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Graphs of Stable Gauss Maps and Quine’s Theorem for Oriented 2-Manifolds

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Abstract: In this paper we will study the topology of the singular set of stable Gauss maps from closed orientable surfaces immersed in the 3-space from a global point of view and we will obtain a relationship between the Euler Characteristic of the regular components and the signs of the cusps.

Keywords: stable Gauss maps; cusps of Gauss maps; parabolic curve degree; Euler Characteristic

MSC: 57R45; 57R65; 37E25



Citation: Mendes de Jesus, C.; Romero, P.D. Graphs of Stable Gauss Maps and Quine’s Theorem for Oriented 2-Manifolds. *Mathematics* **2022**, *10*, 3104. <https://doi.org/10.3390/math10173104>

Academic Editors: Marian Ioan Munteanu and Alexander Felshtyn

Received: 21 July 2022

Accepted: 23 August 2022

Published: 29 August 2022

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1. Introduction

A geometrical description of surface curvature is provided by the Gauss map. This concept has several applications, we will highlight the relationship with visibility or accessibility maps that are used to develop algorithms for geometric applications to robotics path planning, numerical matching of a surface, mould design, and computer vision as shape operators [1–3].

From the point of view of Singularities Theory, a classical problem is to search invariants to classify stable maps, which allows variable changes in the domain and the image of the maps through diffeomorphisms.

The singularities of a stable Gauss map of a surface generically immersed in 3-space, in Whitney’s sense, was described in [4]. The fold curves with isolated cusp points, are the parabolic set that configures the singular set. Each of these parabolic curves divides a hyperbolic region from an elliptic region of the surface.

In order to investigate the global classification of the topological type of its regular set of stable Gauss maps, graphs of stable Gauss maps were introduced in [5] to provide a combinatorial description of the topology of the singular set, separating the regular regions. This graph, with weights on its vertices, describes the position of the singular and regular sets on the surface (see Figure 1). In [6], the authors studied the singularities of a stable Gauss map of a surface generically immersed in Euclidean 3-space.

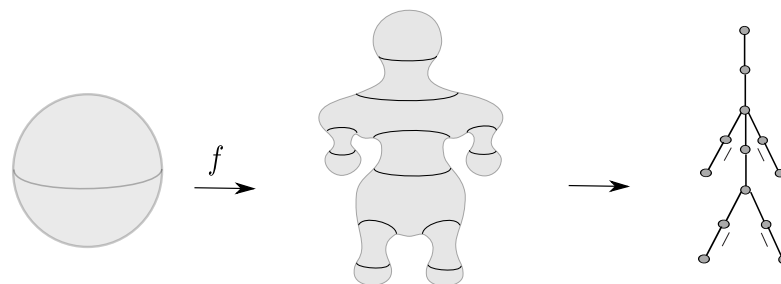


Figure 1. Example of graphs of stable Gauss maps from sphere.

Quine used differential topology techniques to prove the Theorem 4 for the case of stable maps between two closed surfaces, using differential topology techniques. We present a new proof of Theorem 4 using graphs of maps between closed and oriented surfaces, introduced in [7]. In this paper we extend the results obtained in [8] to the case of stable Gauss maps, achieving results comparable to those obtained for stable maps in the sphere.

The aim of this paper is to study the Gauss maps of closed orientable surfaces immersed in the 3-space, and to obtain and derive a relation between the Euler Characteristic, cusps, and the degree of stable Gauss maps (Theorem 4). We will use an inductive constructive process to prove this result, beginning with simple graphs of well-known examples and applying suitable codimension one transition (see [9]). The codimension one transitions of their singularities are determined by studying the height function families associated with generic 1-parameter embedding families.

To properly understand the action of such a transition, we will separate beak transitions in different cases: when the cusps number increases, there are changes in the regular components (positive or negative) or genus of the regular components (positive or negative).

This paper is organised as follows: in Section 2, we introduce the notation, definitions and basic preliminary results related with the stable Gauss maps. In Section 3, we will study beak and lip transitions and how it affects to the regular set. Finally in Section 4, we will show the proof of the Theorem 4.

2. Graphs of Stable Gauss Maps

We will denote by M a connected, smooth and closed orientable surface. In the sense of Guillemin–Gulubisk ([10]), we say that two smooth maps $f, g \in C^\infty(M, N)$ are **A-equivalent** if there are orientation-preserving diffeomorphisms, l and k , such that $g \circ l = k \circ f$. We say that a map $f \in C^\infty(M, N)$ is **stable** if all maps $g \in C^\infty(M, N)$ are sufficiently close to f (in the Whitney C^∞ -topology) and are equivalent to f . A point of the source surface M is a **regular** point of a stable map f if f is a local diffeomorphism around that point and **singular** otherwise.

