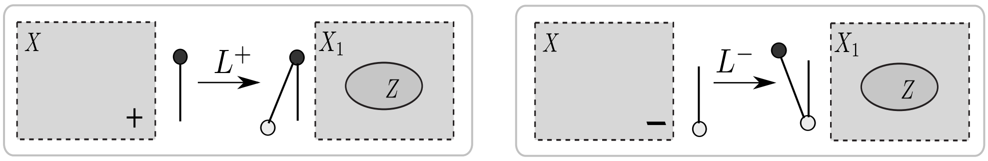

Figure 7. Lips transition.

(ii) **Beaks transitions**: Such a transition occurs when we approach two arcs of the singular set until they join in a common point (beaks point) and break again, giving rise to a new pair of arcs and as a result a couple of cusp points that was introduced in the branch set. This process, from the global point of view, is to increase the cusp points. We can classify the beaks transitions into eight different cases, as illustrated (locally) in Figure 8, where:

1. $B_v^{+,\pm}$: increases by 1 the number of vertices in $V^\mp$, edges E and it increases by 2 the number cusps $C^\pm$.
2. $B_v^{-,\pm}$: decreases by 1 the number of vertices in $V^\pm$ and edges E and it increases by 2 the number cusps $C^\pm$.
3. $B_w^{+,\pm}$: increases by 1 the weight in $W^\pm$ and decreases by 1 the number of edges E and it increases by 2 the number cusps $C^\pm$.
4. $B_w^{-,\pm}$: transition decreases by 1 the weight in $W^\mp$ and increases by 1 the number of edges E and increases by 2 the number cusps $C^\pm$.

Table 1 shows the increments of the numbers of edges, cuspids, vertices and weight related to beaks and lips transitions, in the sense that the number of cusps increases.

Remark 2. *The importance of beaks decomposition transition B (see Figure 8 and [6]) lies in the variations of the number of edges of the graph, which affects the number of the graph vertices or its weights (ver Table 1).*

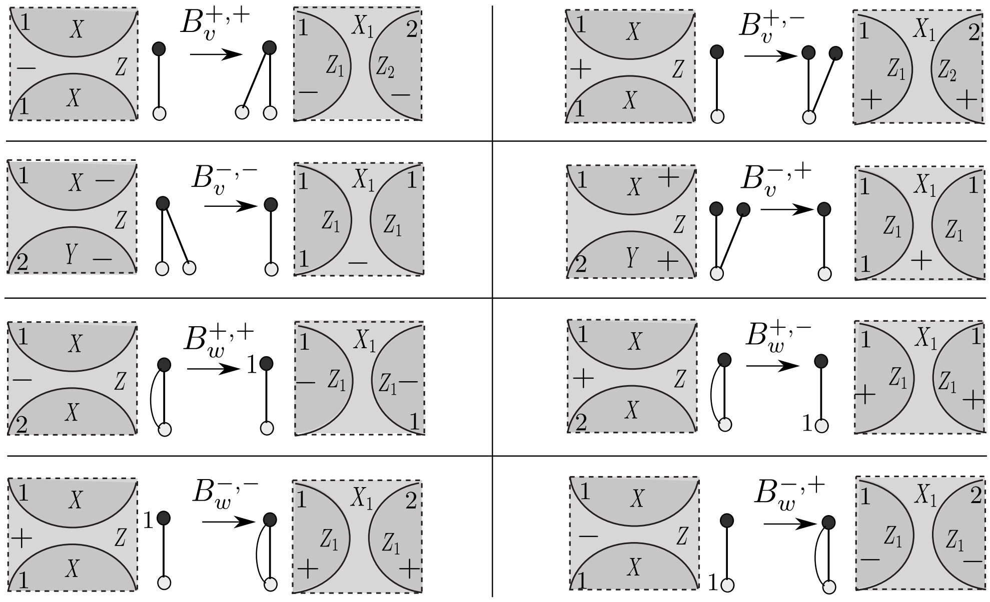

Figure 8. Local effect: decomposition of the beaks transitions.

Table 1. Local effect: decomposition of the beaks transitions.

