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A CHARACTERIZATION OF THE KNEADING PAIR FOR BIMODAL DEGREE ONE CIRCLE MAPS

by Ll. ALSEDÀ and A. FALCÓ

1. INTRODUCTION

The goal of this paper is to characterize at a symbolic level the bimodal degree one circle maps. As it is usual, instead of working with the circle maps themselves we will rather use their liftings to the universal covering space $\mathbb{R}$. To this end, we introduce the following class $\mathcal{A}$ of maps. First we define $\mathcal{L}$ to be the class of all continuous maps F from $\mathbb{R}$ into itself such that

$$F(x+1) = F(x) + 1 \quad \text{for all } x \in \mathbb{R}.$$

That is, $\mathcal{L}$ is the class of all liftings of degree one circle maps. Then we will say that $F \in \mathcal{A}$ if (see Figure 1):

1) $F \in \mathcal{L}$;

2) there exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

We note that every map $F \in \mathcal{A}$ has a unique local maximum and a unique local minimum in $[0,1)$. To define the class $\mathcal{A}$ we restricted ourselves to the case in which F has the minimum at 0. Since each map from $\mathcal{L}$ is conjugate by a translation to a map from $\mathcal{L}$ having the minimum at 0, the fact that in (2) we fix that F has a minimum in 0 is not restrictive.

The study of these maps arises naturally in different contexts in dynamical systems. For instance, a three parameter family of such maps

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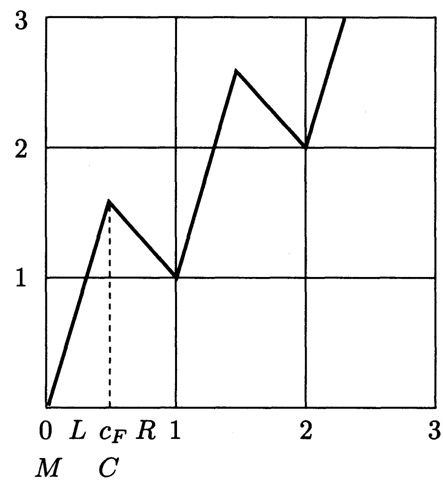

Figure 1. An example of a map F in class $\mathcal{A}$.

has been introduced by Levi [13] and used to study the Van der Pol equation (see [13] and [2]). On the other hand, the standard maps family defined as

$$F_{b,w}(x) = x + w + b\,\frac{\sin(2\pi x)}{2\pi}$$

where $x, w \in \mathbb{R}$ and $b \in [0, \infty)$ belongs to the class $\mathcal{A}$ for all $b > 1$. The study of this two parameter family displays a correspondence with periodically forced chick-heart cells (see [11]) and the plot of the phase-locking zones as a function of b and w gives the Arnold tongues (see [6]). Also, the class $\mathcal{A}$ is relevant in the description of the transition to chaos for contracting annulus maps.

We shall use the extension of the Kneading Theory of Milnor and Thurston [16] given by Alsedà and Mañosas [4] to maps from $\mathcal{A}$. The key point of this Kneading Theory is a suitable definition of itinerary. With this notion they extended some basic results of the kneading theory for unimodal maps to the class $\mathcal{A}$. Moreover, they showed that for a map from class $\mathcal{A}$, the set of itineraries of all points (that is, all the dynamics of the map) can be characterized by the *kneading pair* which is the pair formed by the itinerary of the maximum and of the minimum of the map under consideration (see the next section for a precise definition). Thus, in the study of bifurcations of parameterized families in $\mathcal{A}$ these two sequences play a crucial role. This is our motivation to characterize the set of kneading pairs of maps from $\mathcal{A}$. This will be done in the main result of this paper. Unfortunately, to state this characterization we need some background on kneading theory for maps of the class $\mathcal{A}$ and some notation. We will introduce this background and state the main result of the paper in the next section. The rest of the paper is organized as follows. In Section 3 we introduce some more notation and the preliminary results to prove the Main Theorem and in Section 4 we prove it. Finally, in Section 5 we extend the Main Theorem to the orientation preserving homeomorphisms of the circle and we make some remarks and derive some consequences on the existence of full families for maps from $\mathcal{A}$.

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2. DEFINITIONS AND STATEMENT OF THE MAIN RESULT

We start this section by recalling the kneading theory developed by Alsedà and Mañosas in [4].

First we recall the notion of itinerary of a point. In what follows we shall denote the integer part function by $E(\cdot)$. For $x \in \mathbb{R}$ we set

$$D(x) = x - E(x).$$

For $F \in \mathcal{A}$ and $x \in \mathbb{R}$ let

$$s(x) = \begin{cases} R & \text{if } D(x) \in (c_F, 1), \\ C & \text{if } D(x) = c_F, \\ L & \text{if } D(x) \in (0, c_F), \\ M & \text{if } D(x) = 0; \end{cases}$$

$$d(x) = E(F(x)) - E(x).$$

Then the *reduced itinerary of* x, denoted by $\widehat{\underline{I}}_F(x)$, is defined as follows. For $i \in \mathbb{N}$, set

$$s_i = s\big(F^i(x)\big), \quad d_i = d\big(F^{i-1}(x)\big).$$

Then $\widehat{\underline{I}}_F(x)$ is defined by

$$\widehat{\underline{I}}_F(x) = \begin{cases} d_1^{s_1} d_2^{s_2} \cdots & \text{if } s_i \in \{L, R\} \text{ for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n} & \text{if } s_n \in \{M, C\} \text{ and } s_i \in \{L, R\} \\ & \text{for all } i \in \{1, \dots, n-1\}. \end{cases}$$

Since $F \in \mathcal{L}$ we have that $\widehat{\underline{I}}_F(x) = \widehat{\underline{I}}_F(x+k)$ for all $k \in \mathbb{Z}$.

Let $x, y \in \mathbb{R}$ be such that $D(x) \neq D(y)$. We say that x and y are *conjugate* if and only if $F(D(x)) = F(D(y))$. Note that if x and y are conjugate then they have the same reduced itinerary.

Let $\mathcal{S} = \{M, L, C, R\}$ and let

$$\underline{\alpha} = \alpha_0\alpha_1 \cdots$$

be a sequence of elements $\alpha_i = d_i^{s_i}$ of $\mathbb{Z} \times \mathcal{S}$. We say that $\underline{\alpha}$ is *admissible* if one of the following two conditions is satisfied:

1) $\underline{\alpha}$ is infinite, $s_i \in \{L, R\}$ for all $i \geq 1$ and there exists $k \in \mathbb{N}$ such that $|d_i| \leq k$ for all $i \geq 1$;

2) $\underline{\alpha}$ is finite of length n, $s_n \in \{M, C\}$ and $s_i \in \{L, R\}$ for all $i \in \{1, \ldots, n-1\}$.

Notice that any reduced itinerary is an admissible sequence. Now we shall introduce some notation for admissible sequences (and hence for reduced itineraries).

The cardinality of an admissible sequence $\underline{\alpha}$ will be denoted by $|\underline{\alpha}|$ (if $\underline{\alpha}$ is infinite we write $|\underline{\alpha}| = \infty$).

We denote by S the shift operator which acts on the set of admissible sequences of length greater than one as follows:

$$S(\underline{\alpha}) = \alpha_2\alpha_3 \cdots \quad \text{if} \quad \underline{\alpha} = \alpha_1\alpha_2\alpha_3 \cdots .$$

We will write S^k for the k-th iterate of S. Obviously S^k is only defined for admissible sequences of length greater than k. Clearly, for each $x \in \mathbb{R}$ we have

$$S^n\big(\widehat{\underline{I}}_F(x)\big) = \widehat{\underline{I}}_F\big(F^n(x)\big) \quad \text{if} \quad \big|\widehat{\underline{I}}_F(x)\big| > n.$$

Let $\underline{\alpha} = \alpha_1\alpha_2 \cdots \alpha_n$ and $\underline{\beta} = \beta_1\beta_2 \cdots$ be two sequences of symbols in $\mathbb{Z} \times \mathcal{S}$. We shall write $\underline{\alpha}\,\underline{\beta}$ to denote the concatenation of $\underline{\alpha}$ and $\underline{\beta}$ (*i.e.* the sequence $\alpha_1\alpha_2 \cdots \alpha_n\beta_1\beta_2 \cdots$). We also shall use the symbols

$$\underline{\alpha}^n = \overbrace{\underline{\alpha}\,\underline{\alpha} \cdots \underline{\alpha}}^{n \text{ times}} \quad \text{and} \quad \underline{\alpha}^\infty = \underline{\alpha}\,\underline{\alpha} \cdots .$$

Let $\underline{\alpha} = \alpha_1\alpha_2 \cdots \alpha_n$, be a sequence of symbols in $\mathbb{Z} \times \mathcal{S}$. Set $\alpha_i = d_i^{s_i}$ for $i = 1, 2, \cdots, n$. We say that $\underline{\alpha}$ is *even* if $\operatorname{Card}\{i \in \{1, \cdots, n\}\,;\; s_i = R\}$ is even; otherwise we say that $\underline{\alpha}$ is *odd*.

Now we endow the set of admissible sequences with a total ordering. First set

$$M < L < C < R.$$

Then we extend this ordering to $\mathbb{Z} \times S$ lexicographically. That is, we write $d^s < t^m$ if and only if either $d < t$ or $d = t$ and $s < m$. Let now $\underline{\alpha} = \alpha_1\alpha_2\cdots$ and $\underline{\beta} = \beta_1\beta_2\cdots$ be two admissible sequences such that $\underline{\alpha} \neq \underline{\beta}$. Then there exists n such that $\alpha_n \neq \beta_n$ and $\alpha_i = \beta_i$ for $i = 1, 2, \cdots, n-1$. We say that $\underline{\alpha} < \underline{\beta}$ if either $\alpha_1\alpha_2\cdots\alpha_{n-1}$ is even and $\alpha_n < \beta_n$ or $\alpha_1\alpha_2\cdots\alpha_{n-1}$ is odd and $\alpha_n > \beta_n$.

The following results show that the above ordering of reduced itineraries is, in fact, the ordering of points in $[0, c_F]$ (see [4], Lemma 3.2).

