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Prospective Teachers' Use of Conceptual Advances of Learning Trajectories to Develop Their Teaching Competence in the Context of Pattern Generalization

Alberto Zapatera Llinares

Department of Educational Sciencies, University CEU Cardenal Herrera, C/Carmelitas 1, CP 03203 Elche, Alicante, Spain; alberto.zapatera@uchceu.es

Abstract: (1) Background: This research shows how the identification of conceptual advances (CA) that determine the transition between the stages of a progression model in a learning trajectory will help prospective primary teachers (PPT) to develop the teaching competence "professional noticing of students' mathematical thinking". Conceptual advances are key moments in the construction of mathematical structures and involve a change in the way students understand mathematical relationships. (2) Methods: A teaching module has been designed in which students of the Teaching Degree will analyze the responses of primary education students to tasks of pattern generalization from the identification and use of conceptual advances. (3) Results: The results of the teaching module show that professional noticing can be developed in suitable teaching environments. (4) Conclusions: The recognition of conceptual advances helps to interpret students' thinking and learning trajectories which are effective tools to structure and develop professional noticing.

Keywords: professional noticing; conceptual advances; progression model; learning trajectory; pattern generalization

MSC: 97B50

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1. Introduction

The study of the knowledge and skills needed by mathematics teachers is currently one of the most important objectives of research in mathematical education. This perspective has led to the emergence of the teaching competence: "professional noticing of students' mathematical thinking", which focuses on the use of teacher knowledge to reflect on teaching and learning situations [1].

One way to conceptualize professional noticing comes from the study of the three skills described by [1]: (1) identifying the relevant aspects, (2) interpreting the students' understanding, and (3) decision making actions. Regarding the first skill, the teacher identifies significant mathematical elements that students use when solving a given mathematical task (mathematical dimension); in the second skill, the teacher interprets the mathematical understanding of students by connecting the significant mathematical elements, identified in their responses, with cognitive aspects (cognitive dimension); and in the third skill, the teacher uses the interpretation of the students' understanding to decide the actions necessary to improve the teaching process (didactic dimension).

Studies that have generated descriptors of the development level of the competence "professional noticing" show the identification of significant mathematical elements as a starting point for interpreting students' understanding [2]. The recognition of mathematical elements is an indicator of the development of professional noticing, and the use of conceptual advances linked to mathematical elements helps determine the progression model in a learning trajectory [3–5].

Within a learning trajectory, the transition from one stage to another represents a key developmental understanding in the ability to think and/or perceive mathematical relationships [6]. In this sense, the understanding of certain mathematical elements represents a conceptual advance, recognizing different development levels of professional noticing competence [7]. Llinares, Fernández, and Sánchez-Matamoros defined the development levels of professional noticing based on how teachers, taking into account the understanding of mathematical elements, are able to identify the key developmental understandings in students' responses [8].

From this perspective, Buforn, when characterizing how future teachers interpret students' responses to problems of proportional reasoning, by using a learning trajectory with several stages, concluded that, although the identification of mathematical elements is a necessary condition to properly interpret students' mathematical thinking, it is not sufficient if they do not understand them as conceptual advances [4]. In this way, the conceptual advance, in addition to helping analyze students' responses, is the key to making progress in understanding mathematical concepts within a learning trajectory [3].

This research expands and complements previous research by studying, in the context of a teaching module on pattern generalization, how PPTs progress in their professional noticing when using a learning trajectory. To study this progress, we analyze how PPTs recognize, in primary education students' responses to pattern generalization problems, mathematical elements as conceptual advances and how they use them to interpret the students' understanding of mathematical thinking.

Professional development based on students' mathematical thinking enables teachers to create teaching environments that foster interest in mathematics and improve academic achievement. For this reason, attention to students' mathematical thinking can be a consistent and constant source of professional development. That is, focusing attention on the mathematical thinking of students helps the teacher in his or her professional development and promotes his knowledge of mathematical content, by dealing with the mathematics present in the strategies that students use to solve mathematical questions [1].

From this perspective, the objective of this research is to characterize how the identification and understanding of mathematical elements as conceptual advances in a learning trajectory, will help PPTs develop the professional noticing of primary students' mathematical thinking in the context of pattern generalization.

1.1. Professional Noticing of Students' Mathematical Thinking

Professional noticing is a construct that is used to indicate the act of observing or recognizing relevant events of a situation and acting on them. From this perspective, professional noticing is not an exclusive competence of teaching but is also part of the learning of any profession [1].

Ref. [9] consider that professional noticing as a teaching competence implies identifying and recognizing the relevant aspects in a classroom situation, connecting the identified aspects with the general principles of teaching-learning and applying context knowledge in order to make decisions. Ref. [1] particularized this perspective and conceptualized the professional noticing of students' mathematical thinking as a set of three interrelated skills: identifying relevant mathematical elements in students' responses, interpreting students' mathematical understanding taking into account identified mathematical aspects, and making decisions based on students' thinking to improve their learning.

Several research studies have shown that professional noticing can be developed by using a framework that provides references to teachers (e.g., [10]) and that learning trajectories can provide information to teachers in order to interpret students' understanding (e.g., [11]).

In this research, to structure the professional noticing of the PPTs and direct their attention towards the relevant aspects of the mathematical thinking of the students, we have used the conceptual advances that will determine the transition between the development stages of a progression model for a learning trajectory on pattern generalization.

1.2. Learning Trajectories and Conceptual Advances

Although the learning trajectories have been conceptualized in different ways, they are based on the hypothetical learning trajectories that Ref. [12] presented as part of his model in the mathematics teaching course.

The literature describes the learning trajectories as "predictable sequences of constructs that capture how knowledge progresses from initial levels to more sophisticated levels" [13]. Ref. [14] (p. 83) refer to them as "related and conjectured trajectories through a set of instructional tasks [. . .] to involve children in a progression of development of thinking levels".

Ref. [14] consider that a learning trajectory is composed of a mathematical learning objective, a model of progression in learning a specific domain, and instructional tasks that may support such progression. Objectives are concepts and skills that generate future learning; progression models are levels of thinking, each more sophisticated than the previous one that will lead to the achievement of the goal; and instructional tasks are situations designed to help children learn the ideas and skills necessary to achieve the goal.