$I \setminus^T$	L^+	L^-	$B_v^{+,+}$	$B_v^{+,-}$	$B_v^{-,+}$	$B_v^{-,-}$	$B_w^{+,+}$	$B_w^{+,-}$	$B_w^{-,+}$	$B_w^{-,-}$
ΔE	1	1	1	1	−1	−1	−1	−1	1	1
ΔC^+	2	0	2	0	2	0	2	0	2	0
ΔC^-	0	2	0	2	0	2	0	2	0	2
ΔV^+	0	1	0	1	−1	0	0	0	0	0
ΔV^-	1	0	1	0	0	−1	0	0	0	0
ΔW^+	0	0	0	0	0	0	1	0	0	−1
ΔW^-	0	0	0	0	0	0	0	1	−1	0

Definition 3. *If I is an invariant of stable maps and f can be obtained from h through a codimension one transition $\eta \in \{l^{\pm}, b_v^{\pm}, b_w^{\pm}\}$, then the difference $I(f) - I(h)$ is called increment of I and is denoted by $\Delta I = I(f) - I(h)$.*

To calculate $\theta_I(f)$, we can choose an initial map h in some homotopy class of f with the $\theta_I(h)$ known. Then, we can compute $\theta_I(f)$ starting from $\theta_I(h)$ and adding the increment ΔI, along the path from h to f.

Lemma 1. *The increments of the number of cusps, number of singular curves, regular regions and the total genus of the regions along a path between two stable map f and h in the same homotopy $C^\infty(M,N)$ are given, respectively, by*

$$\Delta E = (L^+ + L^- + B_v^{+,+} + B_v^{+,-} + B_w^{-,-} + B_w^{-,+}) - (B_v^{-,+} + B_v^{-,-} + B_w^{+,+} + B_w^{+,-}),$$
$$\Delta C^{\pm} = 2(B_v^{+,\pm} + B_v^{-,\pm} + B_w^{+,\pm} + B_w^{-,\pm} + L^{\pm}),$$
$$\Delta V^{\pm} = L^{\mp} + B_v^{+,\mp} - B_v^{-,\pm},$$
$$\Delta W^{\pm} = B_w^{+,\pm} - B_w^{-,\mp}.$$

Consequently,

$$\Delta\theta_C = 2[(L^+ + B_v^{+,+} + B_v^{-,+} + B_w^{+,+} + B_w^{-,+}) - (L^- + B_v^{+,-} + B_v^{-,-} + B_w^{+,-} + B_w^{-,-}),$$
$$\Delta\theta_V = (L^- + B_v^{+,-} + B_v^{-,-}) - (L^+ + B_v^{+,+} + B_v^{-,+}),$$
$$\Delta\theta_W = (B_w^{+,+} + B_w^{-,+}) - (B_w^{+,-} + B_w^{-,-}),$$

$$\Delta\theta_\chi = 2[\Delta\theta_V - \Delta\theta_W] = -\Delta\theta_C.$$

Proof. Note that $\Delta\theta_C = \Delta(C^+ - C^-) = \Delta C^+ - \Delta C^-$, $\Delta\theta_V = \Delta V^+ - \Delta V^-$ e $\Delta\theta_W = \Delta W^+ - \Delta W^-$. The result follows from Table 1 and Proposition 1. □

Theorem 2. *Let $f : M \longrightarrow N$ be a stable map and let h be a fold map in the same homotopy class of f. Then,*

$$\theta_\chi(f) + \theta_C(f) = \theta_\chi(h).$$

Proof. If h is a fold map, then $\theta_C(h) = 0$ and $\theta_C(f) = \Delta\theta_C(f)$. The result follows from equality $\theta_\chi(f) = \theta_\chi(h) + \Delta\theta_\chi(f)$ and by Lemma 1, $\Delta\theta_\chi(f) = -\Delta\theta_C(f)$ □

Corollary 2. *If h and f are two fold maps ($\theta_C(f) = 0$) in the same homotopy class of $C^\infty(M, N)$, then $\theta_\chi(f) = \theta_\chi(h)$.*

Example 1. *We illustrate in Figure 9 the transitions between four stable maps from the sphere to the plane, wherein: (a) the map has three singular curves, no double points, nor cups, whose graph is type $\mathcal{G}(4,3,0)$; in (b), the map with graph $\mathcal{G}(3,2,0)$ can be obtained by applying a beaks transition $B_v^{-,+}$ in (a); in (c), the map with graph $\mathcal{G}(4,3,0)$ can be obtained by applying the transition L^+ on the map (b); the map (d), is the trivial projection of the sphere, with graph $\mathcal{G}(2,1,0)$, can be obtained by applying a lips transition $-L^+$ on the map (b) or two transition $-L^+$ on the map (c).*

