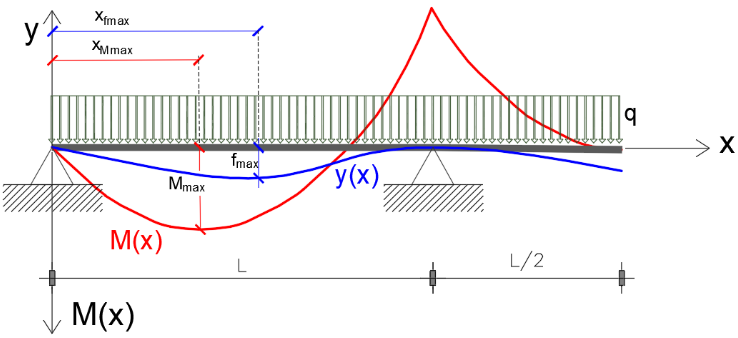
San Pablo CEU University ______ Institute of Technology

Degree in Architecture

EXAMPLES OF MATHEMATICAL FUNCTIONS IN STRUCTURAL ANALYSIS

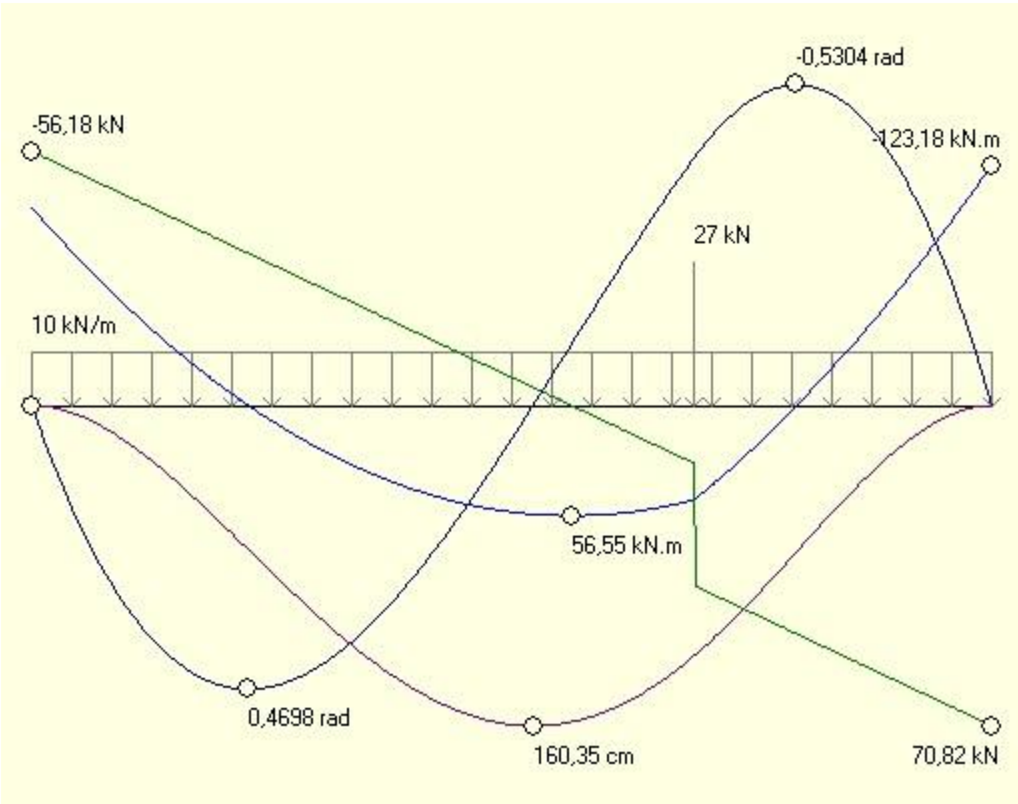


M. Castilla, F. Hernando October 2018

Translation: M. Castilla, J. C. Garro

Both, mathematical knowledge and mastering function analysis lock the door to success when analyzing the structural behavior of simple and continuous beams.

Each of the sections of those structural members shows variable shear forces, bending moments, rotation and deflection. The image below shows some of these changing functions.