Ref. [15] defined the thinking levels of a progression model as successively more complex levels reached by students when they progress in the acquisition of a given mathematical concept. These authors highlighted the importance of the delimitation of thinking levels and, especially, of the reasons that promote changes from one level to another. In this way, a progression model is characterized by the levels of thinking, from now on defined as stages of understanding, and by the conceptual advances that allow the transition of students from a stage to a higher one.

Conceptual advances are fundamental moments in the construction of mathematical structures by students and cause "a change in their ability to think and/or perceive mathematical relationships" [6] (p. 362). This change in the student's mathematical skills is developed through certain tasks, so that teachers, when observing and comparing the different ways with which students solve them, infer conclusions on the construction of a mathematical concept and on conceptual advances [16].

In this research, the conceptual advances linked to the significant mathematical elements that allow the transition between the stages of understanding in a progression model in a learning trajectory are used to characterize the professional noticing of PPTs.

1.3. Learning Trajectory of Pattern Generalization

Generalization is a mathematical construct that involves going from the particular to the general and seeing the general in the particular; that is, generalizing consists of universalizing a property observed in a limited number of cases. Specifically, in pattern generalization problems, the first terms of a sequence are presented graphically, numerically, or verbally, and the student must identify a common property in them, generalize that property to all the terms of the sequence (near and far generalizations) and, often, also invert the process (reverse process).

Research focused on the way in which primary students solve generalization tasks of patterns [17] have pointed out the relevant role of understanding three mathematical elements: numerical and spatial structures, functional relationship, and reverse process. The numerical and spatial structures emerge respectively from the number and distribution of the components of each term of the sequence, the functional relationship associates each term of the sequence with its number of components, and the reverse process allows to identify a term of the sequence from its number of components.

These mathematical elements are key to defining the stages of understanding and the conceptual advances of the progression model of a learning trajectory of pattern generalization, since: (1) to continue a sequence, the students must identify a regularity between the spatial and numerical structures, coordinating both structures; (2) to identify a distant term they must establish a functional relationship between the term of the sequence and the number of elements that are part of it; and (3) to identify the term of the sequence from the number of elements that are part of it, they must establish the inverse functional relationship to the previous one, by reversing the process.

From the extension and modification of the stages established by Ref. [2] and Ref. [18], four stages of primary students' understanding in the learning of pattern generalization have been identified (Table 1).

Table 1. Understanding stages of pattern generalization.

Stages	Characterization of Stages
Stage 0	- The student does not generalize. - The student is unable to continue the sequence because he/she does not respect the spatial and/or numerical structure or does not perceive the growing pattern. - The students do not coordinate spatial and numerical structures, preventing them from progressing.
Stage 1	- The student performs a near generalization. - The student is able to continue the sequence for near terms because he/she identifies the growing pattern by coordinating spatial and numerical structures. - The students do not relate the term of the figure with its number of elements, which prevents them from generalizing far terms.
Stage 2	- The student makes a far generalization. - The students are able to continue the sequence for far terms because they identify the functional relationship between the figure term and its number of elements, and they are able to establish a general rule to find the number of elements of any given element. - The students do not identify the inverse functional relationship, necessary to find out an element of the sequence from the number of elements, which prevents them from reversing the process.
Stage 3	- The student reverses the process. - The students are able to identify the inverse functional relationship, allowing them to find any term in the sequence from its elements.

When moving from Stage 0 to Stage 1, the student needs to coordinate the spatial and numerical structures to find the growth pattern; to move from Stage 1 to Stage 2, they need to establish the functional relationship between the term of the sequence and the number of its elements, to find the number of elements of any term; and to move from Stage 2 to Stage 3, they need to reverse the process to find any term of the sequence from its elements. Three conceptual advances linked to the mathematical elements that allow the transition between the stages of understanding are thus determined: the coordination between spatial and numerical structures, the recognition of the functional relationship, and the reversibility of the process.

In Figure 1, the progression model is presented within a learning trajectory of pattern generalization, which is carried out in the four stages of understanding and in the three conceptual advances that allow the transition from one stage to a higher one. This progression model is a general model, since "not all students will follow a general sequence, but multiple sequences (often interacting)" [19] (p. 220).