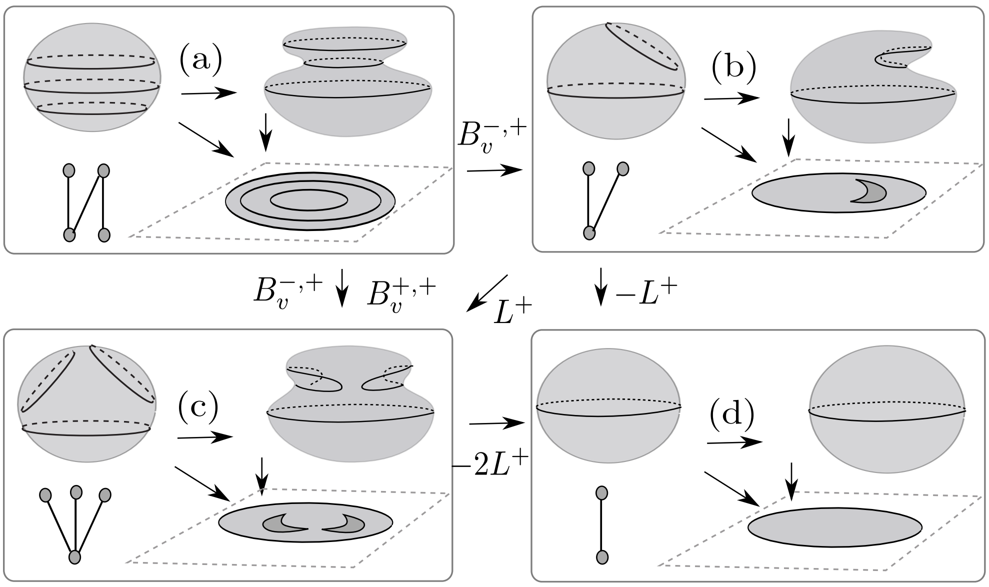

Figure 9. Examples of lips transitions L^+ and beaks transitions $B_v^{\pm,+}$ (see [9]).

Example 2. *Figure 10 illustrates four stable maps from torus to N: in (a), the fold maps with two singular curves, two double points and graph type $\mathcal{G}(2,2,0)$ is composed by an immersion of the torus into 3-space, a projection of 3-space into the plane and an embedding of the plane into N; in (c), the map with one singular curve has two double points and two cusps and it is a graph type $\mathcal{G}(2,1,1)$. This map can be obtained by applying a beaks transition $B_w^{+,+}$ in the map (a); in (d), the map with one singular curve has two double points and four cusps. It is a graph of type $\mathcal{G}(2,2,0)$. This map can be obtained by applying two swallowtail transitions $-S$ on the map (c); in (b), the fold map with two singular curves has no double points, and it can be obtained by applying a beaks transition $B_w^{-,-}$ to the map (d).*