PROPOSITION 2.1. — *Let $F \in \mathcal{A}$. Then*

(a) *If $x, y \in [0, c_F]$, and $x < y$ then $\widehat{\underline{I}}_F(x) \leq \widehat{\underline{I}}_F(y)$.*

(b) *If $x, y \in [c_F, 1)$, and $x < y$ then $\widehat{\underline{I}}_F(x) \geq \widehat{\underline{I}}_F(y)$.*

COROLLARY 2.2. — *Let $F \in \mathcal{A}$. For all $x \in \mathbb{R}$ we have*

$$\widehat{\underline{I}}_F(0) \leq \widehat{\underline{I}}_F(x) \leq \widehat{\underline{I}}_F(c_F).$$

To define the kneading pair of a map $F \in \mathcal{A}$ we introduce the following notation. For a point $x \in \mathbb{R}$ we define the sequences $\widehat{\underline{I}}_F(x^+)$ and $\widehat{\underline{I}}_F(x^-)$ as follows. For each $n \geq 0$ there exists $\delta(n) > 0$ such that $d(F^{n-1}(y))$ and $s(F^n(y))$ take constant values for each $y \in (x, x + \delta(n))$ (resp. $y \in (x - \delta(n), x)$). Denote these values by $d(F^{n-1}(x^+))$ and $s(F^n(x^+))$ (resp. $d(F^{n-1}(x^-))$ and $s(F^n(x^-))$). Then we set

$$\widehat{\underline{I}}_F(x^+) = d(x^+)^{s(F(x^+))} d\big(F(x^+)\big)^{s(F^2(x^+))} \cdots,$$
$$\widehat{\underline{I}}_F(x^-) = d(x^-)^{s(F(x^-))} d\big(F(x^-)\big)^{s(F^2(x^-))} \cdots.$$

Clearly, $\widehat{\underline{I}}_F(x^+)$ and $\widehat{\underline{I}}_F(x^-)$ are infinite admissible sequences and

$$\widehat{\underline{I}}_F(x^+) = \widehat{\underline{I}}_F((x+k)^+), \quad \widehat{\underline{I}}_F(x^-) = \widehat{\underline{I}}_F((x+k)^-)$$

for all $k \in \mathbb{Z}$. Moreover, if $x \notin \mathbb{Z}$ and $|\widehat{\underline{I}}_F(x)| = \infty$ then

$$\widehat{\underline{I}}_F(x^-) = \widehat{\underline{I}}_F(x) = \widehat{\underline{I}}_F(x^+).$$

Let $F \in \mathcal{A}$. The pair

$$(\underline{\widehat{I}}_F(0^+), \underline{\widehat{I}}_F(c_F^-))$$

will be called *the kneading pair of* F and will be denoted by $\mathcal{K}(F)$. Let $\mathcal{AD}$ denote the set of all infinite admissible sequences. Then for each $F \in \mathcal{A}$ we have that $\mathcal{K}(F) \in \mathcal{AD} \times \mathcal{AD}$. As it has been said in the introduction the kneading pair of a map characterizes its dynamics completely (see Proposition 3.1).

To characterize the pairs in $\mathcal{AD} \times \mathcal{AD}$ that can occur as a kneading pair of a map from $\mathcal{A}$ we will define a subset $\mathcal{E}$ of $\mathcal{AD} \times \mathcal{AD}$ and, afterwards, we shall prove that this set consists of all kneading pairs of maps from $\mathcal{A}$. To this end we introduce the following notation.

Let $\underline{\alpha} = d_1^{s_1}\alpha_2 \cdots$, be an admissible sequence. We will denote

$$\underline{\alpha}' = (d_1 + 1)^{s_1}\alpha_2 \cdots .$$

Therefore, since for $F \in A$ we have $d(F(0^+)) = d(F(0^-)) - 1$ we can write

$$(\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(0^-).$$

We will denote by $\mathcal{E}^*$ the set of all pairs $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{AD} \times \mathcal{AD}$ such that the following conditions hold:

1) $\underline{\nu}_1' < \underline{\nu}_2$;

2) $\underline{\nu}_1 \leq S^n(\underline{\nu}_i) \leq \underline{\nu}_2$ for all $n > 0$ and $i \in \{1, 2\}$;

3) if for some $n \geq 0$, $S^n(\underline{\nu}_i) = d^R \cdots$, then $S^{n+1}(\underline{\nu}_i) \geq \underline{\nu}_1'$ for $i \in \{1, 2\}$.

We note that condition 2) says, in particular, that $\underline{\nu}_1$ and $\underline{\nu}_2$ are minimal and maximal, respectively, according to the following definition. Let $\underline{\alpha} \in \mathcal{AD}$, we say that $\underline{\alpha}$ is *minimal* (resp. *maximal*) if and only if $\underline{\alpha} \leq S^n(\underline{\alpha})$ (resp. $\underline{\alpha} \geq S^n(\underline{\alpha})$) for all $n \in \{1, 2, \ldots, |\underline{\alpha}| - 1\}$.

As we will see, the above set contains (among others) the kneading pairs of maps from $\mathcal{A}$ with non-degenerate rotation interval. To deal with some special kneading pairs associated to maps with degenerate rotation interval we introduce the following sets. For $a \in \mathbb{R}$ we set

$$\epsilon_i(a) = E(ia) - E\big((i-1)a\big),$$
$$\delta_i(a) = \widetilde{E}(ia) - \widetilde{E}\big((i-1)a\big),$$

where $\widetilde{E}:\mathbb{R}\to\mathbb{Z}$ is defined as follows:

$$\widetilde{E}(x)=\begin{cases}E(x) & \text{if } x\notin\mathbb{Z},\\ x-1 & \text{if } x\in\mathbb{Z}.\end{cases}$$

Also, we set

$$\widehat{\underline{I}}_\epsilon(a)=\epsilon_1(a)^L\epsilon_2(a)^L\cdots\epsilon_n(a)^L\cdots,$$

$$\widehat{\underline{I}}_\delta(a)=\delta_1(a)^L\delta_2(a)^L\cdots\delta_n(a)^L\cdots.$$

Let

$$\widehat{\underline{I}}^*_\epsilon(a)=\big(\widehat{\underline{I}}_\epsilon(a)\big)'$$

and let $\widehat{\underline{I}}^*_\delta(a)$ denote the sequence that satisfies

$$\big(\widehat{\underline{I}}^*_\delta(a)\big)'=\widehat{\underline{I}}_\delta(a).$$

When $a=p/q$ with $(p,q)=1$ we denote by $\widehat{\underline{I}}_R(a)$ the sequence

$$\big(\delta_1(a)^L\cdots\delta_{q-1}(a)^L\delta_q(a)^R\big)^\infty$$

and by $\widehat{\underline{I}}^*_R(a)$ the sequence which satisfies

$$(\widehat{\underline{I}}^*_R(a))'=\widehat{\underline{I}}_R(a).$$

To simplify the use of the sequences $\widehat{\underline{I}}_\epsilon(a),\widehat{\underline{I}}_\delta(a),\widehat{\underline{I}}^*_\epsilon(a)$ and $\widehat{\underline{I}}^*_\delta(a)$ the following lemma will be helpful (see [4], 4.1–4.3).

LEMMA 2.3. — *Let $a\in\mathbb{R}$. Then the following statements hold:*

(a) *If $a\notin\mathbb{Z}$ then $\delta_1(a)=\epsilon_1(a)+1$. Furthermore, if $a\notin\mathbb{Q}$ then $\delta_i(a)=\epsilon_i(a)$ for all $i>1$. That is, $\widehat{\underline{I}}^*_\delta(a)=\widehat{\underline{I}}_\epsilon(a)$ and $\widehat{\underline{I}}_\delta(a)=\widehat{\underline{I}}^*_\epsilon(a)$. If $a=p/q$ with $(p,q)=1$ and $q>1$ then $\epsilon_i(a)=\delta_i(a)$ for $i=2,\dots,q-1$, $\delta_q(a)=\epsilon_q(a)-1$ and, $\epsilon_{i+q}(a)=\epsilon_i(a)$ and $\delta_{i+q}(a)=\delta_i(a)$ for all $i\in\mathbb{N}$.*

(b) *If $a\in\mathbb{Z}$ then $\epsilon_i(a)=\delta_i(a)=a$ for all $i>0$.*

Now we set

$$\mathcal{E}_a=\begin{cases}\big\{\big(\widehat{\underline{I}}_\epsilon(a),\widehat{\underline{I}}^*_\epsilon(a)\big),\big(\widehat{\underline{I}}^*_\delta(a),\widehat{\underline{I}}_\delta(a)\big),\big(\widehat{\underline{I}}^*_R(a),\widehat{\underline{I}}_R(a)\big)\big\} & \\ & \text{if } a=p/q\in\mathbb{Q}, \text{ with } (p,q)=1,\\ \big\{\big(\widehat{\underline{I}}^*_\delta(a),\widehat{\underline{I}}_\delta(a)\big)\big\} & \text{if } a\notin\mathbb{Q}.\end{cases}$$

We note that for each $a\in\mathbb{R}$, $\mathcal{E}_a\cap\mathcal{E}^*=\emptyset$ because all elements from $\mathcal{E}_a$ do not satisfy condition 1) of the definition of $\mathcal{E}^*$.

Finally we denote by $\mathcal{E}$ the set $\mathcal{E}^*\cup\big(\bigcup_{a\in\mathbb{R}}\mathcal{E}_a\big)$.

The following result characterizes the kneading pairs of the maps from class $\mathcal{A}$ and is the main result of this paper.

THEOREM A. — *For $F \in \mathcal{A}$ we have that $\mathcal{K}(F) \in \mathcal{E}$. Conversely, for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}$ there exists $F \in \mathcal{A}$ such that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$.*

Remark 2.1. — From the proof the first statement of Theorem A (see Proposition 4.1) one sees that if $\mathcal{K}(F) \in \mathcal{E}_a$ then the rotation interval of F is $\{a\}$. However, we note that there are maps in $\mathcal{A}$ with degenerate rotation interval whose kneading pair is contained in $\mathcal{E}^*$. Indeed, it is easy to check that the pair

$$((0^L 0^L 1^L 0^L 1^L 1^L)^\infty, (1^L 1^L 0^L 0^L)^\infty)$$

belongs to $\mathcal{E}^*$. Hence, from Theorem A there is a map $F \in \mathcal{A}$ having it as its kneading pair. Moreover, from Theorem 3.6 it follows that the rotation interval of F is $\{\frac{1}{2}\}$.

On the other hand, since $\mathcal{E} \setminus \mathcal{E}^*$ is the boundary of $\mathcal{E}^*$ one would expect that if $\mathcal{K}(F) \notin \mathcal{E}^*$ for some $F \in \mathcal{A}$, then the topological entropy of F is zero. This can be proved in a straightforward way by using the techniques from [1] (see also Proposition 4.3.3 of [10]). However, there are also maps $F \in \mathcal{A}$ such that $\mathcal{K}(F) \in \mathcal{E}^*$ and the topological entropy of F is zero, as the following example shows. Let F be the map shown in Figure 2.

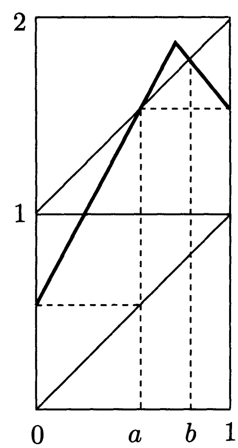

Figure 2. A map $F \in \mathcal{A}$ such that $\mathcal{K}(F) \in \mathcal{E}^*$ and has topological entropy zero.

Then, clearly, $\widehat{\underline{I}}_F(c_F^-) = (1^R)^\infty$ and $\widehat{\underline{I}}_F(0^+) = 0^L(1^L)^\infty$. Therefore,

$$\big(\widehat{\underline{I}}_F(0^+)\big)' = (1^L)^\infty < (1^R)^\infty = \widehat{\underline{I}}_F(c_F^-)$$

and so $\mathcal{K}(F) \in \mathcal{E}^*$. On the other hand, the non-wandering set of the circle map which has F as a lifting is just two fixed points: $\exp(2\pi i a)$ and $\exp(2\pi i b)$. Therefore, the topological entropy of F is zero (see for instance [9] or [3]).

3. DEFINITIONS AND PRELIMINARY RESULTS

In the first subsection we study the symbolic properties of the kneading pair. In Subsection 3.2 we give some standard definitions and preliminary results about circle maps that we will use in the proof of Theorem A.

3.1. Some properties of the kneading pair.

In Section 2, to each map $F \in \mathcal{A}$, we assigned a pair from $\mathcal{AD} \times \mathcal{AD}$ called the kneading pair. The following definitions were introduced in [4].