These functions are related to one another: The rotation function (represented with a dark blue curve is the derivative of the elastic deflection (purple). Similarly, the bending moment function (light blue) is the derivative of the rotation function; the shear-force function (green) is the derivative of the bending moment one, and the loading function (grey) is the derivative of the shear-force function.

If the opposite way is followed, starting with the applied load's function, the shear force, bending moment, rotation and deflections functions can be obtained through sequential integrations. Here it is where understanding different simplified integration methods become more than useful.

In the image above, it can be observed that the maximum and minimum values, and points of inflection of each of the functions appear where the corresponding first or second derivative vanishes. If we study the shear diagram, an abrupt change in the slope of the bending moment function happens where the shear diagram shows a discontinuity. The ability to obtain all these critical values in functions is based on functions analysis.

The examples that have been gathered in this collection are aimed to first- or second-year students, in an attempt to foster the students' understanding of the practical application -in the area of structural design- of the knowledge they will be obtaining during their elemental training in functions.

Solving these examples requires, solely, mathematical skills. No previous knowledge related to structural design is required. The explanations linked to this area are already included in each of the statements.

The previous mentioned mathematical skills will be widely demanded during the years of training that follow the initial courses – mainly in Physics or Structural Design courses -, thus this training in the use of functions must be as thorough and concise as possible.

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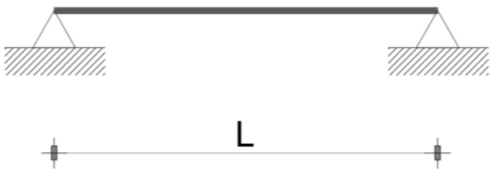
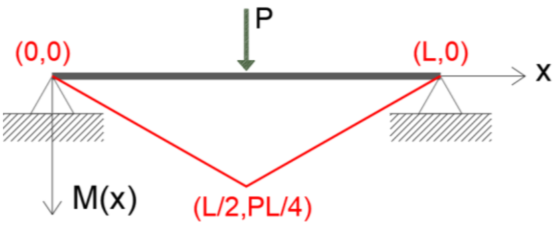
Problem V01

The beam in the image is simply supported at its ends. Its length is **L**; its cross-section is even - with an **I** value of the moment of inertia -. The value of the Young's Modulus of the material is **E**.



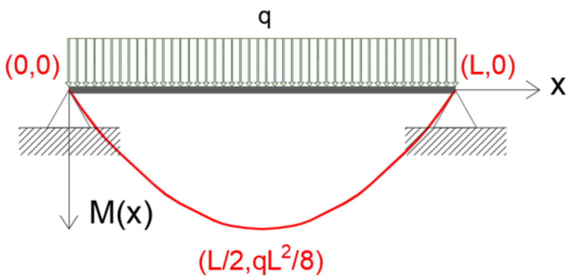
When the beam is loaded with a vertical, point load **P** in the middle of the span, each of its sections is subjected to a different value of internal bending force. The figure below represents the function of this internal bending force **M(x)**, that affects every section located at **x** distance from the origin.



The variation of this function is linear on both intervals. The left interval **(0, L/2)** M(x) passes through the points **(0, 0)** y **(L/2, PL/4).** The right one passes through **(L/2, PL/4)** y **(L, 0)**

a) *Obtain the explicit expression of M(x) for each of the intervals, using the provided coordinate system (ordinates axis oriented downwards).*

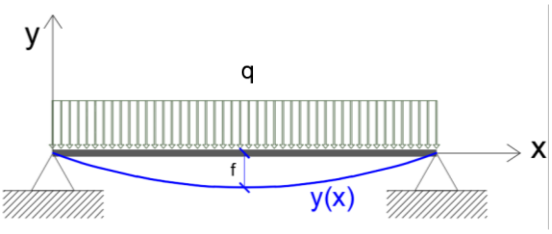
When the beam is subjected to an evenly distributed load **q**, the **M(x)** function is a 2nd-degree polynomial whose graph is a parabola, with vertical axis, that goes through points **(0,0)** and **(0, L)**. Its vertex is located on point **(L/2, qL^2/8)**



b) *Obtain the explicit expression of M(x), using the provided coordinate system (y-axis oriented downwards).*