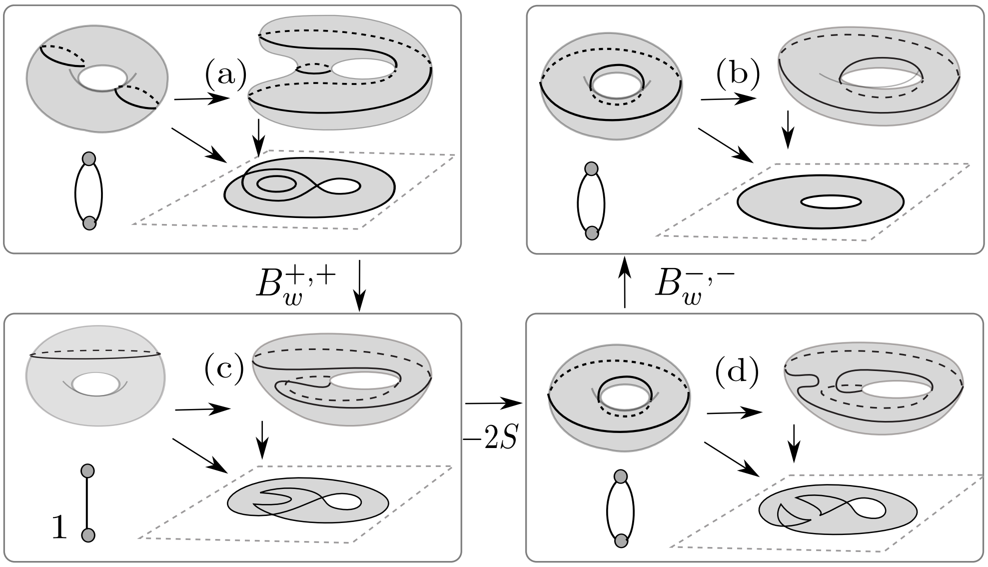

Figure 10. Examples of transitions $B_w^{+,+}$ and $B_w^{-,-}$.

4. Global Invariant of Stable Maps

We now see the relation between θ_χ, θ_C and d for a stable map between two closed orientable surfaces.

Theorem 3. *If the stable map $f : M \longrightarrow N$ has degree $d(f)$, then*

$$\theta_\chi(f) + \theta_C(f) = d(f)\chi(N).$$

Proof. Let h be a fold map in the same homotopy class f. Then, $d(f) = d(h)$. According to Proposition 2, $\theta_\chi(h) = d(h)\chi(N)$. By Theorem 2, $\theta_\chi(f) + \theta_C(f) = \theta_\chi(h)$. Then, the result follows from these three equalities. Note that the result does not depend on the choice of h in the homotopy class of f. □

Note that $\theta_\chi(f) = \chi(M) - 2\chi(M_f^-)$ and $\theta_C(f)$ is the total sum of the signs of the cusps of f. Thus, we can see that the Theorem 3 is equivalent to Quine's theorem, which was proved in [11]. Here, we give a new proof based on the previous results.

Proposition 3. *If the stable map $f : M \longrightarrow N$ has degree $d(f)$, then,*

$$\theta_V - \theta_W + \frac{\theta_C}{2} = d(f)(1 - n).$$

Proof. By Theorem 3, we have $\theta_\chi(f) = d(f)\chi(N) - \theta_C(f)$. By Propostion 1, we have $\theta_\chi(f) = 2[\theta_V - \theta_W]$. Then, it follows from these two equalities that $\theta_V - \theta_W = d(f)(1 - n) - \theta_C(f)/2$. □

The next result is a consequence of Proposition 3 and it was proven in [5].

Corollary 3. *If $\mathcal{G}(2,1,W)$ is the graph of a stable map $f : M \longrightarrow N$ with degree d, then $\theta_W = d(f)(n-1) + \theta_C/2$.*

Consequently, if $\theta_W = 0$, then

$$\theta_C(f) = \begin{cases} 2d, & \textit{if} \quad n = 0, \\ 0, & \textit{if} \quad n = 1, \\ 2d(f)(1-n), & \textit{if} \quad n > 1. \end{cases}$$

An apparent contour of a stable map $f \in C^\infty(M, N)$ it is said to be irreducible if Σf has only connected component (see [14]). If N is a closed orientable surface, according to Eliashberg [12], for each map $f : M \longrightarrow N$ of class C^∞, there is a stable map $h : M \longrightarrow N$ that is homotopic to f, whose contour is irreducible.

Example 3. *Figure 11 illustrates a sequence of transitions between stable maps from the 7-torus to 3-torus, where the first has six singular curves and the latter has an irreducible contour: in (a), the fold map with degree one,no double points nor cusps, six singular curves and graph type $\mathcal{G}(7,6,7)$; in (b), the map, with three singular curves and graph type $\mathcal{G}(5,4,7)$, can be obtained by applying a beak transition $B_v^{-,+}$ and a lip transition $-L^+$ in the map (a); in (c), the map with three singular curves with four cusps points (no double points), can be obtained by applying two beaks transitions $B_v^{-,+}$ on the map (b); in (d), the map with one singular curve, without double points and two cusps points, can be obtained by applying a beak transition $-B_v^{+,+}$ on (c).*