Let $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ be admissible sequences such that $\underline{\beta} < \underline{\gamma}$. We will say that $\underline{\alpha}$ is *quasidominated* by $\underline{\beta}$ and $\underline{\gamma}$ if and only if the following statements hold:

1) $\underline{\beta} \leq S^n(\underline{\alpha}) \leq \underline{\gamma}$ for all $n \in \{0, 1, \dots, |\underline{\alpha}| - 1\}$,

2) if for some $n \in \{0, 1, \dots, |\underline{\alpha}| - 1\}$ we have $S^n(\underline{\alpha}) = d^R \cdots$ then $S^{n+1}(\underline{\alpha}) \geq \underline{\beta}'$.

We will say that $\underline{\alpha}$ is *dominated* by $\underline{\beta}$ and $\underline{\gamma}$ if and only if 1) and 2) hold with strict inequalities.

Let $F \in \mathcal{A}$. We say that $\underline{\alpha}$ is *quasidominated* (resp. *dominated*) by F if $\underline{\alpha}$ is quasidominated (resp. dominated) by $\widehat{\underline{I}}_F(0^+)$ and $\widehat{\underline{I}}_F(c_F^-)$.

The next result, due to Alsedà and Mañosas (see [4], Proposition A), characterizes the set of reduced itineraries (and hence the dynamics) of a map $F \in \mathcal{A}$ in terms of the kneading pair.

PROPOSITION 3.1. — *Let $F \in \mathcal{A}$. Then the following hold:*

(a) *Let $x \in (0,1)$ with $x \neq c_F$. Then $\widehat{\underline{I}}_F(x)$ is quasidominated by F.*

(b) *Let $\underline{\alpha}$ be an admissible sequence dominated by F. Then there exists $x \in [0, c_F]$ such that $\widehat{\underline{I}}_F(x) = \underline{\alpha}$.*

The following result, which is an immediately corollary of the above proposition, will be used in the study of the kneading pair.

COROLLARY 3.2. — *Let $F \in \mathcal{A}$. Then the following hold:*

(a) *Let $x \in (0, c_F)$. Then $\widehat{\underline{I}}_F(0^+) \le \widehat{\underline{I}}_F(x) \le \widehat{\underline{I}}_F(c_F^-)$.*

(b) *Let $x \in (c_F, 1)$. Then $\widehat{\underline{I}}_F(0^-) \le \widehat{\underline{I}}_F(x) \le \widehat{\underline{I}}_F(c_F^-)$.*

The following proposition gives the main symbolic properties of the kneading pair.

PROPOSITION 3.3. — *For each $F \in \mathcal{A}$ we have that*

$$\big(\widehat{\underline{I}}_F(0^+)\big)' \le \widehat{\underline{I}}_F(c_F^-)$$

and $\widehat{\underline{I}}_F(0^+)$ and $\widehat{\underline{I}}_F(c_F^-)$ are quasidominated by F.

Before proving Proposition 3.3 we introduce some more notation.

For $k \in \mathbb{Z}$ we denote by $(\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$ the set of sequences $\underline{\alpha} = d_1^{s_1} d_2^{s_2} \cdots \in (\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}$ such that $|d_i| \le k$ for all $i \ge 1$ (recall that $\mathcal{S}$ denotes the set $\{M, L, C, R\}$). Let $\underline{\alpha} = d_1^{s_1} d_2^{s_2} \cdots$ and $\underline{\beta} = t_1^{r_1} t_2^{r_2} \cdots$ be two sequences in $(\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$. We consider in $(\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}$ the topology defined by the metric

$$d(\underline{\alpha}, \underline{\beta}) = \sum_{i=0}^{\infty} 2^{-i} d(d_i^{s_i} t_i^{r_i})$$

where

$$d(d_i^{s_i} t_i^{r_i}) = \begin{cases} 1 & \text{if } d_i^{s_i} \ne t_i^{r_i}, \\ 0 & \text{if } d_i^{s_i} = t_i^{r_i}. \end{cases}$$

With this topology, $(\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$ is a compact metric space and the shift transformation $S : (\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k \longrightarrow (\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$ defined by

$$S(d_1^{s_1} d_2^{s_2} \cdots) = d_2^{s_2} d_3^{s_3} \cdots$$

is continuous. Moreover, we can extend in a natural way the ordering defined for the admissible sequences to the sequences from $(\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$.

Let $\underline{\alpha}$, $\underline{\beta}$ be two admissible sequences such that $\underline{\alpha}' \le \underline{\beta}$ and let $\mathcal{AD}_{\underline{\alpha},\underline{\beta}}$ denote the set of all admissible sequences quasidominated by $\underline{\alpha}$ and $\underline{\beta}$ union $\{\underline{\alpha}, \underline{\beta}, \underline{\alpha}'\}$. Now, we define $\Gamma_{\underline{\alpha},\underline{\beta}} : \mathcal{AD}_{\underline{\alpha},\underline{\beta}} \to (\mathbb{Z} \times \mathcal{S})^{\mathbb{N}}_k$ as follows:

- If $|\underline{\gamma}| = \infty$ then $\Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \underline{\gamma}$.
- If $\underline{\gamma}$ is finite and ends with C, then we define

$$\Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \begin{cases} \underline{\gamma}\,\underline{\beta} & \text{if } \underline{\beta} \text{ is infinite,} \\ \underline{\gamma}(\underline{\beta})^{\infty} & \text{if } \underline{\beta} \text{ is finite and ends with } C, \\ \underline{\gamma}\,\underline{\beta}\,\underline{\alpha} & \text{if } \underline{\beta} \text{ is finite and ends with } M \text{ and } \underline{\alpha} \text{ is infinite,} \\ \underline{\gamma}\,\underline{\beta}(\underline{\alpha})^{\infty} & \text{if } \underline{\beta} \text{ is finite and ends with } M \text{ and } \underline{\alpha} \text{ is finite} \\ & \text{and ends with } M, \\ \underline{\gamma}(\underline{\beta}\,\underline{\alpha})^{\infty} & \text{otherwise.} \end{cases}$$

• If $\underline{\gamma}$ ends with M then we proceed similarly with the roles of $\underline{\alpha}$ and $\underline{\beta}$, and M and C interchanged.

We note that the map $\Gamma_{\underline{\alpha},\underline{\beta}}$ preserves the ordering of the sequences and that $S^n \circ \Gamma_{\underline{\alpha},\underline{\beta}}(\underline{\gamma}) = \Gamma_{\underline{\alpha},\underline{\beta}} \circ S^n(\underline{\gamma})$ for all $n \in \{0, 1, \cdots |\underline{\gamma}| - 1\}$.

Proof of Proposition 3.3. — The first statement follows from Corollary 3.2 (b) and the fact that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(0^-)$. Now, we prove the second statement. Denote by Γ_F the map

$$\Gamma_{\widehat{\underline{I}}_F(0^+),\widehat{\underline{I}}_F(c_F^-)}.$$

From the part of the proposition already proved it is well defined. It is not difficult to show that

$$\widehat{\underline{I}}_F(x^+) = \lim_{\substack{y \to x \\ y > x}} \Gamma_F\big(\widehat{\underline{I}}_F(y)\big),$$

$$\widehat{\underline{I}}_F(x^-) = \lim_{\substack{y \to x \\ y < x}} \Gamma_F\big(\widehat{\underline{I}}_F(y)\big).$$

Now, we consider several cases. Assume first that

$$S^n\big(\widehat{\underline{I}}_F(0^+)\big) = d^L \cdots \quad (\text{resp. } S^n(\widehat{\underline{I}}_F(c_F^-)) = d^L \cdots)$$

for some $n \geq 0$. Then there exist points $x \in (0, c_F)$ arbitrarily close to 0 (resp. c_F) such that $D(F^{n+1}(x)) \in (0, c_F)$ and $\widehat{\underline{I}}_F(x)$ coincides with $\widehat{\underline{I}}_F(0)$ (resp. $\widehat{\underline{I}}_F(c_F)$) in the first $n+1$ symbols. Then from Corollary 3.2 (a) we have that $\widehat{\underline{I}}_F(0^+) \leq \widehat{\underline{I}}_F(F^{n+1}(x)) \leq \widehat{\underline{I}}_F(c_F^-)$. Thus

$$\widehat{\underline{I}}_F(0^+) \leq \Gamma_F\big(\widehat{\underline{I}}_F(F^{n+1}(x))\big) \leq \widehat{\underline{I}}_F(c_F^-).$$

Since

$$\Gamma_F\big(\widehat{\underline{I}}_F(F^{n+1}(x))\big) = \Gamma_F\big(S^{n+1}(\widehat{\underline{I}}_F(x))\big) = S^{n+1}\big(\Gamma_F(\widehat{\underline{I}}_F(x))\big),$$

letting x tend to 0 from the right we get

$$\widehat{\underline{I}}_F(0^+) \leq S^n(\widehat{\underline{I}}_F(0^+)) \leq \widehat{\underline{I}}_F(c_F^-)$$

(resp. letting x tend to c_F from the left we get

$$\widehat{\underline{I}}_F(0^+) \leq S^n\big(\widehat{\underline{I}}_F(c_F^-)\big) \leq \widehat{\underline{I}}_F(c_F^-)).$$

Now, assume that

$$S^n(\widehat{\underline{I}}_F(0^+)) = d^R \cdots \quad \big(\text{resp. } S^n(\widehat{\underline{I}}_F(c_F^-)) = d^R \cdots\big)$$

for some $n \geq 0$. There exist points $x \in (0, c_F)$ arbitrarily close to 0 (resp. c_F) such that $D(F^{n+1}(x)) \in (c_F, 1)$ and $\widehat{\underline{I}}_F(x)$ coincides with $\widehat{\underline{I}}_F(0)$ (resp. $\widehat{\underline{I}}_F(c_F)$) in the first $n+1$ symbols.

From Corollary 3.2 (b) we have that

$$\widehat{\underline{I}}_F(0^-) \le \widehat{\underline{I}}_F(F^{n+1}(x)) \le \widehat{\underline{I}}_F(c_F^-).$$

Then, in a similar way as above we can show that

$$\widehat{\underline{I}}_F(0^-) \le S^{n+1}(\widehat{\underline{I}}_F(0^+)) \le \widehat{\underline{I}}_F(c_F^-)$$

$$\big(\text{resp. } \widehat{\underline{I}}_F(0^-) \le S^{n+1}(\widehat{\underline{I}}_F(c_F^-)) \le \widehat{\underline{I}}_F(c_F^-)\big)$$

and the proposition follows. □

3.2. The rotation interval, twist periodic orbits and the kneading pair.

We advise to the reader that most of the results we are quoting from other authors will be written in terms of class $\mathcal{L}$ unlike the original versions are stated for circle maps of degree one.

We shall say that a point $x \in \mathbb{R}$ is *periodic* (mod. 1) *of period q with rotation number p/q* for a map $F \in \mathcal{L}$ if

$$F^q(x) - x = p \quad \text{and} \quad F^i(x) - x \notin \mathbb{Z} \quad \text{for} \quad i = 1, \ldots, q-1.$$

A periodic (mod. 1) point of period 1 will be called *fixed* (mod. 1).