Problem V02

The distributed load q on the beam that was analyzed in problem V01 produces the deflection represented on the image below. This deflection is defined by its Deflection Equation **y(x)**:



The y(x) expression can be obtained from the equation $\mathbf{d^2y(x)/dx^2 = M(x)/EI}$, where M(x) is the same found in section b), problem V01. Taking all the previous into consideration:

a) *Work the two indefinite integration stages and: Find **y(x)** as a function of **q, L, E, I** (constants), the variable **x** and both integration constants, C_1 y C_2.*

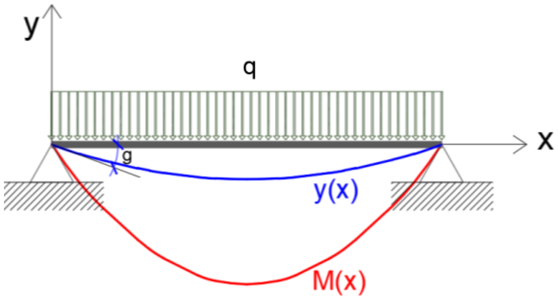
b) *Solve the initial value problem: Obtain the values of C_1 y C_2 imposing the following condition: the graph of y(x) must pass through **(0, 0)** and **(L, 0)**, which are the points where the external supports are located.*

c) *Substitute C_1 y C_2 in **y(x)** and obtain its final expression as a function of **q, L, E, I, x***

d) *Find the value of the maximum displacement **f** (maximum deflection) that, due to symmetry conditions, in this case, happens in the middle of the span.*

Problem V03

The elastic deflection that the distributed load, q, produced in problem V01 shows rotation of the beam on its left end. The value of this rotation is g, as shown in the image below:



This rotation can be calculated dividing the area of the parabola **M(x)** (that was obtained in V01) by the constant value 2EI with a negative sign (counterclockwise rotation is considered positive by sign convention). Answer the following questions:

a) *Find the area* ***A*** *bounded by the parabola and the interval (0, L) in the x-axis (use an integral).*

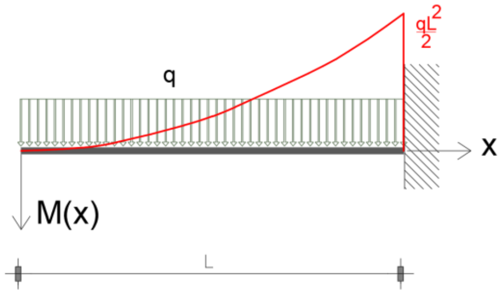
b) *Find the value of the rotation* ***g*** *on the left end of the beam.*

c) *Prove that the rotation corresponds to the derivative of the* ***y(x)*** *function at the origin (this function was obtained in problem V02)*

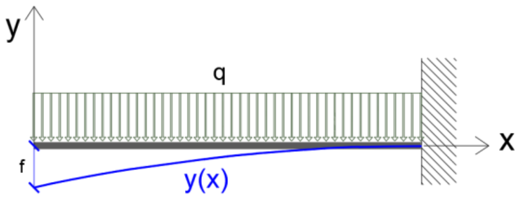
Problem V04

An **L** long fixed supported on its right end, cantilever beam, will be studied. The beam's cross-section is even - with an **I** value of the moment of inertia - and the value of the Young's Modulus of the material is **E**.

The beam is subjected to an evenly distributed load **q**, that produces a rate of the internal bending force of **M(x)** on each of the sections, located at a distance x from the origin. This M(x) function is a 2nd-degree polynomial whose graph is a parabola. At the free end of the cantilever, the value of this polynomial is zero, and the slope of the parabola is 0 (the tangent is horizontal). On the right end, the value is negative, as shown in the image below:



The bending behavior produces the deflection represented on the image below. This deflection is defined by its Deflection Equation **y(x)**. The maximum deflection **f** appears on the free end of the beam.