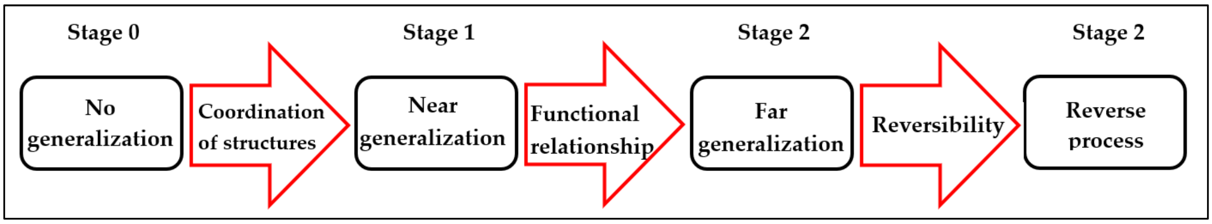
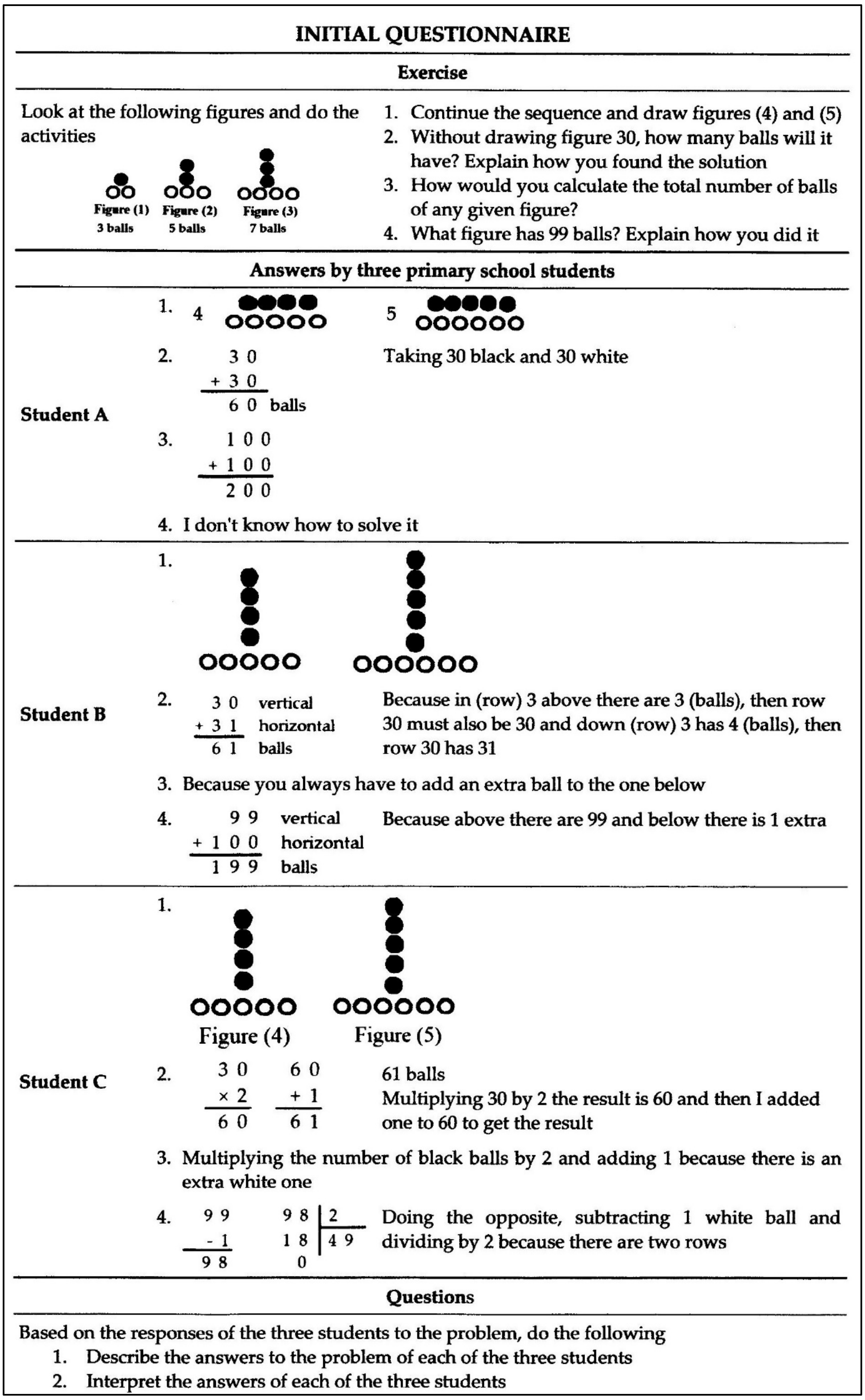


Figure 1. Progression model of pattern generalization.

In this way, the learning trajectory of primary education students in the context of pattern generalization used in this work is defined by:

- Learning objective: the development of algebraic thinking through the generalization of patterns.
- Progression model: composed of four stages of understanding and three conceptual advances.
- Tasks: problems of linear generalization of patterns (near generalization, far generalization, general rule, and process inversion), such as the cases in Figures 2 and 3.



INITIAL QUESTIONNAIRE

Exercise

Look at the following figures and do the activities

Figure (1) 3 balls Figure (2) 5 balls Figure (3) 7 balls

1. Continue the sequence and draw figures (4) and (5)
2. Without drawing figure 30, how many balls will it have? Explain how you found the solution
3. How would you calculate the total number of balls of any given figure?
4. What figure has 99 balls? Explain how you did it

Answers by three primary school students

Student A

1. 4 5
2. 30 + 30 = 60 balls — Taking 30 black and 30 white
3. 100 + 100 = 200
4. I don't know how to solve it

Student B

1.
2. 30 vertical + 31 horizontal = 61 balls — Because in (row) 3 above there are 3 (balls), then row 30 must also be 30 and down (row) 3 has 4 (balls), then row 30 has 31
3. Because you always have to add an extra ball to the one below
4. 99 vertical + 100 horizontal = 199 balls — Because above there are 99 and below there is 1 extra

Student C

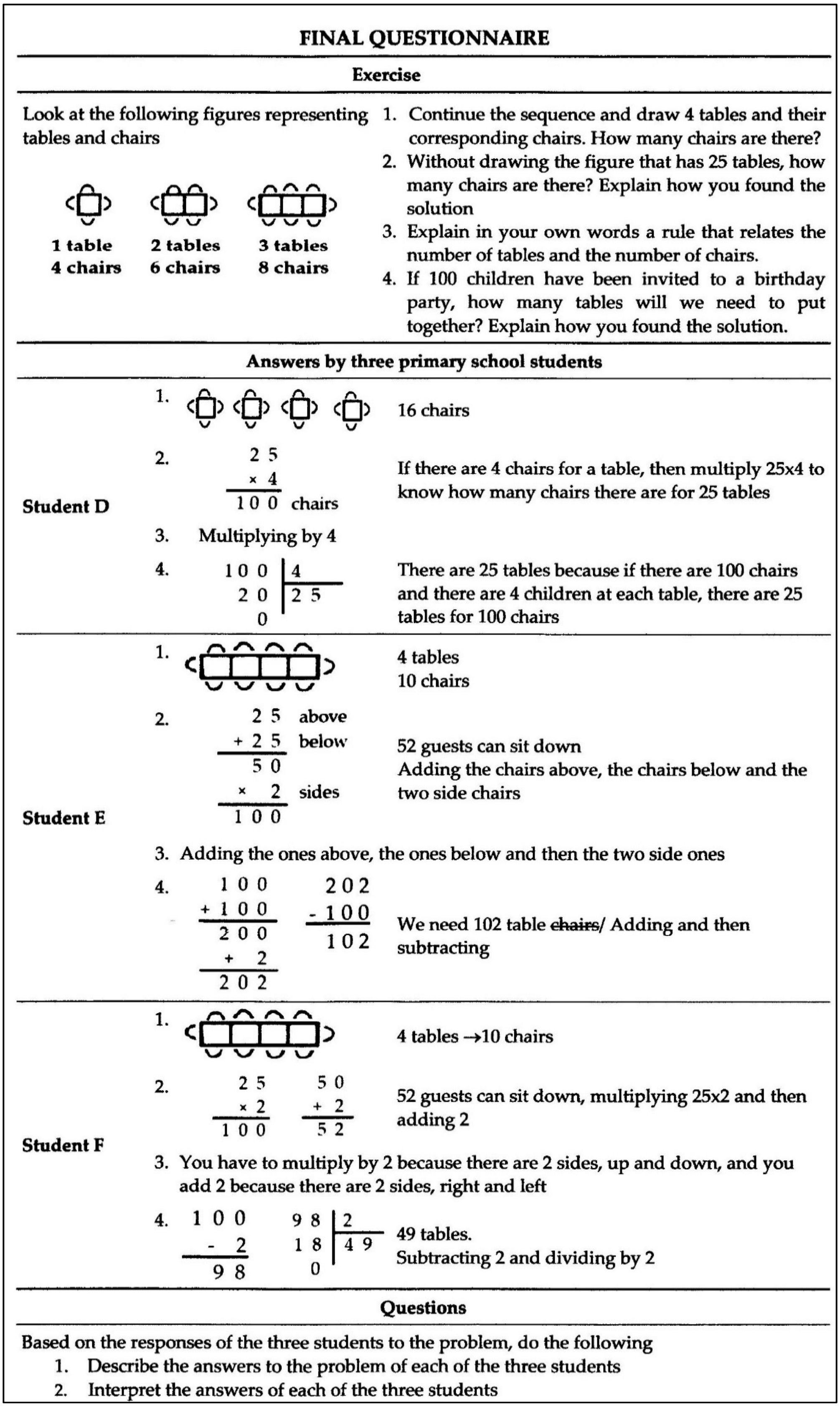
1. Figure (4) Figure (5)
2. 30 × 2 = 60; 60 + 1 = 61 — 61 balls. Multiplying 30 by 2 the result is 60 and then I added one to 60 to get the result
3. Multiplying the number of black balls by 2 and adding 1 because there is an extra white one
4. 99 − 1 = 98; 98 ÷ 2 = 49 (18, 0) — Doing the opposite, subtracting 1 white ball and dividing by 2 because there are two rows