The notion of rotation number was introduced by Poincaré [19] for homeomorphisms of the circle of degree one. This notion will be used to characterize the set of periods of circle maps of degree one. It is well-known (see [17]) that if $F \in \mathcal{L}$ is a non-decreasing map, then

$$\lim_{n\to\infty} \frac{1}{n}\big(F^n(x) - x\big)$$

exists and it is independent of x. From above it follows that to every non-decreasing map $F \in \mathcal{L}$ we can associate a real number

$$\rho(F) = \lim_{n\to\infty} \frac{1}{n}\big(F^n(x) - x\big),$$

which is called *the rotation number of F*. Roughly speaking, $\rho(F)$ is the average angular speed of any point moving around the circle under iteration of the map. We note that $\rho(F)$ is a topological invariant of F. That is, if F and G are topologically conjugate (*i.e.* there exists an increasing map $h \in \mathcal{L}$ such that $F \circ h = h \circ G$) then $\rho(F) = \rho(G)$. Poincaré also proved that F has a periodic orbit if and only if $\rho(F) \in \mathbb{Q}$.

We remark that $\lim_{n\to\infty} \frac{1}{n}(F^n(x) - x)$ may not exist for a general map $F \in \mathcal{L}$ and if it exists it may depend on the choice of the point x. This motivates the following extension of this notion due to Newhouse, Palis and Takens (see [18]) to each map $F \in \mathcal{L}$. For $F \in \mathcal{L}$ and $x \in \mathbb{R}$ we set

$$\rho_F(x) = \rho(x) = \limsup_{n\to\infty} \frac{1}{n}\big(F^n(x) - x\big).$$

We denote by R_F the set of all rotation numbers of F. Ito (see [12]) proved that the set R_F is a closed interval, perhaps degenerate to a single point.

When looking at periodic points of circle maps sometimes it is useful to look at the set of all iterates of the point under consideration. In our framework this means that we have to look at the set of all points projecting on the iterates of the periodic point under consideration. This motivates the following definition.

Let $F \in \mathcal{L}$ and let $x \in \mathbb{R}$. Then the set

$$\{y \in \mathbb{R}\,;\ y = F^n(x) \text{ (mod. 1) for } n = 0, 1, \ldots\}$$

will be called the (mod. 1) *orbit* of x by F. We stress the fact that if P is a (mod. 1) orbit and $x \in P$, then $x + k \in P$ for all $k \in \mathbb{Z}$.

It is not difficult to prove that each point from an orbit (mod. 1) P has the same rotation number. Thus, we can speak about the *rotation number* of P.

If $A \subset \mathbb{R}$ and $x \in \mathbb{R}$, we shall write $x + A$ or $A + x$ to denote the set $\{x + a\,;\ a \in A\}$ and we shall write $A + B$ to denote the set $\{a + b\,;\ a \in A,\ b \in B\}$ if $B \subset \mathbb{R}$.

If x is a periodic (mod. 1) point of F of period q with rotation number p/q then its (mod. 1) orbit is called a *periodic* (mod. 1) *orbit of F of period q with rotation number p/q*. If P is a (mod. 1) orbit of F we denote by P_i the set $P \cap [i, i+1)$ for all $i \in \mathbb{Z}$. Obviously $P_i = i + P_0$. We note that if P is a periodic (mod. 1) orbit of F with period q, then $\mathrm{Card}(P_i) = q$ for all $i \in \mathbb{Z}$.

Let P be a (mod. 1) orbit of a map $F \in \mathcal{L}$. We say that P is a *twist orbit* if F restricted to P is increasing. If a periodic (mod. 1) orbit is twist then we say that P is a *twist periodic orbit* (from now on *TPO*).

The following result gives a geometrical interpretation of a TPO.

LEMMA 3.4. — *Let $P = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ be a TPO with period q and rotation number p/q and assume that $x_i < x_j$ if and only if $i < j$. Then $(p, q) = 1$ and $F(x_i) = x_{i+p}$.*

Remark 3.1. — We note that if P is a twist orbit then the rotation number of P can be computed by $\lim_{n\to\infty} \frac{1}{n}(F^n(x) - x)$ for each $x \in P$.

The following result, given in [8], studies the relation between the rotation number and twist orbits.

LEMMA 3.5. — *Let $F \in \mathcal{L}$. For each $a \in R_F$ there exists a twist orbit P of F with rotation number a. Moreover P is contained in a union of closed intervals on which F is increasing.*

Now, we will study the basic properties of the sequences $\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\delta(a), \widehat{\underline{I}}^*_\epsilon(a)$ and $\widehat{\underline{I}}^*_\delta(a)$. These sequences give the characterization of the rotation interval by means of the kneading pair. The following result is due to Alsedà and Mañosas (see [4], Theorem B).

THEOREM 3.6. — *Let $F \in \mathcal{A}$. Then $R_F = [a, b]$ if and only if*

$$\widehat{\underline{I}}^*_\delta(a) \leq \widehat{\underline{I}}_F(0^+) \leq \widehat{\underline{I}}_\epsilon(a) \quad \textit{and} \quad \widehat{\underline{I}}_\delta(b) \leq \widehat{\underline{I}}_F(c_F^-) \leq \widehat{\underline{I}}^*_\epsilon(b).$$

Remark 3.2. — Since $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(0^-)$, by the definition of the sequences $\widehat{\underline{I}}^*_\delta(a)$, $\widehat{\underline{I}}_\epsilon(a)$, $\widehat{\underline{I}}_\delta(a)$ and $\widehat{\underline{I}}^*_\epsilon(a)$, we have that

$$\widehat{\underline{I}}^*_\delta(a) \leq \widehat{\underline{I}}_F(0^+) \leq \widehat{\underline{I}}_\epsilon(a)$$

is equivalent to

$$\widehat{\underline{I}}_\delta(a) \leq \widehat{\underline{I}}_F(0^-) \leq \widehat{\underline{I}}^*_\epsilon(a).$$

We end this section by stating three results which give some properties of the maps $F \in \mathcal{A}$ such that the kneading pair satisfies that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. In view of Theorem 3.6 and Remark 3.2 we get:

LEMMA 3.7. — *Let $F \in \mathcal{A}$ be such that $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$. Then R_F is degenerate to a point.*

Lastly, from Lemma 4.4 of [4], the proof of Theorem 2 of [8] and Lemma 3.5 we have the following result. For a map $F \in \mathcal{L}$ we will denote by F_u the map defined as follows:

$$F_u(x) = \sup\{F(y)\,;\; y \leq x\}.$$

It is known that F_u is a non–decreasing map from $\mathcal{L}$.

PROPOSITION 3.8. — *Let $F \in \mathcal{A}$ be such that $(\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$ and $R_F = \{a\}$ with $a \in \mathbb{R}$. Then the map F has a twist orbit P of rotation number a such that $P \cap [0, 1) \subset [0, c_F]$ and $F|_P = F_u|_P$. Moreover, if $a = p/q \in \mathbb{Q}$ with $(p, q) = 1$, then P is a twist periodic orbit of period q. Set $\mu_P = \min P \cap [0, c_F]$ and $\nu_P = \max P \cap [0, c_F]$. Then the following statements hold:*

(a) $\{0, c_F\} \not\subset \{\mu_P, \nu_P\}$.

(b) *Assume that $\nu_P \neq c_F$. If $\mu_P \neq 0$ then $\underline{\widehat{I}}_F(\mu_P) = \underline{\widehat{I}}_\epsilon(a)$. Otherwise*

$$\underline{\widehat{I}}_F(0) = \epsilon_1(a)^L \cdots \epsilon_{q-1}(a)^L \epsilon_q(a)^M \quad \text{and} \quad \underline{\widehat{I}}_F(0^+) = \underline{\widehat{I}}_\epsilon(a).$$

(c) *Assume that $\mu_P \neq 0$. If $\nu_P \neq c_F$ then $\underline{\widehat{I}}_F(\nu_P) = \underline{\widehat{I}}_\delta(a)$. Otherwise*

$$\underline{\widehat{I}}_F(c_F) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^C \quad \text{and} \quad \underline{\widehat{I}}_F(c_F^-) = \underline{\widehat{I}}_\delta(a).$$

4. PROOF OF THEOREM A.

We split the proof of Theorem A into two parts. In the first subsection we will prove the first statement and the second one in Subsection 4.2.

4.1. Proof of the first statement of Theorem A.

We start by noting that if for $F \in \mathcal{A}$ we have $(\underline{\widehat{I}}_F(0^+))' < \underline{\widehat{I}}_F(c_F^-)$ then, in view of Proposition 3.3 and the definition of $\mathcal{E}$, $\mathcal{K}(F) \in \mathcal{E}^* \subset \mathcal{E}$. Thus, to prove the first statement of Theorem A, we only have to prove that if $(\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$, then $\mathcal{K}(F) \in \mathcal{E}_a$ for some $a \in \mathbb{R}$. This follows from the following result.

PROPOSITION 4.1. — *Let $F \in \mathcal{A}$ be such that $(\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$. Then there exists $a \in \mathbb{R}$ such that $R_F = \{a\}$ and $\mathcal{K}(F) \in \mathcal{E}_a$.*

Proof. — From Lemma 3.7 we have that $R_F = \{a\}$. Assume that $a \notin \mathbb{Q}$. From Lemma 2.3 (a) and Theorem 3.6 we see that $\mathcal{K}(F) \in \mathcal{E}_a$. Now, assume that $a = p/q$ with $(p, q) = 1$. Let P be the twist periodic orbit of period q and rotation number p/q given by Proposition 3.8. If $\mu_P = 0$, from Proposition 3.8 (a), we have $\nu_P \neq c_F$ (here we use the notation from the statement of Proposition 3.8). Therefore, from Proposition 3.8 (b), $\underline{\widehat{I}}_F(0^+) = \underline{\widehat{I}}_\epsilon(a)$. Hence, $\underline{\widehat{I}}_F(c_F^-) = (\underline{\widehat{I}}_\epsilon(a))' = \underline{\widehat{I}}_\epsilon^*(a)$ and so, $\mathcal{K}(F) \in \mathcal{E}_a$. If $\nu_P = c_F$ then, as above, $\mu_P \neq 0$. By Proposition 3.8 (c), $\underline{\widehat{I}}_F(c_F^-) = \underline{\widehat{I}}_\delta(a)$ and, consequently, $\underline{\widehat{I}}_F(0^+) = \underline{\widehat{I}}_\delta^*(a)$. So, $\mathcal{K}(F)$ also belongs to $\mathcal{E}_a$. We are left with the case $\mu_P \neq 0$ and $\nu_P \neq c_F$.

From the definition of P and F_u we see that for all $y \in P$ and $z \leq y$ we have

$$F(z) \leq F_u(z) \leq F_u(y) = F(y).$$

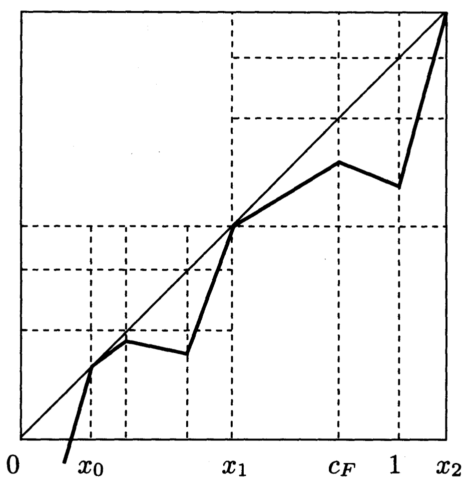

Figure 3. The map $G = F^q - p$ in case $q = 2$, $p = 1$.