Considering the Deflection Equation $\mathbf{d^2y(x)/dx^2 = M(x)/EI}$, answer the following:

a) *Obtain the explicit expression of the M(x) parabola, observing the given conditions.*

b) *Work the two indefinite integration stages and find **y(x)** as a function of **q, L, E, I,** constants, the variable **x** and both integration constants, C_1 y C_2.*

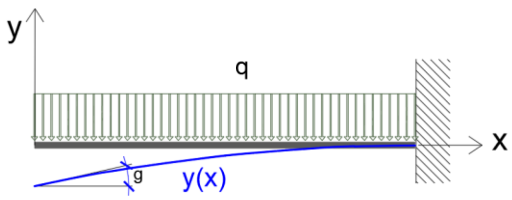
c) *Obtain the values of C_1 y C_2 imposing the following conditions: y(x) must pass through **(L, 0)** and its derivative must be zero on **(L, 0)***

e) *Substitute C_1 y C_2 in **y(x)** and obtain its final expression as a function of **q, L, E, I, x***

f) *Find the value of the maximum displacement **f** (maximum deflection) that happens on the left end of the cantilever.*

Problem V05

The elastic deflection that the distributed load, q, produced in problem V04 shows rotation of the beam on its left end. The value of this rotation is g, as shown in the image below:



This rotation can be calculated dividing the area below the parabola **y=M(x)** (that was obtained in V04) by the constant value **EI** with a negative sign (counterclockwise rotation is considered positive by sign convention). Answer the following questions:

a) *Find the area **A** of the parabola by integration of the (0, L) interval.*

b) *Find the value of the rotation **g** on the left end of the beam.*

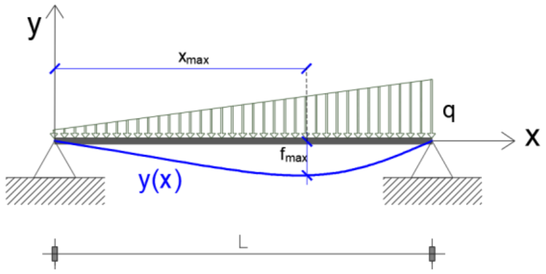
c) *Prove that the rotation corresponds to the derivative of the **y(x)** function at the origin (this function was obtained in problem V04)*

Problem V06

The beam in the image is simply supported at its ends. Its length is **L**; its cross-section is even - with an **I** value of the moment of inertia -. The value of the Young's Modulus of the material is **E**.

A triangular, distributed load is applied to it. The value of the load on the beam's left end is **0**; the value on the right end is **q**. Each of the sections of the beam is subjected to a different value of internal bending force. The figure below represents the function of this internal bending force **M(x)**, that affects every section located at **x** distance from the origin. The 3rd degree polynomial function that express this bending behavior is **$M(x) = q (Lx – x^3/L) / 6$.**

The image below represents the deflection produced by the triangular loading situation.



The load is not symmetrical. Thus neither M(x) nor y(x) are symmetrical. Hence, the maximum internal bending force and the maximum deflection will not appear in the middle of the beam. Answer the following:

a) *Find the **x_{mmax}** position where function **M(x)** presents its relative maximum. Value of this **M_{max}***

b) *Deflection Equation **y(x)** obtained from **$d^2y(x)/dx^2 = M(x)/EI$** and values of the integration constants imposing the following condition: the graph of y(x) must pass through **(0, 0)** and **(L, 0)***

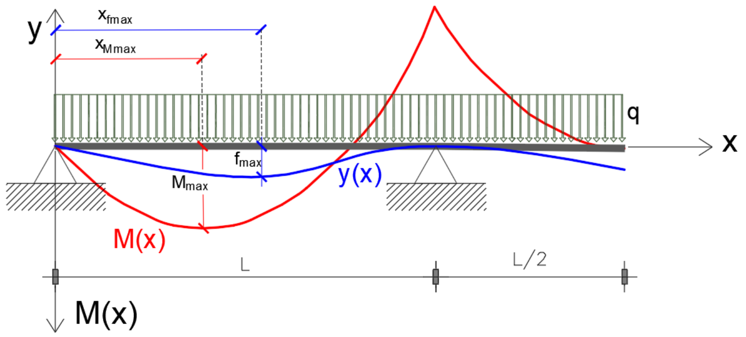
c) *Find the **x_{mmax}** position where function **y(x)** presents its relative maximum. Do the maximum bending, and the maximum deflection coincide on the same x position?*

d) *Value of the maximum deflection **f_{max}**.*

Problem V07

The beam in the image is simply supported at its ends. Its length is **L**, with an additional cantilever. The length of this element is **L/2**. The cross-section of the beam is even - with an **I** value of the moment of inertia -. The value of the Young's Modulus of the material is **E**.