Questions

Based on the responses of the three students to the problem, do the following

1. Describe the answers to the problem of each of the three students
2. Interpret the answers of each of the three students

Figure 2. Initial questionnaire. The students' handwritten answers are transcribed and translated into English for better understanding by the reader.



FINAL QUESTIONNAIRE

Exercise

Look at the following figures representing tables and chairs

1 table, 4 chairs; 2 tables, 6 chairs; 3 tables, 8 chairs

1. Continue the sequence and draw 4 tables and their corresponding chairs. How many chairs are there?
2. Without drawing the figure that has 25 tables, how many chairs are there? Explain how you found the solution
3. Explain in your own words a rule that relates the number of tables and the number of chairs.
4. If 100 children have been invited to a birthday party, how many tables will we need to put together? Explain how you found the solution.

Answers by three primary school students

Student D

1. 16 chairs
2. 25 × 4 = 100 chairs. If there are 4 chairs for a table, then multiply 25x4 to know how many chairs there are for 25 tables
3. Multiplying by 4
4. 100 ÷ 4 = 25 (20, 0). There are 25 tables because if there are 100 chairs and there are 4 children at each table, there are 25 tables for 100 chairs

Student E

1. 4 tables, 10 chairs
2. 25 above + 25 below = 50; × 2 sides = 100. 52 guests can sit down. Adding the chairs above, the chairs below and the two side chairs
3. Adding the ones above, the ones below and then the two side ones
4. 100 + 100 = 200; + 2 = 202; 202 − 100 = 102. We need 102 table ~~chairs~~/ Adding and then subtracting

Student F

1. 4 tables →10 chairs
2. 25 × 2 = 100; 50 + 2 = 52. 52 guests can sit down, multiplying 25x2 and then adding 2
3. You have to multiply by 2 because there are 2 sides, up and down, and you add 2 because there are 2 sides, right and left
4. 100 − 2 = 98; 98 ÷ 2 = 49 (18, 0). 49 tables. Subtracting 2 and dividing by 2

Questions

Based on the responses of the three students to the problem, do the following

1. Describe the answers to the problem of each of the three students
2. Interpret the answers of each of the three students

Figure 3. Final questionnaire. The students' handwritten answers are transcribed and translated into English for better understanding by the reader.

From this perspective, the research question posed in this study is: how do PPTs use, in a teaching module, the understanding of mathematical elements as conceptual advances in a progression model of a learning trajectory, to interpret the mathematical thinking of primary students in the context of pattern generalization?

2. Materials and Methods

2.1. Participants and Context

This study consisted of 18 participants, all PPTs in their third year of the Primary Education Degree taking the subject "Learning and didactics of mathematics". One of the objectives of this course is the development of the professional noticing of students' mathematical thinking. To this end, a teaching module was carried out in the context of pattern generalization.

The objectives of this teaching module were: (1) to provide PPTs with information, based on the results of previous research, on the understanding development of pattern generalization in primary students and (2) to develop the professional noticing of PPTs in relation to the mathematical thinking of students, by using a progression model in a learning trajectory.

The teaching module consists of three different parts and a total of seven one-hour sessions each. The first part was developed in two sessions and focused on solving problems of pattern generalization; the second part, in which four sessions were used, focused on the development of the three skills of the professional noticing; and in the third part, which was developed in one session, the evaluation was carried out. The tasks of the sessions have been adapted from other previous research in which professional noticing is developed in the context of pattern generalization [2,5,18]. Table 2 shows the development of the sessions specifying contents and professional tasks.

Table 2. Contents and tasks of the teaching module sessions.

Phases	Sessions	Contents	Tasks
Problem solving	1	Pattern generalization	- Individual resolution of generalization problems
	2	Pattern generalization: - Problem solving strategies	- Pattern generalization problem solving for large groups and study of problem solving strategies
Development of professional noticing	3	Professional noticing - Identify - Interpret - Decide	- Initial questionnaire: professional questions on the responses of three primary school students to a pattern generalization problem
	4	Identification of skills - Mathematical elements	- Large group analysis of the skill identified in the resolution of some problems by primary school students - Individual reconstruction of the first question of the initial questionnaire (identify)
	5	Interpretation of skills - Stages of understanding - Conceptual advances - Progression model	- Large group analysis of the skill identified in the resolution of some problems by primary school students - Individual reconstruction of the second question of the initial questionnaire (interpret)
	6	Decision of skills - Instructional tasks - Learning trajectory	- Large group analysis of the actions proposed by PPTs that interpreted the answers to some problems by primary school students - Individual reconstruction of the third skill (decide)
Evaluation	7	Pattern generalization Professional noticing	- Final questionnaire: professional questions on the responses by three primary school students to two pattern generalization problems.