Let $G = F^q - p$ (see Figure 3 for an example). Then $G(z) \leq G(y) = y$ for all $y \in P$ and $z \leq y$. Set $P = \{x_i\}_{i \in \mathbb{Z}}$ with $x_i < x_j$ if and only if $i < j$, and $x_0 = \mu_P$. Then, since P has period q we have $x_{q-1} = \nu_P$ and $x_{i+q} = x_i + 1$ for each $i \in \mathbb{Z}$. From Lemma 3.4 we get $F(x_i) = x_{i+p}$ for each $i \in \mathbb{Z}$. Thus, since $P \cap [0,1] \subset [0, c_F)$, each interval $[x_i, x_{i+1}]$ is mapped homeomorphically (preserving ordering) into $[x_{i+p}, x_{i+1+p}]$ for $i = 0, 1, \ldots, q-2$. On the other hand $[x_{q-1}, x_q]$ contains $\{c_F, 1\}$ in its interior (recall that $x_{q-1} = \nu_P < c_F$ and $x_q = \mu_P + 1 > 1$). Since $F|_{[x_{q-1}, c_F]}$ is increasing and $c_F < x_q$ we obtain

$$x_{q-1+p} = F(x_{q-1}) \leq F(z) \leq F(c_F) \leq F(x_q) = x_{q+p}$$

for each $z \in [x_{q-1}, c_F]$. Since $(p, q) = 1$, for each $i \in \{1, 2, \ldots, q-1\}$, we have $ip \not\equiv 0$ (mod. q). Therefore, $q - 1 + ip \neq q - 1 + mq$ with $m \in \mathbb{Z}$ and so, $x_{q-1+ip} \neq x_{q-1} + m$. Consequently, $F|_{[x_{q-1+ip}, x_{q+ip}]}$ is strictly increasing for $i = 1, 2, \ldots, q-1$. Therefore, for each $z \in [x_{q-1}, c_F]$,

$$G(z) \in [x_{q-1+qp} - p, G(c_F)] = [x_{q-1}, G(c_F)] \subset [x_{q-1}, x_q].$$

Moreover, $G|_{[x_{q-1}, c_F]}$ is strictly increasing. By Proposition 3.8 (c) we see that

$$\widehat{\underline{I}}_F(\nu_P) = \widehat{\underline{I}}_F(x_{q-1}) = \widehat{\underline{I}}_\delta(a) = \left(\delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^L\right)^\infty.$$

So, from above it follows that, for each $z \in [x_{q-1}, c_F]$,

$$\underline{\widehat{I}}_F(z) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L {d_q}^{s(G(z))} \underline{\widehat{I}}_F(G(z))$$

where

$$d_q = \begin{cases} \delta_q(a) & \text{if } G(z) < 1, \\ \delta_q(a) + 1 & \text{otherwise,} \end{cases}$$

(recall that $\underline{\widehat{I}}_F(x) = \underline{\widehat{I}}_F(x+m)$ for each $m \in \mathbb{Z}$).

Now we consider three cases.

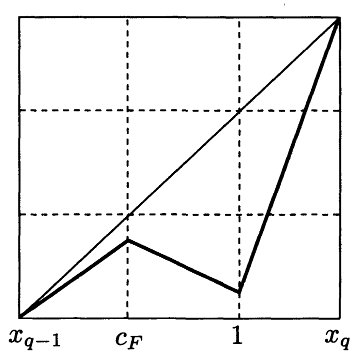

Figure 4. The map $G|_{[x_{q-1}, x_q]}$ in Case 1.

- Case 1: $G(c_F) \in [x_{q-1}, c_F]$ (see Figures 4 and 3).

Then $G([x_{q-1}, c_F]) \subset [x_{q-1}, c_F]$ and, if we take $z < c_F$ close enough to c_F, we have

$$\begin{aligned} \underline{\widehat{I}}_F(c_F^-) &= \underline{\widehat{I}}_F(z) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^L \underline{\widehat{I}}_F(G(z)) \\ &= (\delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^L)^2 \underline{\widehat{I}}_F(G^2(z)) \\ &= \cdots \\ &= \underline{\widehat{I}}_\delta(a). \end{aligned}$$

- Case 2: $G(c_F) \in (c_F, 1]$ (see Figure 5 next page).

We claim that $G(1) \in (c_F, G(c_F))$. To prove the claim we start showing that $G(1) > c_F$. Otherwise, either $G(1) \in [0, c_F]$ or $G(1) < 0$. In

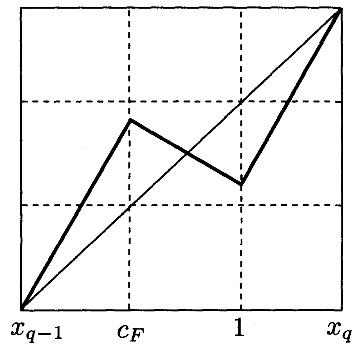

Figure 5. The map $G|_{[x_{q-1},x_q]}$ in Case 2.

the first case $\underline{\widehat{I}}_F(1^-)$ is of the form $d_1^{s_1}d_2^{s_2}\cdots d_q^L\cdots$ while $s(G(c_F)^-) = R$. This contradicts the fact that

$$\underline{\widehat{I}}_F(1^-) = \big(\underline{\widehat{I}}_F(0^+)\big)' = \underline{\widehat{I}}_F(c_F^-).$$

In the second case, take $x < 1$ close enough to 1 so that $\underline{\widehat{I}}_F(x)$ and $\underline{\widehat{I}}_F(1^-)$ coincide in the first q symbols and $G(x) < 0$. From above it follows that either $\underline{\widehat{I}}_F(c_F)$ and $\underline{\widehat{I}}_F(c_F^-)$ coincide in the first q symbols when $G(c_F) < 1$ or $\underline{\widehat{I}}_F(c_F)$ and $\underline{\widehat{I}}_F(c_F^-)$ coincide in the first $(q-1)$ symbols and $d(F^q(c_F)) = d(F^q(c_F^-)) + 1$ when $G(c_F) = 1$. Set

$$s_G = \begin{cases} 1 & \text{if } G(c_F) = 1, \\ 0 & \text{if } G(c_F) < 1. \end{cases}$$

Hence, since $\underline{\widehat{I}}_F(1^-) = (\underline{\widehat{I}}_F(0^+))' = \underline{\widehat{I}}_F(c_F^-)$ we have that

$$\begin{aligned} 0 > G(x) \geq E\big(G(x)\big) &= E\big(F^q(x)\big) - p \\ &= \Big(\sum_{i=1}^{q} E\big(F^i(x)\big) - E\big(F^{i-1}(x)\big)\Big) - p \\ &= \Big(\sum_{i=1}^{q} d\big(F^{i-1}(x)\big)\Big) - p \\ &= \Big(\sum_{i=1}^{q} d\big(F^{i-1}(c_F)\big)\Big) - p - s_G \\ &= E\big(G(c_F)\big) - s_G = 0; \end{aligned}$$

a contradiction.

In short, we have proved that $G(1) > c_F$. Now we prove that $G(1) < G(c_F)$. Note that if $F(1) \leq F(x_{q-1})$ then

$$G(1) \leq G(x_{q-1}) = x_{q-1} < c_F.$$

Hence $F(1) > F(x_{q-1})$. So, there exists $z_1 \in [x_{q-1}, c_F)$ such that $F(z_1) = F(1)$. Since $c_F < x_q$ we have $F(1) = F(z_1) \leq F(c_F) < F(x_q)$. Thus, from above it follows that $G(1) < G(c_F)$. This ends the proof of the claim.

From the claim and its proof it follows that $G_{|[c_F,1]}$ is decreasing and $G([c_F, 1]) \subset (c_F, 1]$. We note that from all said above, for each $x \in [c_F, 1]$, there exists $x^* \in [x_{q-1}, c_F]$ such that $G(x^*) = G(x)$. So,

$$\widehat{\underline{I}}_F(x) = \widehat{\underline{I}}_F(x^*) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^R \widehat{\underline{I}}_F\big(G(x)\big).$$

Now take $z < c_F$ close enough to c_F. Since $G^i(z) \in (c_F, 1)$ for each $i \geq 1$, we have

$$\begin{aligned}\widehat{\underline{I}}_F(c_F^-) &= \widehat{\underline{I}}_F(z) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^R \widehat{\underline{I}}_F\big(G(z)\big) \\ &= (\delta_1(a)^L \cdots \delta_{q-1}(a)^L \delta_q(a)^R)^2 \widehat{\underline{I}}_F\big(G^2(z)\big) \\ &= \cdots \\ &= \widehat{\underline{I}}_R(a).\end{aligned}$$

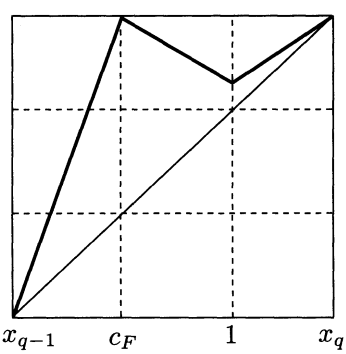

Figure 6. The map $G_{|[x_{q-1},x_q]}$ in Case 3.

- Case 3: $G(c_F) \in (1, x_q]$ (see Figure 6).

In a similar way as in Case 2 we get that $G(1) \in [1, G(c_F))$. Therefore, $G([1, x_q]) \subset [1, x_q]$. As in Case 2, for each $x \in [1, G(c_F)]$ there exists

$x^* \in [x_{q-1}, c_F]$ such that $G(x^*) = G(x)$ and so,

$$\big(\widehat{\underline{I}}_F(x)\big)' = \widehat{\underline{I}}_F(x^*) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \big(\delta_q(a)+1\big)^L \widehat{\underline{I}}_F\big(G(x)\big).$$

As in the previous two cases, for $z < c_F$ close enough to c_F we have

$$\begin{aligned}
\widehat{\underline{I}}_F(c_F^-) &= \widehat{\underline{I}}_F(z) = \delta_1(a)^L \cdots \delta_{q-1}(a)^L \big(\delta_q(a)+1\big)^L \widehat{\underline{I}}_F\big(G(z)\big) \\
&= \delta_1(a)^L \cdots \delta_{q-1}(a)^L (\delta_q(a)+1)^L \\
&\qquad \big(\delta_1(a)-1\big)^L \cdots \delta_{q-1}(a)^L \big(\delta_q(a)+1\big)^L \widehat{\underline{I}}_F\big(G^2(z)\big) \\
&= \cdots \\
&= \delta_1(a)^L \cdots \delta_{q-1}(a)^L \big(\delta_q(a)+1\big)^L \\
&\qquad \big(\big(\delta_1(a)-1\big)^L \cdots \delta_{q-1}(a)^L \big(\delta_q(a)+1\big)^L\big)^\infty
\end{aligned}$$

and from Lemma 2.3 (a) we get that $\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_\epsilon^*(a)$. This ends the proof of the proposition. □

Proof of the first statement of Theorem A. — Let $F \in \mathcal{A}$. If $(\widehat{\underline{I}}_F(0^+))' < \widehat{\underline{I}}_F(c_F^-)$ then, as it has been said before, $\mathcal{K}(F) \in \mathcal{E}^* \subset \mathcal{E}$ by Proposition 3.3. Otherwise, $(\widehat{\underline{I}}_F(0^+))' = \widehat{\underline{I}}_F(c_F^-)$ and, by Proposition 4.1, $\mathcal{K}(F) \in \mathcal{E}_a$ for some $a \in \mathbb{R}$. □

4.2. Proof of the second statement of Theorem A.

The next theorem already proves the second statement of Theorem A in the case $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$.