The beam is subjected to an evenly distributed load **q**, that produces an internal bending force. This bending is expressed by the parabolic function **M(x) = q (3Lx/8 – x²/2)** along the **(0, L)** interval. The image shows the graphs of M(x) and y(x) functions:



Answer the following:

a) *Find the* x_{mmax} *position where function* ***M(x)*** *presents its relative maximum. Value of this* M_{max}

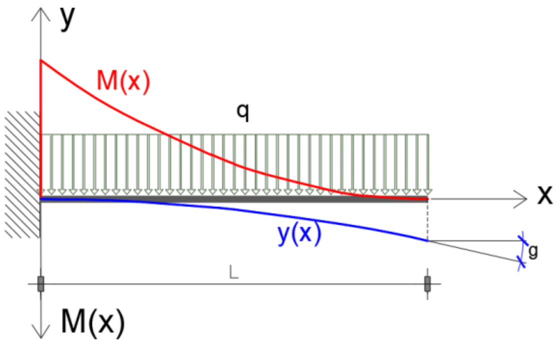
b) *Deflection Equation* ***y(x)*** *obtained from* $d^2y(x)/dx^2 = M(x)/EI$ *and values of the integration constants imposing the following condition: y(x) must pass through* ***(0, 0)*** *and* ***(L, 0)***

c) *Rotation values* g_i *and* g_d *on both left and right supports.*

Problem V08

An **L** long fixed supported on its left end, cantilever beam, will be studied. The beam's cross-section is even - with an **I** value of the moment of inertia - and the value of the Young's Modulus of the material is **E**.

The beam is subjected to an evenly distributed load **q**, that produces a rate of the internal bending force of **M(x)** on each of the sections, located at an x distance from the origin. This bending can be expressed by the parabolic function $M(x) = -q\,(L - x)^2/2$ which has been plotted on the image below, as well as the elastic deflection and the maximum values of deflection and rotation.



The maximum **g** rotation appears on the right end of the cantilever. Its value can be obtained with a definite integral between 0 and L of the function that results from dividing **M(x)** by the **EI** constant.

*a) Determine the definite integral and the value of the maximum rotation **g***

The maximum deflection **f** appears on the right end of the cantilever, too. Its value can be obtained with a definite integral between 0 and L of the product **M(x) (L-x)**, divided by the **EI** constant.

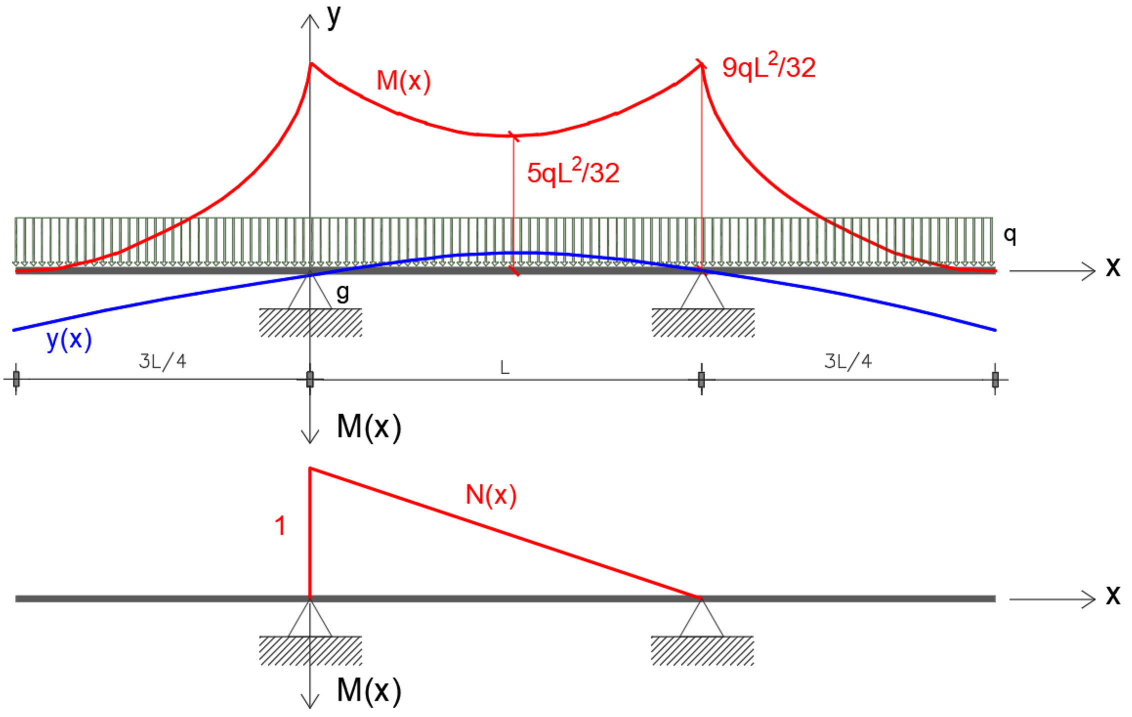
*b) Determine the definite integral and the value of the maximum deflection **f***