During the teaching module, PPTs were provided with information on professional noticing skills, pattern generalization problem solving strategies, meaningful mathematical elements, the learning trajectory of primary students (objective, progression model, and tasks), and the progression model (stages of understanding and conceptual advances).

In sessions 4, 5, and 6 the PPTs reconstructed their responses to the initial questionnaire. These reconstructions allowed them to analyze their own responses, reflect on them, and make the modifications they deem appropriate. Reconstruction, understood in this way, is a conscious and reflective activity achieved through the exchange of ideas and teaching experiences with other participants and that allows building and strengthening knowledge, attitudes, and ways of acting [2].

2.2. Data Collection

In this research, the first two skills of professional noticing are analyzed; that is, the identification of mathematical elements and their subsequent use for the interpretation of the students' understanding, postponing the analysis of the third skill for future research.

The research data were collected at two different times of the teaching test: initial stage (initial questionnaire, session 3) and final stage (final questionnaire, session 7). At the initial moment, the PPTs only had information on the strategies for solving the problems of pattern generalization, while at the final moment, the PPTs analyzed the responses of three other students to a second problem of pattern generalization, based on the information received in the teaching module.

2.2.1. Data Collection at Initial Stage

The data collected at the initial stage are the PPTs' responses to the initial questionnaire proposed in the third session (Figure 2).

The problem statement (Figure 2) presents a situation in which the first terms of an arithmetic progression defined by the affinity function $f(n) = 2n + 1$ are provided. Next, we ask: (1) to calculate the number of balls for a small figure (near generalization), (2) to calculate the number of balls for a large figure (far generalization), (3) to explain a general rule that relates the two variables (functional relationship), and (4) to find the number of the figure that contains a specific number of balls (reverse process).

The professional tasks of the initial questionnaire, proposed to the PPTs, consisted in the analysis of the responses given by three students from the fifth and sixth year of primary education (aged 10–12) to the problem described above. These students had not previously carried out any tasks of pattern generalization. Their answers to this problem correspond to three different stages of understanding:

- Student A is in Stage 0 because, although he/she maintains the numerical structure and establishes the growth pattern, he/she does not respect the spatial structure because the balls are placed in two parallel rows instead of perpendicular rows, which prevents him/her from coordinating the structures; nor does the student establish the functional relationship by considering that the two rows have the same number of balls.
- Student B is in Stage 2 by coordinating structures and establishing the functional relationship between the number of the figure and the total number of balls, allowing him/her to continue the sequence for far terms (far generalization), but the process is not reversed.
- Student C is in Stage 3 by coordinating structures, establishing the functional relationship, and reversing the process, allowing him/her to find the figure number from the total number of balls.

2.2.2. Data Collection at Final Stage

The data collected at the final moment are the PPTs' responses to the final questionnaire proposed in the last session, in which the responses of three other primary students are shown. The final questionnaire, similar to the initial questionnaire, consists of a problem defined by the affinity function $f(n) = 2n + 2$, and the two same professional issues (Figure 3).

The responses of the three new students to the problem of the final questionnaire also correspond to three different stages:

- Student D of the final questionnaire is in Stage 0 by not continuing the sequence and drawing separate tables; this representation does not allow him/her to continue the generalization correctly, although he/she recognizes, in a manner consistent with the erroneous representation, the functional relationship and the inverse process.
- Student E is in Stage 2 by coordinating structures and establishing the functional relationship between the number of tables and the number of chairs, allowing him/her to continue the sequence for distant terms (far generalization), without reversing the process.
- Student F is in Stage 3 by coordinating structures, establishing the functional relationship, and reversing the process; the use of these elements allows him/her to calculate the number of tables from the number of chairs.

2.3. Data Analysis

The data analysis has been carried out in four phases:

- Phase 1: Criteria are characterized, and levels are coded to determine if PPTs are able to identify mathematical elements and use them as conceptual advances to interpret students' understanding at both initial and final stages.
- Phase 2: The results obtained in the two stages are compared.
- Phase 3: PPTs profiles are defined based on the use of mathematical elements as conceptual advances.
- Phase 4: Representative examples of each of the defined profiles are selected and a case study is carried out.

2.3.1. Phase 1: Criteria and Levels of Identification and Use of Mathematical Elements as Conceptual Advances

The following criteria were considered to analyze whether PPTs use mathematical elements as conceptual advances:

- One of the PPTs uses the understanding of the coordination between spatial and numerical structures as conceptual advance when he/she realizes that the student has recognized a regularity between the number of balls and their distribution in problem 1, or the number of chairs and their distribution in problem 2.
- One of the PPTs uses the understanding of the functional relationship as conceptual advance when he/she realizes that the student establishes a relationship that associates the number of balls and the number of the figure in problem 1, or the number of tables with the number of chairs in problem 2.
- One of the PPTs uses the understanding of the inverse process as conceptual advance when he/she realizes that the student is able to find the number of the figure from the number of balls in problem 1, or the number of tables from the number of chairs in problem 2.

Table 3 shows examples of how PPTs give evidence of the use of mathematical elements as conceptual advances in the final moment.

The following levels were established to assess how PPTs identified and used mathematical elements as conceptual advances:

- Level 0: The PPT makes no reference to the mathematical element.
- Level 1: The PPT identifies the mathematical element but does not use it as conceptual advance.
- Level 2: The PPT identifies the mathematical element and uses it as conceptual advance.

Table 4 shows examples of level coding by the PPTs when they analyze the responses of student D in the final questionnaire. The response by one of the PPTs and its characterization are shown for each mathematical element: structure coordination, functional relationship, and reversibility.

Table 3. Examples of evidence of understanding mathematical elements as conceptual advances.