THEOREM 4.2. — *Let $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$. Then there exists $F \in \mathcal{A}$ such that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$.*

Proof. — Set $\underline{\nu}_i = d_{i,1}^{s_{i,1}} d_{i,2}^{s_{i,2}} \cdots d_{i,k}^{s_{i,k}} \cdots$ for $i = 1, 2$. Since $\underline{\nu}_1$ and $\underline{\nu}_2$ are admissible there exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \le d_{i,j} \le k_2$ for all $j \ge 1$ and $i = 1, 2$. Let $F \in \mathcal{A}$ be such that $F(0) = k_1 - 1$ and $F(c_F) = k_2 + 1$. Clearly

$$\mathcal{K}(F) = \big(\big((k_1-1)^L\big)^\infty, (k_2+1)^R\big((k_1-1)^L\big)^\infty\big)$$

and $\underline{\nu}_i$ is dominated by F for $i = 1, 2$. From Proposition 3.1 (b) there exists $x_i \in [0, c_F]$ such that $\widehat{\underline{I}}_F(x_i) = \underline{\nu}_i$ for $i = 1, 2$. By Proposition 2.1 (a) we have that $0 < x_1 < x_2 < c_F$ because $\underline{\nu}_1 < \underline{\nu}_2$. Let $x_1^*, x_2^* \in [c_F, 1]$ be such that $F(x_1^*) = F(x_1) + 1$ and $F(x_2) = F(x_2^*)$. Thus,

$$\widehat{\underline{I}}_F(x_1^*) = \underline{\nu}_1' \quad \text{and} \quad \widehat{\underline{I}}_F(x_2^*) = \underline{\nu}_2.$$

Since $\underline{\nu}_1' < \underline{\nu}_2$, from Proposition 2.1 (b), we obtain that $c_F < x_2^* < x_1^* < 1$. We note that

$$\widehat{\underline{I}}_F\big(F^n(x_i)\big) = S^n\big(\widehat{\underline{I}}_F(x_i)\big) = S^n(\underline{\nu}_i)$$

for $i = 1, 2$. Therefore, if $F^n(x_i) \in [0, c_F]$ (resp. $F^n(x_i) \in [c_F, 1]$), by Proposition 2.1, we see that $F^n(x_i) \in [x_1, x_2]$ (resp. $F^n(x_i) \in [x_2^*, x_1^*]$) because $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$. So,

$$P_{x_1} \cup P_{x_2} \subset [x_1, x_2] \cup [x_2^*, x_1^*]$$

where P_{x_i} denotes the intersection of the (mod. 1) orbit of x_i by F with $[0, 1)$ for $i = 1, 2$. Set

$$K = (P_{x_1} \cup P_{x_2}) \cup \{x_2^*, x_1^*\}.$$

Let

$$d_x = E\big(F(x)\big) \quad \text{and} \quad \pi(x) = F(x) - d_x.$$

We note that $\pi(x_i) = \pi(x_i^*)$ for $i = 1, 2$, $d_{x_1} = d_{x_1^*} + 1$ and $d_{x_2} = d_{x_2^*}$.

We choose an auxiliary map $h : \mathbb{R} \to \mathbb{R}$ satisfying:

- $h(x + 1) = h(x) + 1$ for all $x \in \mathbb{R}$;
- $h(0) = 0$;
- $h_{|\mathbb{R}\setminus(K+\mathbb{Z})}$ is continuous and strictly increasing;
- if $x \in K$ then $h(x) = \lim_{\substack{y \to x \\ y < x}} h(y) < \lim_{\substack{y \to x \\ y > x}} h(y)$.

Let $g \in \mathcal{L}$ be the nondecreasing map obtained from h^{-1} by extending it to the whole real line. We note that g is strictly increasing on $h(\mathbb{R} \setminus (K + \mathbb{Z}))$, for each $x \in K$ there exists a closed interval $[a_x, b_x] \subset (0, 1)$ such that $g([a_x, b_x]) = x$ and if $x, x' \in K$ then, $x < x'$ if and only if $b_x < a_{x'}$. In particular, since $(x_2, x_2^*) \cap K = \emptyset$, $h_{|(x_2, x_2^*)}$ is strictly increasing and $g^{-1}(c_F) \in (b_{x_2}, a_{x_2^*})$. Then we define $G \in \mathcal{L}$ as follows:

- $G_{|[b_{x_1}, a_{x_2}]}$ is strictly increasing and, for each $a_x, b_x \in [b_{x_1}, a_{x_2}]$,

$$G(a_x) = a_{\pi(x)} + d_x, \quad G(b_x) = b_{\pi(x)} + d_x\,.$$

- $G_{|[b_{x_2^*}, a_{x_1^*}]}$ is strictly decreasing and, for each $a_x, b_x \in [b_{x_2^*}, a_{x_1^*}]$,

$$G(a_x) = b_{\pi(x)} + d_x, \quad G(b_x) = a_{\pi(x)} + d_x\,.$$

• $G(g^{-1}(c_F)) \in (a_{\pi(x_2)} + d_{x_2}, b_{\pi(x_2)} + d_{x_2})$, $G|_{[a_{x_2}, g^{-1}(c_F)]}$ is strictly increasing and $G|_{[g^{-1}(c_F), b_{x_2^*}]}$ is strictly decreasing.

• $G(0) \in (a_{\pi(x_1)} + d_{x_1}, b_{\pi(x_1)} + d_{x_1})$, $G|_{[0,b_{x_1}]}$ is strictly increasing and $G|_{[b_{x_1^*},1]}$ is strictly decreasing.

We note that $G \in \mathcal{A}$ and $c_G = g^{-1}(c_F)$. Moreover, for each $x \in K$ we have that $G([a_x, b_x]) \subset [a_{\pi(x)} + d_x, b_{\pi(x)} + d_x]$.

Now, we only have to prove that $\underline{\widehat{I}}_G(0^+) = \underline{\widehat{I}}_G(0) = \underline{\widehat{I}}_F(x_1)$ and $\underline{\widehat{I}}_G(c_G^-) = \underline{\widehat{I}}_G(c_G) = \underline{\widehat{I}}_F(x_2)$. From all said above we see that

$$E\big(F^n(x_1)\big) = E\big(G^n(0)\big) \quad \text{and} \quad E\big(F^n(x_2)\big) = E\big(G^n(c_G)\big).$$

Since $g(0) = 0$, $g(c_G) = c_F$, g is non-decreasing and $g|_{(b_{x_2}, a_{x_2^*})}$ is strictly increasing we have that $g(D(x)) \in (0, c_F)$ (resp. $g(D(x)) \in (c_F, 1)$) if and only if $D(x) \in (0, c_G)$ (resp. $D(x) \in (c_G, 1)$). Therefore,

$$\underline{\widehat{I}}_G(0) = \underline{\widehat{I}}_F(x_1) = \underline{\nu}_1 \quad \text{and} \quad \underline{\widehat{I}}_G(c_G) = \underline{\widehat{I}}_F(x_2) = \underline{\nu}_2.$$

In short, $\mathcal{K}(G) = (\underline{\nu}_1, \underline{\nu}_2)$ and the theorem follows. □

Another strategy to prove the above theorem is the one used by de Melo and van Strien in the proof of Theorem II 3.3 of [15]. However our approach, suggested by F. Mañosas, is considerably more simple in the case of maps with two critical points. It seems to us that this approach, which uses strongly the characterization of the itineraries of a map given by Proposition 3.1 (b), could also simplify the proof in their case and could be used to deal with similar problems for multimodal circle maps of degree one.

To end the proof of the second statement of Theorem A we still have to prove that if $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}_a$ for some $a \in \mathbb{R}$ then there exists $F \in \mathcal{A}$ such that $R_F = \{a\}$ and $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$. We note that the strategy used in the proof of Theorem 4.2 also works in this case. However, we prefer a constructive approach which characterizes better the allowed kneading pairs in $\mathcal{E} \setminus \mathcal{E}^*$. We consider separately the rational and the irrational case. To deal with the rational case we need the following technical lemma.

LEMMA 4.3. — *Let $F \in \mathcal{A} \cap C^1(\mathbb{R}, \mathbb{R})$ and let $p/q \in \mathbb{Q}$ with $(p, q) = 1$. Then the following statements hold:*

(a) *Assume that* $\underline{\widehat{I}}_F(c_F) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$. *Then there exists* U, *a neighborhood of* F *in* $\mathcal{A} \cap C^1(\mathbb{R},\mathbb{R})$, *such that for each* $G \in U$, $\underline{\widehat{I}}_G(c_G^-)$ *is either* $\underline{\widehat{I}}_R(p/q)$ *or* $\underline{\widehat{I}}_\delta(p/q)$.

(b) *Assume that* $\underline{\widehat{I}}_F(0) = \epsilon_1(p/q)^L \cdots \epsilon_{q-1}(p/q)^L \epsilon_q(p/q)^M$. *Then there exists* U, *a neighborhood of* F *in* $\mathcal{A} \cap C^1(\mathbb{R},\mathbb{R})$, *such that for each* $G \in U$, $\underline{\widehat{I}}_G(0^+)$ *is either* $\underline{\widehat{I}}^*_R(p/q)$ *or* $\underline{\widehat{I}}_\epsilon(p/q)$.

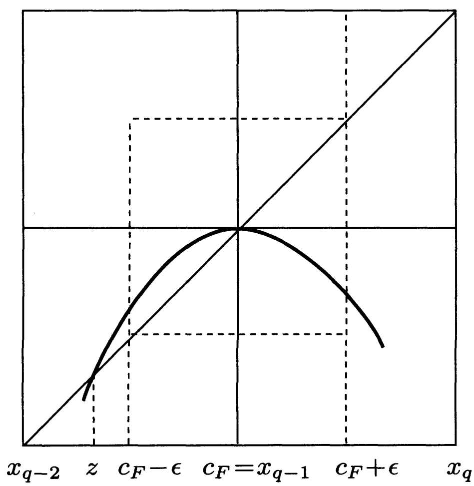

Figure 7. The graph of $(F^q - p)$ *near* c_F.

Proof. — We only prove statement (a). Statement (b) follows in a similar way. Assume that $\underline{\widehat{I}}_F(c_F) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$. Let $P = \{x_i\}_{i\in\mathbb{Z}}$ be the twist periodic orbit of period q and rotation number p/q such that $x_{q-1} = c_F$. Clearly, we can take $F|_{[x_{q-2},c_F]}$ (see Figure 7) in such a way that F has a periodic (mod. 1) point $z \in (x_{q-2}, c_F)$ close to c_F, of period q, such that $(F^q - p)|_{[z,c_F]}$ is strictly increasing, $(F^q - p)(x) > x$ for each $x \in (z, c_F)$ and $\underline{\widehat{I}}_F(z) = \underline{\widehat{I}}_\delta(p/q)$ (in particular $F(z) > 1 = F(1)$). Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(F^q - p)(c_F) = 0$$

there exists $0 < \epsilon < c_F - z$ such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}(F^q - p)(x)\right| < \frac{1}{4}$$

for each $x \in (c_F - \epsilon, c_F + \epsilon)$. Now we take U, a neighborhood of F in $C^1(\mathbb{R},\mathbb{R}) \cap \mathcal{A}$, such that for each $G \in U$ the following conditions hold:

a) $\widehat{\underline{I}}_G(c_G) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(c_G))} \ldots$;

b) G has a periodic (mod. 1) point $z_G < c_G$ close to z, of period q, such that $G(z_G) > \max\{1, G(1)\}$ and $\widehat{\underline{I}}_G(z_G) = \widehat{\underline{I}}_\delta(p/q)$;

c) $c_G \in (c_F - \epsilon, c_F + \epsilon)$, $(G^q - p)_{|[z_G, c_G]}$ is strictly increasing and $(G^q - p)_{|[c_G, c_F+\epsilon]}$ is strictly decreasing;

d) $(G^q - p)(c_G) \in (c_F - \epsilon, c_F + \epsilon)$;

e) $\left|\frac{\mathrm{d}}{\mathrm{d}x}(G^q - p)(x)\right| < \frac{1}{2}$ for each $x \in (c_F - \epsilon, c_F + \epsilon)$.