Problem V09

The beam in the image is simply supported at its ends. Its length is **L**, with two additional cantilevers. The length of these elements is 3**L/4**. The cross-section of the beam is even - with an **I** value of the moment of inertia -. The value of the Young's Modulus of the material is **E**.

The beam is subjected to an evenly distributed load **q**, that produces an internal bending force. This bending is expressed by a parabolic function. The value of this function on points **(0, 0)** and **(L, 0)** is -**9qL²/32**. On point **(L/2, 0)** the value is **-5qL²/32** in **(L/2, 0)**.

Also, an **N(x)** linear function between **(0, -1)** y **(L, 0)** represents the power of bending over the rotation of the section located on point **(0, 0)**.



In this case, the **g** rotation on the left support can be obtained with a definite integral between 0 and L of the function that results from dividing the product of **M(x) N(x)** by the **EI** constant.

*Find the value of the **g** rotation, using a geometrical procedure based on the following mathematical property:*

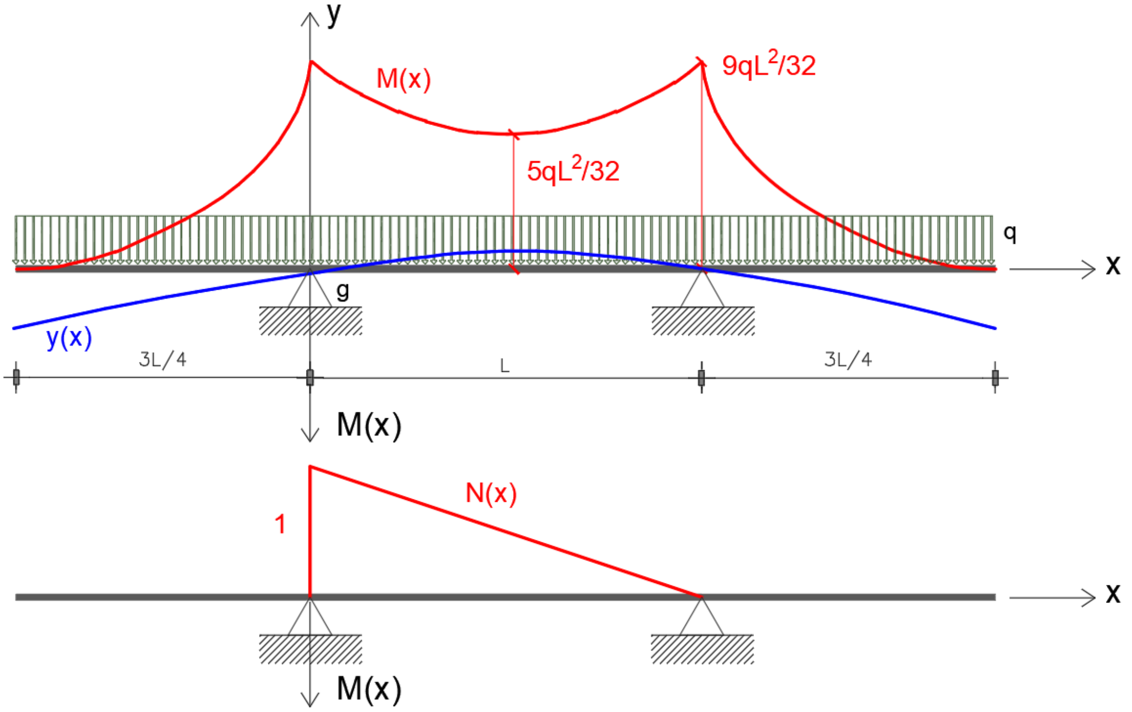
"The definite integral on an interval of the product of two functions, a generic one and a linear one, equals the product of the area under the generic function in the interval times the value of the linear function at the position where the centroid of the generic function is located"