Mathematical Elements	Examples of Using Mathematical Elements as Conceptual Advances
Coordination between structures	Student D "has drawn separate tables so he/she does not adequately follow the distribution of chairs; that is, he/she has not been able to coordinate the spatial with the numerical structures. By failing in the spatial structure, he/she cannot continue with the process of generalization".
Functional relationship	Student E "takes into account that for each table two chairs are added and also takes into account the two permanent chairs on the sides. He/she is able to relate tables to chairs by establishing a functional relationship by adding the chairs above and below and two extra chairs which are the side ones".
Reverse process	Student F "after using the general rule he/she does the opposite, that is, performs the reverse process; in the previous question he/she multiplied by two and added two and now subtracts two and divides by two. He/she is able to realize that he/she is asked to do the opposite so he/she does the reverse operations in reverse order".

Table 4. Examples of level coding.

Level	Element	PPT Response to Professional Task	Level Characterization
0	Structure coordination	The student draws tables poorly	The student only describes or comments generically on the response, but he/she does not refer to the corresponding mathematical element
	Functional relationship	The student multiplies by 4	
	Reverse process	The result is incorrect because he/she divides by 4	
1	Structure coordination	The student draws separate tables, so the spatial structure is not taken into account	The PPT mentions the mathematical element, but it is not used as conceptual advance to interpret the student's understanding
	Functional relationship	Tables and chairs are not properly related	
	Reverse process	The student does not know how to do the reverse process	
2	Structure coordination	He/she does not coordinate the spatial and numerical structures because he/she draws separate tables and there is an ongoing error throughout the problem, so he/she cannot generalize well	The PPT identifies the element and uses it as a conceptual advance in the learning trajectory to interpret the student's understanding
	Functional relationship	As the student drew badly the tables he/she has not been able to generalize well, although he/she has been able to see the relationship between tables and chairs, but from his/her drawing	
	Reverse process	Although the problem is incorrectly resolved because he/she did not respect the spatial structure, he/she has realized that to find the number of tables the process must be reversed and that is why he/she divides by 4 because he/she drew four chairs for each table	

2.3.2. Phase 2: Evaluation of Progress: Comparison of Results at the Initial and Final Stages

To evaluate the progress of the PPTs during the teaching module, we compared the results obtained at the initial and final moments, taking into account the use made by the PPTs of each mathematical element as conceptual advance. The averages and medians of the levels obtained made it possible to observe the progress of the PPTs in the development of professional noticing.

2.3.3. Phase 3: Definition of Profiles

Based on the analysis of the PPTs' responses to the professional tasks of the final questionnaire, four PPT profiles were established, based on the levels reached in the identification of mathematical elements and their use as conceptual advances.

2.3.4. Phase 4: Case Study: Selection of Representative Examples of Each Profile

Once the PPTs were classified into profiles, an instrumental case study of the search and verification of theories was carried out [20], in which PPTs' responses representative of each of the profiles were selected, in which the identification and use of mathematical elements as conceptual advances is verified.

3. Results

The results have been organized into four sections that follow the process marked in the data analysis: in the first section the levels obtained by the PPTs are collected when analyzing the answers of the three students in the two questionnaires, in the second section the results obtained in the two moments are analyzed and compared, in the third section the PPTs are classified into profiles according to the level reached at the final moment, and in the fourth section representative examples of each of the profiles are shown.

3.1. Levels of PPT Identification

Table 5 shows the levels obtained by the PPTs at the initial and final moments in each of the aspects linked to the mathematical elements (coordination between structures, functional relationship, and reversibility) when analyzing the responses of each of the primary students.

Table 5. PPT levels in the identification and use of mathematical elements.

PPT	Coordination of Structures						Functional Relationship						Reversibility					
	A	D	B	E	C	F	A	D	B	E	C	F	A	D	B	E	C	F
	I	F	I	F	I	F	I	F	I	F	I	F	I	F	I	F	I	F
1	1	2	0	2	0	2	0	2	0	2	0	1	0	1	0	1	0	1
2	1	2	0	2	0	2	1	2	0	2	0	2	1	2	0	2	0	2
3	1	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
4	1	2	1	2	1	2	0	1	0	2	1	2	0	1	0	2	1	2
5	1	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
6	0	2	0	2	0	2	0	2	1	2	0	2	0	2	1	2	0	2
7	1	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	0	1
8	1	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
9	1	2	0	2	0	2	0	1	1	2	1	2	0	1	1	2	1	2
10	1	2	1	2	0	2	0	1	0	2	0	2	1	1	1	2	0	2
11	1	2	1	2	1	2	1	2	0	2	0	2	0	2	0	2	0	2
12	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2
13	1	2	1	2	1	2	0	1	1	2	1	2	0	1	0	2	1	2
14	0	2	1	2	0	2	0	1	0	2	0	2	0	1	1	2	1	2
15	1	2	1	2	1	2	1	2	1	2	1	2	0	1	1	2	0	2
16	0	2	1	2	0	2	0	2	1	2	1	2	1	1	1	2	1	2
17	1	2	0	2	0	2	1	2	0	2	0	2	0	2	0	2	0	2
18	0	2	0	2	0	2	0	1	0	1	0	1	0	1	0	1	0	1

By way of example, the PPT-1, when analyzing the coordination of the structures of the student A, is at level 1 at the initial moment and at level 2 when analyzing student D at the final moment. This PPT identified the coordination of structures at the initial moment, but did not show evidence of his/her understanding as conceptual advance (level 1), instead, at the final moment, the PPT was able to use coordination as conceptual advance (Level 2).

3.2. Results from Initial and Final Stages

As seen in Table 1, the results obtained by all the PPTs at the beginning of the teaching module were classified between levels 0 and 1. The PPTs did not identify mathematical elements in students' responses or, in the case of identifying them, they did not show evidence of their use as conceptual advances to interpret students' thinking. At the end of the module, all PPTs were at levels 1 or 2 as they identified the mathematical elements in the responses of the three students and most of the PPTs used them as conceptual advances.

Table 6 shows the arithmetic means and the medians of the levels obtained by the PPTs, taking into account the moment (initial or final), the students (A/D, B/E, and C/F), and the mathematical elements (coordination between structures, functional relationship, and reversibility). The comparison of these values at different times allowed us to define progress in the professional noticing of the PPTs. This progress is also seen in the median values shown in Table 2 in parentheses, which, in addition, coincide with the mean values.

Table 6. Means and medians of the levels obtained by the PPTs at the beginning and at the end.