We note that for each $G \in U$ and $x \in [z_G, c_G]$ we have that

$$\widehat{\underline{I}}_G(x) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \ldots .$$

Let $z_G^* \in (c_G, 1)$ be such that $G(z_G) = G(z_G^*)$ (such z_G^* exists because, in view of b), $G(z_G) > G(1)$). Clearly, for all $x \in [z_G, z_G^*]$ we also have that

$$\widehat{\underline{I}}_G(x) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \ldots .$$

If $(G^q - p)(c_G) \leq c_G$, then for each $x \in [z_G, c_G]$ we have that $(G^q - p)^i(x) \in [z_G, c_G]$ for each $i \in \mathbb{N}$. Hence, $\widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_\delta(p/q)$. Now, assume that $(G^q - p)(c_G) > c_G$. From c) and d) we see that $c_G < (G^q - p)(c_G) \in (c_F - \epsilon, c_F + \epsilon)$ and $(G^q - p)(c_G)$ is the maximum of $G^q - p$ in $(c_F - \epsilon, c_F + \epsilon)$. So $(G^q - p)^2(c_G) < (G^q - p)(c_G)$. On the other hand

$$(G^q - p)(c_G) - (G^q - p)^2(c_G) = \left|\frac{\mathrm{d}}{\mathrm{d}x}(G^q - p)(\xi)\right| \big((G^q - p)(c_G) - c_G\big)$$

with ξ between c_G and $(G^q - p)(c_G)$. In view of e) we have that $\left|\frac{\mathrm{d}}{\mathrm{d}x}(G^q - p)(\xi)\right| < \frac{1}{2}$ and hence

$$(G^q - p)(c_G) - (G^q - p)^2(c_G) < (G^q - p)(c_G) - c_G.$$

Therefore $c_G < (G^q - p)^2(c_G)$ and, consequently,

$$(G^q - p)\big([c_G, (G^q - p)(c_G)]\big) \subset [c_G, (G^q - p)(c_G)].$$

From all said above we see that, in this case, $\widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_R(p/q)$. □

The next result already proves the second statement of Theorem A in the degenerate rational case.

PROPOSITION 4.4. — *Let* $(\underline{\nu}_1,\underline{\nu}_2) \in \mathcal{E}_{p/q}$ *with* $p \in \mathbb{Z}$, $q \in \mathbb{N}$ *and* $(p,q) = 1$. *Then there exists* $F \in \mathcal{A}$ *such that* $R_F = \{p/q\}$ *and* $\mathcal{K}(F) = (\underline{\nu}_1,\underline{\nu}_2)$.

Proof. — We will deal first with the case $p/q \in \mathbb{Z}$ (that is, $q = 1$). From Lemma 2.3 (b) we have

$$\mathcal{E}_p = \Big\{ \big((p^L)^\infty, (p+1)^L(p^L)^\infty\big), \big((p-1)^R(p^R)^\infty, (p^R)^\infty\big), \\ \big((p-1)^L(p^L)^\infty, (p^L)^\infty \Big\}.$$

Assume that $(\underline{\nu}_1,\underline{\nu}_2) = ((p-1)^R(p^R)^\infty, (p^R)^\infty)$. Then we take $F \in \mathcal{A}$ such that $a \in (0, c_F)$ is a fixed point of $(F-p)$ with the property that $(F-p)_{|[a,1]}$ is a unimodal map satisfying that $c_F < (F-p)(1)$ (see Figure 8).

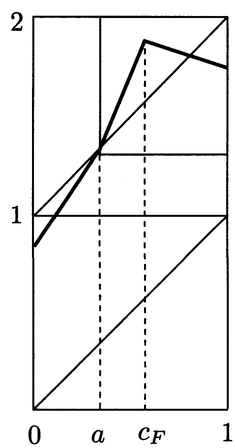

Figure 8. The map F.

In consequence

$$\widehat{\underline{I}}_F(c_F^-) = \widehat{\underline{I}}_F(1^-) = (\widehat{\underline{I}}_F(0^+))' \quad \text{and} \quad \widehat{\underline{I}}_F(c_F) = (p^R)^\infty.$$

Thus $\mathcal{K}(F) = ((p-1)^R, (p^R)^\infty(p^R)^\infty)$. The rest of the cases follows in a similar way.

Now we consider the case $q \neq 1$. Assume first that

$$(\underline{\nu}_1,\underline{\nu}_2) \in \big\{ (\widehat{\underline{I}}^*_\delta(p/q), \widehat{\underline{I}}_\delta(p/q)), (\widehat{\underline{I}}^*_R(p/q), \widehat{\underline{I}}_R(p/q)) \big\}.$$

Set $P = \{x_i\}_{i\in\mathbb{Z}}$ with $x_i = i/q + 1/(2q)$ for each $i \in \mathbb{Z}$ and let $F \in \mathcal{C}^1(\mathbb{R},\mathbb{R}) \cap \mathcal{A}$ be such that

1) $F(0) = 0$ and $c_F = x_{q-1}$;
2) $F(x) = x + p/q$ for each $x \in P$;
3) F is affine in the interval $[x_0, x_{q-2}]$.

Note that P is F-invariant and $F^i(x) = x + i\frac{p}{q}$ for each $x \in P$ and $i \in \mathbb{N}$. Hence, $s(F^i(c_F)) = s(F^i(x_{q-1})) = L$ for $i = 1, 2, \dots, q-1$. Moreover, since $F^q(c_F) = F^q(x_{q-1}) = x_{q-1} + q\frac{p}{q} = c_F + p$ we see that $s(F^q(c_F)) = s(F^q(x_{q-1})) = C$. On the other hand,

$$\begin{aligned} d(c_F) = d(x_{q-1}) &= E\big(F(x_{q-1})\big) - E(x_{q-1}) \\ &= E\Big(\frac{2q-1}{2q} + \frac{p}{q}\Big) = E\Big(\frac{p}{q}\Big) + 1 \\ &= \epsilon_1(p/q) + 1 = \delta_1(p/q), \end{aligned}$$

and, for $i = 2, \dots, q-2$,

$$\begin{aligned} d\big(F^i(c_F)\big) &= d\big(F^i(x_{q-1})\big) \\ &= E\big(F^{i+1}(x_{q-1})\big) - E\big(F^i(x_{q-1})\big) \\ &= E\Big(\frac{2q-1}{2q} + (i+1)\frac{p}{q}\Big) - E\Big(\frac{2q-1}{2q} + i\frac{p}{q}\Big) \\ &= \Big(E\Big((i+1)\frac{p}{q}\Big) + 1\Big) - \Big(E\Big(i\frac{p}{q}\Big) + 1\Big) \\ &= \epsilon_i(p/q) = \delta_i(p/q). \end{aligned}$$

Lastly,

$$\begin{aligned} d\big(F^{q-1}(c_F)\big) &= d\big(F^{q-1}(x_{q-1})\big) \\ &= E\Big(\frac{2q-1}{2q} + p\Big) - E\Big(\frac{2q-1}{2q} + (q-1)\frac{p}{q}\Big) \\ &= E(p) - \Big(E\Big(\frac{(q-1)p}{q}\Big) + 1\Big) \\ &= \epsilon_q(p/q) - 1 = \delta_q(p/q). \end{aligned}$$

In consequence $\widehat{\underline{I}}_F(c_F) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^C$.

Now we are ready to construct maps $H_\delta, H_R \in \mathcal{A}$ such that

$$\begin{aligned} R_{H_\delta} = R_{H_R} &= \{p/q\}, \\ \mathcal{K}(H_\delta) &= \big(\widehat{\underline{I}}^*_\delta(p/q), \widehat{\underline{I}}_\delta(p/q)\big), \\ \mathcal{K}(H_R) &= \big(\widehat{\underline{I}}^*_R(p/q), \widehat{\underline{I}}_R(p/q)\big). \end{aligned}$$

From Lemma 4.3 (a) we have that there exists U, a neighborhood of F in $\mathcal{A} \cap C^1(\mathbb{R},\mathbb{R})$, such that for each $G \in U$, $\underline{\widehat{I}}_G(c_G^-)$ is either $\underline{\widehat{I}}_R(p/q)$ or $\underline{\widehat{I}}_\delta(p/q)$. Moreover, from the proof of Lemma 4.3, G has a periodic (mod. 1) point $z_G < c_G$ of period q such that $G(z_G) > \max\{1, G(1)\}$ and $\underline{\widehat{I}}_G(z_G) = \underline{\widehat{I}}_\delta(p/q)$. Let $z_G^* \in (c_G, 1)$ be such that $G(z_G) = G(z_G^*)$. Clearly, for all $x \in [z_G, z_G^*]$ we also have that

$$\underline{\widehat{I}}_G(x) = \delta_1(p/q)^L \cdots \delta_{q-1}(p/q)^L \delta_q(p/q)^{s(G^q(x))} \cdots.$$

To construct H_δ take $G \in U$ such that $(G^q - p)(c_G) \le c_G$ and let $c^* \in (1, c_G + 1)$ be such that $G(c_G) = G(c^*)$. We take $H_\delta \in C^1(\mathbb{R},\mathbb{R}) \cap \mathcal{A}$ such that $c_{H_\delta} = c_G$, $G_{|[c^*-1, c_{H_\delta}]} = H_{\delta|[c^*-1, c_{H_\delta}]}$ and $H_\delta(x) > G(z_G)$ for all $x \in (c_G, c^*)$ (see Figure 9).

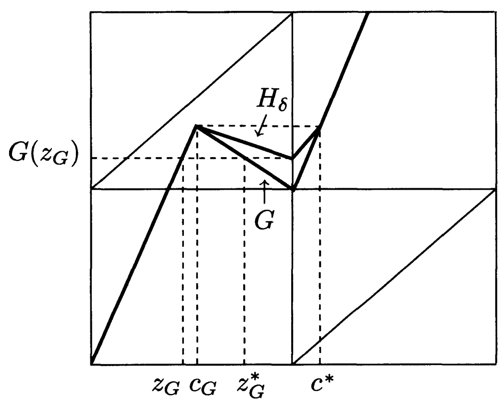

Figure 9. The maps H_δ and G.