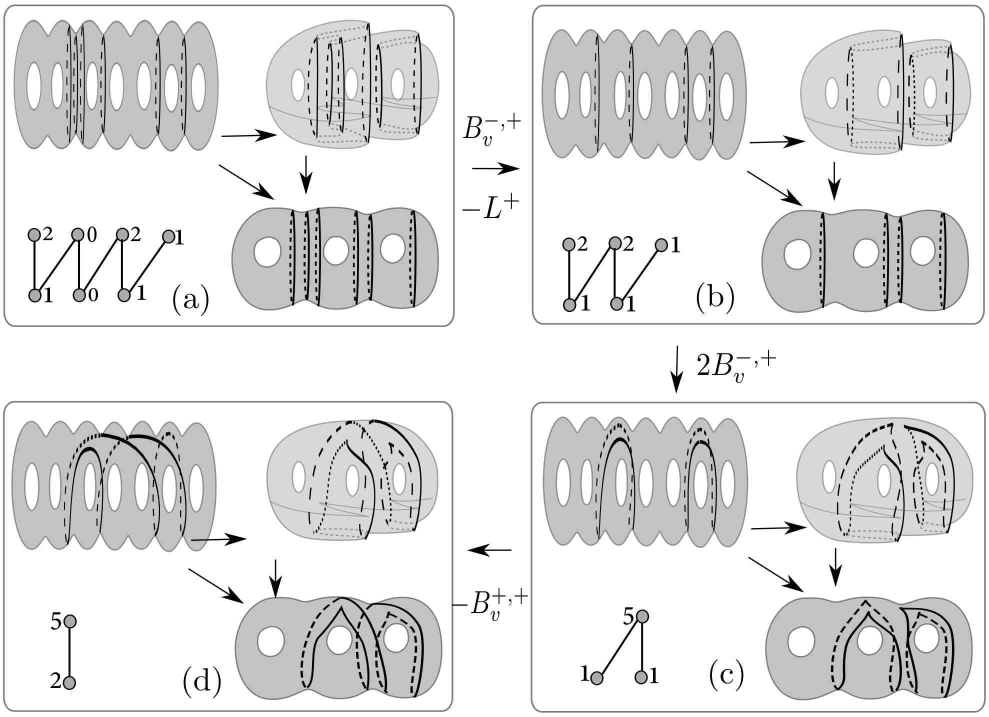

Figure 11. Examples of transitions $B_v^{\pm,+}$ and S.

Remark 3. *If $f : M \longrightarrow N$ is a fold map with zero degree with irreducible apparent contours, then M^- is a copy of M^+. Consequently, by Definitions 1 and 2, we have $\theta_V = \theta_W = \theta_\chi = 0$ (see Figure 12).*

Example 4. *Figure 12 illustrates five irreducible apparent contours of fold maps with zero degree, no double points nor cusps over* 3*-torus. A fold map of* 4*-torus in (a) is homotopic to (b), but it is not homotopic to (c). Also, are homotopics the fold maps of* 6*−torus to*3*-torus in (d) and (e).*

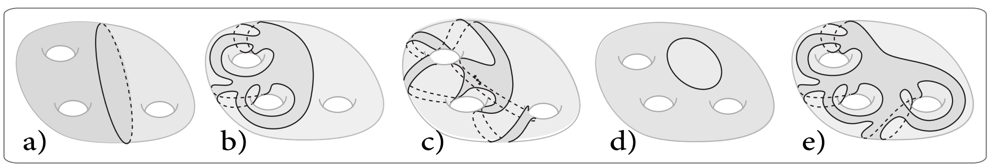

Figure 12. Examples of irreducible apparent contours of fold maps with zero degree over 3-torus.

Proposition 4. *Let $f : M \longrightarrow N$ be a stable map with degree $d(f)$ and irreducible contour. Then, the genus of M is given by $g(M) = 2W^{\mp} \pm d(f)(n-1) \pm \theta_C/2$.*

Proof. If $f : M \longrightarrow N$ is a stable map with degree $d(f)$ and irreducible contour, then $E = V^+ = V^- = 1$. By Corollary 3, $W^+ - W^- = d(f)(n-1) + \theta_C/2$. By Theorem 2, $W^+ = g(M) - W^-$.

By replacing the last equality in the previous equality, we have $g(M) - 2W^- = d(f)(n-1) + \theta_C/2$. The same is true for $W^- = g(M) - W^+$. □

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