The singular set of a stable map f , denoted by Σf , according to Whitney (see [10]), consists of curves of fold points, possibly containing isolated cusp points and the branch set $f(\Sigma f)$, consists of a collection of curves immersed in N with possible isolated cusps and whose self-intersections (double points).

The immersion $f : M \rightarrow \mathbb{R}^3$ and its perturbation defines the concept of stability for a Gauss map.

Let be M a closed orientable surface in \mathbb{R}^3 and let $\mathcal{N}_f, \mathcal{N}_g : M \rightarrow \mathbb{S}^2$ be two Gauss maps associated to the maps $f, g \in C^\infty(M, \mathbb{R}^3)$.

In terms of Guillemin–Gulubisk ([10]), when there is a correspondence between the height function of the maps \mathcal{N}_f and \mathcal{N}_g then we say that the functions are A-equivalents ([11]).

The **regular points** of \mathcal{N}_f are elliptic or hyperbolic points of M and **singular points** of \mathcal{N}_f are parabolic points of M , i.e., points where the height function in the normal direction has a non-stable singularity. In this case, we may have: **fold point** of \mathcal{N}_f , corresponding to a A_2 singularity of the height function in the normal direction or **cusp point** of \mathcal{N}_f when the height function in the normal direction has a singularity of type A_3 (see [6,9]).

Definition 1 ([6]). **A Gauss fold map** is a stable Gauss map with zero cusp points.

The singular set of \mathcal{N}_f , denoted by $\Sigma \mathcal{N}_f$, is the parabolic set of M associated to the immersion f . The non-singular set (elliptic or hyperbolic points), which is immersed in the surface \mathbb{S}^2 by the map \mathcal{N}_f , consists of a finite number of **regions**.

Figure 2 displays three stable Gauss maps of the torus with two non-singular regions: (a) the map shows the elliptical region with genus 1 and the parabolic curve has 6 cusp points: in (b), the elliptical and hyperbolic regions have zero genus and the parabolic curves

have 4 cusp points; in (c) the hyperbolic region has genus 1 and the parabolic curve has 6 cusp points (see [4]).

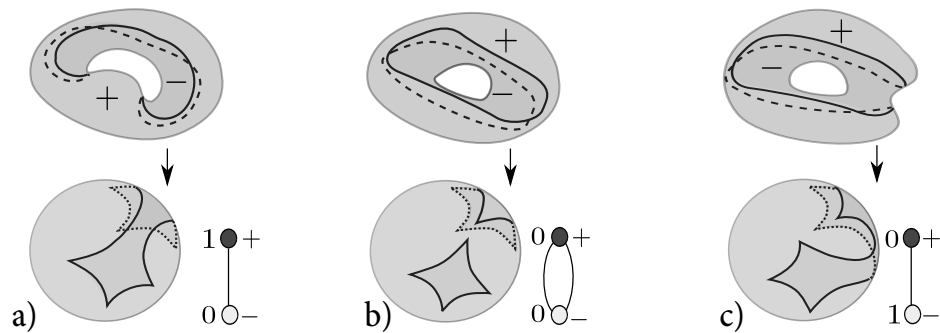


Figure 2. Example of graphs of stable Gauss maps from torus.

If M is a closed surface, then the singular set $\Sigma\mathcal{N}_f$, of a stable Gauss map $\mathcal{N}_f : M \rightarrow \mathbb{S}^2$, is a finite number of closed parabolic curves which separate the elliptical regions from hyperbolic regions in M .

The singular sets of two equivalent maps are equivalent in the sense that there is a diffeomorphism carrying one singular set onto the other and similarly for the branch sets. Thus, any diffeomorphism invariant of singular sets or branch sets will automatically be a topological invariant of the map. Clearly, both the number of connected components of the singular set and the topological types of the regions are topological invariants.

This information was coded in a weighted graph in [5,6], where the pair $(M, \Sigma\mathcal{N}_f)$ may be reconstructed (up to diffeomorphism).

We build a weighted graph from a stable Gauss map \mathcal{N}_f (see Figure 2) (following [6]) identifying the following characteristics which are from a graph (see Figure 2):

1. The edges correspond to the path-components of the parabolic set of M .
2. The vertices correspond to the different regions of the surface with a non-vanishing Gaussian curvature.
3. A weight is defined as the genus of the region that it represents and it is attached to each vertex.
4. A vertex has positive (or negative) label depending on whether the region that it represents has positive (or negative) Gaussian curvature.