Student	Coordination				Functional Relationship				Reversibility				Total			
	I		F		I		F		I		F		I		F	
A/D	0.72	(1)	1.94	(2)	0.22	(0)	1.61	(2)	0.17	(0)	1.44	(1)	0.33	(0)	1.66	(2)
B/E	0.39	(0)	1.94	(2)	0.28	(0)	1.89	(2)	0.39	(0)	1.83	(2)	0.35	(0)	1.89	(2)
C/F	0.22	(0)	1.94	(2)	0.28	(0)	1.83	(2)	0.28	(0)	1.83	(2)	0.26	(0)	1.87	(2)
Total	0.44	(0)	1.94	(2)	0.26	(0)	1.78	(2)	0.28	(0)	1.70	(2)	0.33	(0)	1.81	(2)

The progress of the PPTs throughout the experiment is evidenced in the means of the levels obtained at the two moments: 0.33, out of a maximum of 2, at the initial moment and 1.81 at the final moment. In the values of the medians, this progress can be also observed, ranging from 0 at the initial moment to 2 at the final moment.

In relation to the use of mathematical elements as conceptual advances, the following results are inferred:

- At the initial moment, the best identified element was the coordination of structures (1.94 versus 1.78 and 1.70 regarding functional relationship and reversibility, respectively) and the students best analyzed were students A and B (0.33 and 0.35, respectively, versus 0.26 of student C).
- At the end, the best identified element was the coordination of structures (0.44 versus 0.26 and 0.28 regarding functional relationship and reversibility, respectively) and the students best analyzed were students E and F (1.89 and 1.87, respectively, versus 1.66 of student D).

3.3. Characterization of PPT Profiles Linked to Conceptual Advances

From the final moment results reflected in Table 5, four PPT profiles linked to conceptual advances were established:

- Profile 0: PPTs that do not use any mathematical elements as conceptual advances in any of the three students.
- Profile I: PPTs that only use the coordination of structures as a conceptual advance in the three students.
- Profile II: PPTs that use the coordination of structures and the functional relationship as conceptual advances in the three students.
- Profile III: PPTs that use the coordination of structures, functional relationship, and reversibility as conceptual advances in the three students.

These profiles are progressively linked to the conceptual advances that determine the transition between the stages of understanding. In this way, Profile I is associated with the conceptual advance of coordination between structures, Profile II is associated with the advance of the functional relationship, and Profile III is associated with the advance of the reversibility of the functional relationship.

Table 7 shows the classification of PPTs in the four established profiles.

Table 7. PPTs with level 2 in the three students.

Profile	Mathematical Elements in the Three Students	PPT	PPT No.
0	None	7	1
I	Coordination of structures	1, 4, 9, 10, 13, 14, 18	7
II	Coordination of structures and functional relationship	15, 16	2
III	Coordination of structures, functional relationship, and reversibility	2, 3, 5, 6, 8, 11, 12, 17	8
		Total	18

PPT-7 was the only one assigned to Profile 0 because in the analysis of the responses of the three students he/she did not use as conceptual advance any of the mathematical elements; the seven PPTs of Profile I used the coordination of structures as conceptual advance in the three students, but they did not use any of the other two elements as conceptual advance in any of the students; the two PPTs of Profile II used the coordination of structures and the functional relationship as conceptual advances in the three students, but they did not use reversibility in student D; and the eight PPTs of Profile III used the three mathematical elements as conceptual advances in the three students.

Table 8 shows the levels obtained at the end of the module for each of the PPTs, sorted by profiles.

Table 8. Levels obtained by PPTs sorted by profiles.

		Profile 0					Profile 0				Profile 0			Profile 0	
PPT		7			1			4, 9, 10, 13, 14, 18			15, 16			2, 3, 5, 6, 8, 11, 12, 17	
Student-a	D	E	F	D	E	F	D	E	F	D	E	F	D	E	F
Coordination	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
Functional relationship	1	1	1	2	2	1	1	2	2	2	2	2	2	2	2
Reversibility	1	1	1	1	1	1	1	2	2	1	2	2	2	2	2

3.4. Representative Examples of Each Profile

3.4.1. Example Profile 0: PPT-7

PPT-7 is the only one that at the final moment has not reached Level 2 in any of the mathematical elements when analyzing the responses of the three students. PPT-7 identifies the three mathematical elements, but he/she does not use them as conceptual advances to interpret students' thinking (Table 9).

Table 9. PPT-7's responses from Profile 0.

Student	PPT-7's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
D	*The student is in the lowest stage as he/she does not coordinate the numerical and spatial structures and cannot find the functional relationship or the opposite relationship*	1	1	1
E	*The students coordinate spatial and numerical structures and recognize the functional relationship, but not the opposite relationship*	1	1	1

Table 9. *Cont.*

Student	PPT-7's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
F	*The student coordinates structures, recognizes the functional relationship and also the opposite relationship*	1	1	1

3.4.2. Example Profile I: PPT-4

PPT-4, while identifying the three mathematical elements, only uses coordination between structures as conceptual advance to interpret the thinking of the three students (Table 10).

Table 10. PPT-4's responses from Profile I.

Student	PPT-4's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
D	*The student has drawn the tables and chairs incorrectly so he/she is not able to coordinate the spatial and numerical structures and this makes him/her unable to progress or establish the functional relationship and inverse relationship*	2	1	1
E	*The student draws the tables and chairs correctly, so he/she coordinates the two structures [...] he/she is able to establish the functional relationship that allows him/her to find the number of chairs by knowing the number of tables, but he/she is not able to do it in reverse, that is, he/she does not know how to find the number of tables knowing the number of chairs, in other words he/she does not know how to do the reverse process*	2	2	2
F	*As they student draws the tables and chairs well, he/she coordinates the two structures [...] the student can find the function that relates tables and chairs and also the other way around, he/she is able to do the reverse process to find the number of tables from the number of chairs*	2	2	2

3.4.3. Example Profile II: PPT-15

A representative example of Profile II is PPT-15 who identifies the three mathematical elements in the three students and, except for the reversibility of student D, uses them as conceptual advances (Table 11).

Table 11. PPT-15's responses from Profile II.