We note that

$$H_\delta([c_G, 1]) \subset H_\delta([z_G, c_{H_\delta}]) = G([z_G, c_G]).$$

Hence, from above we have that

$$\underline{\widehat{I}}_{H_\delta}(0^-) = \underline{\widehat{I}}_{H_\delta}(c_{H_\delta}^-) = \underline{\widehat{I}}_G(c_G^-) = \underline{\widehat{I}}_\delta(p/q).$$

Thus

$$\mathcal{K}(H_\delta) = (\underline{\widehat{I}}_\delta^*(p/q), \underline{\widehat{I}}_\delta(p/q)).$$

Furthermore, by Lemma 3.7 and Theorem 3.6 we see that $R_{H_\delta} = \{p/q\}$. To construct H_R we take $G \in U$ such that $(G^q - p)(c_F) > c_F$. Let $a = (G^q - p)(c_G)$ and let $b \in (c_G, z_G^*)$ be such that $(G^q - p)(b) = c_G$. Since

$$(G^q - p)(b) = c_G < (G^q - p)^2(c_G) = (G^q - p)(a)$$

and $a, b \in (c_G, 1)$ we have that $b > a$. Finally, let $c^* \in (1, c_G + 1)$ be such that $G(c_G) = G(c^*)$. Then we take $H_R \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{A}$ such that $c_{H_R} = c_G$, $H_{R|[c^*-1,a]} = G_{|[c^*-1,a]}$ and $H_R(x) > G(b)$ for all $x \in (a, c^*)$ (see Figure 10).

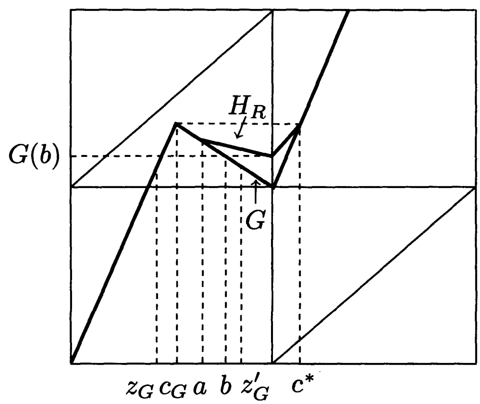

Figure 10. The maps H_R and G.

In consequence, since $b < z_G^*$ we have that $G(b) > G(z_G^*) = G(z_G)$ and hence,

$$H_R([c_{H_R}, 1]) \subset H_R([z_G, c_{H_R}]) = G([z_G, c_G]).$$

Therefore, from above we get that

$$\widehat{\underline{I}}_{H_R}(0^-) = \widehat{\underline{I}}_{H_R}(c_{H_R}^-) = \widehat{\underline{I}}_G(c_G^-) = \widehat{\underline{I}}_R(p/q).$$

Thus, $\mathcal{K}(H_R) = (\widehat{\underline{I}}_R^*(p/q), \widehat{\underline{I}}_R(p/q))$ and $R_{H_R} = \{p/q\}$.

To end the proof of the proposition it remains to construct a map $H_\epsilon \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{A}$ such that $R_{H_\epsilon} = \{p/q\}$ and $\mathcal{K}(H_\epsilon) = (\widehat{\underline{I}}_\epsilon(p/q), \widehat{\underline{I}}_\epsilon^*(p/q))$. To do it we proceed as in the above construction of the map H_δ by using Lemma 4.3 (b) and, instead of the map F, the map $\widetilde{F} \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{A}$ defined as follows. Set $\widetilde{P} = \{\tilde{x}_i\}_{i \in \mathbb{Z}}$ with $\tilde{x}_i = i/q$ for each $i \in \mathbb{Z}$. Then $\widetilde{F}$ is such that:

i) $\widetilde{F}$ is affine in the interval $[\tilde{x}_1, \tilde{x}_{q-1}]$;

ii) $\widetilde{F}(\tilde{x}_i) = \tilde{x}_i + p/q$;

iii) $c_{\widetilde{F}} \in (x_{q-1}, 1)$ and $\widetilde{F}(c_{\widetilde{F}}) = c_{\widetilde{F}} + E(p/q) + 1$. □

Proof of the second statement of Theorem A. — If $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^*$ then the theorem follows from Theorem 4.2. Otherwise, $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}_a$ with $a \in \mathbb{R}$. If $a \in \mathbb{Q}$ then the theorem follows from Proposition 4.4 If $a \notin \mathbb{Q}$ then, from the proof of Proposition 1 of [5] it follows that there exists $F \in \mathcal{A}$ such that $R_F = \{a\}$. Now, from Lemma 2.3 (a) we see that $\widehat{\underline{I}}_\delta(a) = \widehat{\underline{I}}^*_\epsilon(a)$ and $\widehat{\underline{I}}^*_\delta(a) = \widehat{\underline{I}}_\epsilon(a)$. So, from Theorem 3.6 we obtain that $\mathcal{K}(F) = (\widehat{\underline{I}}^*_\delta(a), \hat{I}_\delta(a))$. Hence, by the definition of $\mathcal{E}_a$ we see that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$. □

5. APPENDIX: EXTENSIONS AND CONSEQUENCES OF THEOREM A

In this section we comment some extensions and consequences of Theorem A. In Subsection 5.1 we extend the main result of this paper to the case of orientation preserving circle homeomorphisms. In Subsection 5.2 we will consider some questions about the existence of full families of maps from $\mathcal{A}$.

5.1. The kneading pair for the orientation preserving circle homeomorphisms.

In this subsection we extend Theorem A to the orientation preserving circle homeomorphisms. To this end we will denote the class of all orientation preserving circle homeomorphisms by $\mathcal{H}$. More precisely, $F \in \mathcal{H}$ if and only if $F \in \mathcal{L}$ and it is strictly increasing. Now, we extend the notion of reduced itinerary to the maps from $\mathcal{H}$ as follows. For $F \in \mathcal{H}$ and $x \in \mathbb{R}$ let

$$\hat{s}(x) = \begin{cases} M & \text{if } D(x) = 0, \\ L & \text{if } D(x) \neq 0. \end{cases}$$

For $i \in \mathbb{N}$, set

$$s_i = \hat{s}\big(F^i(x)\big) \quad \text{and} \quad d_i = d\big(F^{i-1}(x)\big)$$

(recall that $d(x) = E(F(x)) - E(x)$). Then $\widehat{\underline{I}}_F(x)$ is defined as

$$\begin{cases} d_1^{s_1} d_2^{s_2} \cdots & \text{if } s_i = L \text{ for all } i \geq 1, \\ d_1^{s_1} d_2^{s_2} \cdots d_n^{s_n} & \text{if } s_n = M \text{ and } s_i = L \text{ for all } i \in \{1, \dots, n-1\}. \end{cases}$$

In this context we define *the kneading pair* of a map $F \in \mathcal{H}$ as $(\widehat{\underline{I}}_F(0^+), \widehat{\underline{I}}_F(0^-))$. As above it will be denoted by $\mathcal{K}(F)$.

With these notations all results from [4] and this paper can be extended to this case in the natural way. In particular Propositions 3.1

and 3.3, Theorem 3.6, Corollary 3.2 and Theorem A. However, since the characterization of the set of kneading pairs of maps from $\mathcal{H}$ is not the straightforward extension of Theorem A, we are going to state this characterization in detail.

For $a \in \mathbb{R}$ we set

$$\widehat{\mathcal{E}}_a = \begin{cases} \{(\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}^*_\epsilon(a)), (\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\delta(a)), (\widehat{\underline{I}}_\epsilon(a), \widehat{\underline{I}}_\delta(a))\} & \\ & \text{if } a = p/q \in \mathbb{Q}, \text{ with } (p,q) = 1, \\ \{(\widehat{\underline{I}}^*_\delta(a), \widehat{\underline{I}}_\delta(a))\} & \text{if } a \notin \mathbb{Q}, \end{cases}$$

and $\widehat{\mathcal{E}} = \bigcup_{a \in \mathbb{R}} \widehat{\mathcal{E}}_a$. The characterization of the kneading pairs of maps from $\mathcal{H}$ is now given by the following.

THEOREM 5.1. — *If $F \in \mathcal{H}$ then $\mathcal{K}(F) \in \widehat{\mathcal{E}}$. Conversely, for each $(\underline{\nu}_1, \underline{\nu}_2) \in \widehat{\mathcal{E}}$ there exists $F \in \mathcal{H}$ such that $\mathcal{K}(F) = (\underline{\nu}_1, \underline{\nu}_2)$. Moreover $R_F = \{a\}$ if and only if $\mathcal{K}(F) \in \widehat{\mathcal{E}}_a$.*

5.2. A remark on the full families for maps in $\mathcal{A}$.

In the context of Theorem A the following question arises in a natural way: Does there exist a family $F_\mu \in C^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{A}$, depending continuously on μ, such that for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}$ there exists μ_0 in the parameter space such that $\mathcal{K}(F_{\mu_0}) = (\underline{\nu}_1, \underline{\nu}_2)$? In the literature, such a parameter family of maps is usually called a *full family* (see [7] and [15]). It is well known that, in the unimodal case, the family $f_\mu(x) = \mu x(1-x)$ with $x \in [0,1]$ and $\mu \in [1,4]$ (among others) is full (see [7]).

The simplest non-invertible degree one circle maps are the ones with two critical points. That is, the maps from class $\mathcal{A}$. Among the families of such maps, the standard maps family defined as

$$F_{b,w}(x) = x + w + b\,\frac{\sin(2\pi x)}{2\pi}$$

where $x \in \mathbb{R}$ and $(b, w) \in (1, \infty) \times \mathbb{R}$ is known to display all dynamical features. Therefore, it is natural to think that this family is full. To discuss this problem we need to introduce some notation

Let $F \in \mathcal{L} \cap C^1(\mathbb{R}, \mathbb{R})$. We shall say that $x \in \mathbb{R}$ is a *non-flat critical point* if it is a critical point and there exists an integer $k > 1$ such that F is C^k in a neighborhood of x and $\frac{\mathrm{d}^k}{\mathrm{d}x^k} F(x) \neq 0$. We say that $x \in \mathbb{R}$ is a *turning point* if the map F has a local extremum in x.

PROPOSITION 5.2. — *Let $F \in \mathcal{A}$ be analytic. Then $\mathcal{K}(F) \notin \bigcup_{a \notin \mathbb{Q}} \mathcal{E}_a$. That is, R_F is not degenerate to an irrational.*

Proof. — Assume that $\mathcal{K}(F) \in \mathcal{E}_a$ for some $a \notin \mathbb{Q}$. From Lemma 2.3 (a) and Theorem 3.6 we have $R_F = \{a\}$. In [14] Malta proves the following statement which depends heavily on a result of Yoccoz [20]: *Let $F \in \mathcal{L} \cap C^2(\mathbb{R}, \mathbb{R})$ and suppose that all non-turning critical points are non-flat. If F has a turning critical point, then F has periodic points.* So, from the fact that if $a \notin \mathbb{Q}$ we have that F has no periodic points, we get that F has flat non-turning critical points. Since F is analytic it follows that $F = 0$; a contradiction. □

Proposition 5.2 tells us that there is no analytic full family in $\mathcal{A}$. In particular, the standard maps family is not full. This suggests that the "good" families from $\mathcal{A}$ will only be *weakly full* in the following sense. We say that a family $F_\mu \in C^1(\mathbb{R}, \mathbb{R}) \cap \mathcal{A}$, depending continuously on μ, is *weakly full* if for each $(\underline{\nu}_1, \underline{\nu}_2) \in \mathcal{E}^* \cup (\cup_{a \in \mathbb{Q}} \mathcal{E}_a)$ there exists μ_0 in the parameter space such that $\mathcal{K}(F_{\mu_0}) = (\underline{\nu}_1, \underline{\nu}_2)$. Thus, we propose the following.

CONJECTURE 5.3. — *The standard maps family is weakly full.*

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