Moreover, it can be considered that the area beneath a parabola similar to the one stated as M(x) is one-third of the rectangle that circumscribes it, plus the area of the rectangle with L width and height equal to the height of the parabola's vertex.

Problem V10

Let us consider the beam with two cantilevers that was defined in problem V09, subjected to the same distributed load.

The functions that were defined in that problem, **M(x)** y **N(x)**, and that have been plotted in the image below are to be considered too, as well as the expression that was applied to find the **g** rotation on the left support.



*Find the value of the **g** rotation, using this time the Simpson's method to solve the definite integral:*

"The definite integral of a less than 4th degree polynomial function in an interval between coordinates x_a and x_b, equals the product of one-sixth of the length of the interval times the sum of the values of the function at its ends plus four times the value of the function in the middle of the interval".

Definite integral of f(x) in a given interval = $(x_b - x_a)\,(f(x_a) + f(x_b) + 4f((x_a+x_b)/2))\,/\,6$

Keep in mind that the polynomial function must be equivalent to the product of both M(x) and N(x) functions.

Answers

Problem	Section	Answers
V01	A	En [0, L/2] $M(x) = P x / 2$; En [L/2, L] $M(x) = P (L – x) / 2$
	B	$M(x) = q (Lx – x^2) / 2$
V02	A	$M(x) = q (2Lx^3 – x^4) / 24 EI + C_1x + C_2$
	B	$C_1 = - qL^3 / 24 EI$ $C_2 = 0$
	C	$M(x) = q (2Lx^3 - L^3x – x^4) / 24 EI$
	D	$f = - 5 qL^4 / 384 EI$
V03	A	$A = qL^3 / 12$
	B	$g = - qL^3 / 24 EI$
	C	$g = dy(x)/dx_{(x= 0)} = C_1 = - qL^3 / 24 EI$
V04	A	$M(x) = - q x^2 / 2$
	B	$M(x) = - q x^4 / 24 EI + C_1x + C_2$
	C	$C_1 = qL^3 / 6 EI$ $C_2 = - qL^4 / 8 EI$
	D	$M(x) = - q (x^4 – 4L^3x + 3L^4) / 24 EI$
	E	$f = - qL^4 / 8 EI$
V05	A	$A = qL^3 / 6$
	B	$g = qL^3 / 6 EI$
	C	$g = dy(x)/dx_{(x= 0)} = qL^3 / 6 EI$
V06	A	$X_{mmax} = L / \sqrt{3}$ $M_{max} = qL^2 / 9 \sqrt{3}$
	B	$M(x) = q (10Lx^3 – 3x^5/L - 7L^3x) / 360 EI$
	C	$X_{fmax} = 0.5193$ (no coincide con X_{mmax})
	D	$f_{max} = - 0.006522 qL^4 / EI$
V07	A	$X_{mmax} = 3L / 8$ $M_{max} = 9qL^2 / 128$
	B	$M(x) = q (3Lx^3 – 2x^4 - L^3x) / 48 EI$
	C	$g_i = - qL^3 / 48 EI$ $g_d = 0$
V08	A	$g = - qL^3 / 6 EI$
	B	$f = - qL^4 / 8 EI$
V09	A	$g = - 19 qL^3 / 192 EI$
V10	A	$g = - 19 qL^3 / 192 EI$