Student	PPT-15's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
D	*The student has not drawn the tables well so he/she has not respected the spatial structure and this has led him/her to the error of not calculating the number of chairs well (numerical structure) and it has caused him/her to do the whole problem wrong [...] he/she has established a wrong functional relationship that has also led him/her to the error since he/she multiplies by 4 when he/she should have multiplied by 2 (above and below each table) and should have added the two side chairs [...] he/she has divided by 4 to find the reverse process".*	2	2	1

Table 11. *Cont.*

Student	PPT-15's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
E	*As the student draws the tables well and counts the chairs well, he/she coordinates the spatial and numerical structures, which allows him/her to find a rule that relates the tables and the chairs, which means adding the tables above and below and the two side ones, but he/she is not able to reverse the process to find the number of tables when he/she knows the number of chairs.*	2	2	2
F	*The student draws well the tables and chairs, therefore he/she respects and coordinates the structures, which allows him/her to continue the process of generalization, he/she is also able to find the relationship between tables and chairs for what he/she multiplies by two, up and down and adds two, right and left [. . .] The student reverses the process because he/she is able to understand that he/she must do it backwards, subtract the two side chairs and then divide by two".*	2	2	2

3.4.4. Example Profile III: PPT-12

PPT-12 belongs to Profile III since he/she identifies and uses the three mathematical elements as conceptual advances in the three students (Table 12).

Table 12. PPT-12's responses from Profile III.

Student	PPT-12's Responses	Elements		
		Structure Coordination	Functional Relationship	Reverse Process
D	*The student has drawn separate tables, so we see that there is no spatial structure [. . .] so he/she cannot reach a correct result nor a correct formula to find the number of chairs for any number of tables [. . .] he/she is not able to do this inverse process and does not identify the inverse relationship [. . .] he/she is in stage 0*	2	2	2
E	*The student controls the spatial and numerical structure, he/she knows how to continue the series and draw it [...] although he/she does not explicitly find the functional relationship he/she has reached a relationship between tables and chairs since he/she adds the ones above, those below and those on the sides [...] he/she is not able to carry out the reverse process, because he/she does not know that he/she is asked to do the opposite and he/she uses the same functional relationship [...] the student is in stage 2".*	2	2	2
F	*The student establishes the spatial and numerical structure, correctly representing the series [...] he/she has reached a functional relationship that is used to calculate the chairs for any number of tables, by the formula of multiplying the number of tables by 2 and adding the 2 side chairs [...] "the student reverses the operations, does the opposite ones, first subtracts 2 and then divides by 2, that is, is able to identify the inverse relationship [...] the student is in the last stage of generalization".*	2	2	2

4. Discussion and Conclusions

The objective of this research is to characterize how the identification and understanding of mathematical elements as conceptual advances in a learning trajectory, will help PPTs develop the professional noticing of primary students' mathematical thinking in the context of pattern generalization.

The results obtained have led to three sets of conclusions: (1) the teaching modules improve the professional noticing of the participants, (2) the recognition of conceptual advances helps interpret the mathematical thinking of the students, and (3) the learning trajectories are effective tools for the development of the professional noticing. Finally, a future prospective is made.

4.1. The Teaching Modules Improve Professional Noticing

The starting hypothesis was that professional noticing can be improved with experience [9] and can be learned in appropriate teaching environments [21]. The results of the teaching module confirm this hypothesis having observed remarkable progress in the development of the professional noticing of the PPTs. This progress has shown that the average level of identification and use of mathematical elements as conceptual advances has gone from 0.33, over a maximum of 2, at the initial moment, to 1.81 at the end.

At the beginning, PPTs did not identify mathematical elements, or identified them implicitly, but in no case did they use them to interpret students' understanding. However, in the final moment, all PPTs explicitly identified mathematical elements and most of them used them as conceptual advances to interpret student understanding. Some PPTs, however, considered that the understanding of student D (Stage 0) was sufficiently defined with the non-coordination of the spatial and numerical structures that prevented him/her from advancing in the progression model of the learning trajectory.

4.2. The Recognition of Conceptual Advances Helps Teachers Progress in the Interpretation of Students' Mathematical Thinking

The teaching module has made it possible to verify that to advance in professional noticing and interpret the mathematical thinking of students, it is important to identify the mathematical elements and use them as conceptual advances. This implies that PPTs must have knowledge of mathematical content [4,7,8].

The PPTs, in the final moment, have recognized more the conceptual advances linked to the coordination of structures than those linked to the functional relationship and the reversibility of the process.

It has also been observed that the recognition of mathematical elements as conceptual advances that determine the transition between stages of the learning trajectory helps PPTs to better understand mathematical concepts and contents [1,5,22].

4.3. Learning Trajectories Are Effective Tools for the Development of Professional Noticing

The gradation of mathematical elements and the conceptual advances linked to them justify the importance of the progression model of the learning trajectory. It has been proven that the progression model on which this study is based has helped PPTs develop their professional noticing, directing their attention towards the relevant aspects of students' mathematical thinking [10,11].

In this way, the PPTs that have best interpreted the mathematical thinking of the students have recognized the conceptual advances that allow them to improve in the progression model and many of them have managed to place the primary students in each of the stages of understanding. This allows us to infer that knowledge of progression models and learning trajectories help interpret students' understanding.

4.4. Prospects for the Future

This study has shown that professional noticing can be improved with experience and can be developed in appropriate learning environments. Therefore, in order to increase the teaching competence of future teachers and in-service teachers, training and professional development programs should include tasks that develop professional noticing skills.

The results obtained in the use of the progression model of the learning trajectory and in the recognition of conceptual advances motivate interest in research in mathematical education regarding these aspects.

For future research on the evolution of the professional noticing in a teaching module on the pattern generalization, it is proposed: (1) to increase the sample of PPTs for a more effective generalization; (2) to analyze the reconstructions that the PPTs carry out from their initial interpretations; (3) to analyze the third skill of professional noticing, that is, the decision-making to improve the teaching-learning process; (4) to deepen some drawbacks and doubts that have arisen in the module, such as teaching the progression model to students who do not coordinate the structures but are able to obtain a functional relationship and invert the process based on a wrong representation; and (5) to carry out individual interviews with PPTs to qualify some interpretations.

The documents and constructs used in the teaching module (questionnaires, mathematical elements, stages of understanding, conceptual advances, progression model, learning trajectory, etc.) can be used as support for tasks related to the development of the professional noticing competence in prospective teachers and in-service teachers.

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