Notation: We refer to this graph as $\mathcal{G}(V, E, W)$, where E, V , and W correspond, respectively, the number of edges, the number of vertices, and the total sum of the weights in the vertices.

The maps in Figure 2 correspond to the following graphs: (a) and (c) are $\mathcal{G}(2, 1, 1)$. In (b) is $\mathcal{G}(2, 2, 0)$. If $\mathcal{N} : M \rightarrow \mathbb{S}^2$ is a stable Gauss map without singular point then \mathcal{N} is equivalent to a trivial map. As a result, the graph of this map has a unique vertex with weight null.

In [5], the following result was proved.

Theorem 1. Any weighted bipartite graph $\mathcal{G}(V, E, W)$ can be realized by a stable Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$. where M is a (connected) closed orientable surface in \mathbb{R}^3 with genus $g(M) = 1 - V + E + W$.

As it was seen in [5], if M is orientable, each point of the parabolic set is on the border of both a positive region of $M_{\mathcal{N}}^+$ and of a negative region in $M_{\mathcal{N}}^-$, and consequently, the associate graph is bipartite.

Corollary 1. The Euler Characteristic of M is given by $\chi(M) = 2(\chi(\mathcal{G}) - W)$, where $\chi(\mathcal{G})$ denotes the Euler number of \mathcal{G} .

Given a Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$, we denote by $M_{\mathcal{N}}^+$ ($M_{\mathcal{N}}^-$, respectively) the union of all the elliptical (respectively hyperbolic) regions including their boundaries. Clearly, $M_{\mathcal{N}}^+$ and $M_{\mathcal{N}}^-$ meet at their common boundary, the singular set of \mathcal{N}_f , i.e., any singular curve of $\Sigma\mathcal{N}_f$ lies on the border of a component of M^+ and a component of M^- .

Let us denote by $V_{\mathcal{N}}^+$ ($V_{\mathcal{N}}^-$, respectively) the number of vertices, corresponding to the components of $M_{\mathcal{N}}^+$ ($M_{\mathcal{N}}^-$, respectively), by W^+ (W^- , respectively) the total sum of the weights in the vertices with positive (negative, respectively) label, correspond to the genus of $M_{\mathcal{N}}^+$ ($M_{\mathcal{N}}^-$, respectively).

Definition 2. Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ be a stable Gauss map. The Euler Characteristic of $M_{\mathcal{N}}^{\pm}$ will be denoted by $\chi(M_{\mathcal{N}}^{\pm})$ and the difference will be denoted by

$$\theta_{\chi}(\mathcal{N}) = \chi(M_{\mathcal{N}}^+) - \chi(M_{\mathcal{N}}^-).$$

Definition 3. Let $\mathcal{G}(V, E, W)$ be a graph associated to a stable Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$, where M is a closed orientable connected surface in \mathbb{R}^3 . By using the above notations we denoted by

1. $\theta_V(\mathcal{N}) = V^+ - V^-$, is the difference between the number of positive vertices and the number of negative vertices.
2. $\theta_W(\mathcal{N}) = W^+ - W^-$, is the difference between the total positive and total negative weight,
3. $\beta_1(\mathcal{G}) = 1 - V + A$, is the number of cycles of $\mathcal{G}(V, E, W)$.

Lemma 1. If $\mathcal{N} : M \rightarrow \mathbb{S}^2$ is a Gauss fold map with degree $d(\mathcal{N})$ then $\theta_{\chi}(\mathcal{N}) = 2d(\mathcal{N})$.

Proof. The degree of a Gauss fold map $\mathcal{N} : M \rightarrow \mathbb{S}^2$ is given by $d(\mathcal{N}) = 1 - g(M) \leq 1$ ([5,12]). Since M is closed, then \mathcal{N} is surjective. Firstly, if $\Sigma\mathcal{N} = \emptyset$, then $M = M^+$ and $\theta_{\chi}(\mathcal{N}) = \chi(M^+) = \chi(\mathbb{S}^2) = 2$. Let us suppose that $\Sigma\mathcal{N} \neq \emptyset$. Then the sets M^- and M^+ are non-empty. We denote by $S_0 = \mathbb{S}^2 \setminus \mathcal{B}\mathcal{N}$, The set formed by the connected regions in the complement of $\mathcal{N}(\Sigma\mathcal{N})$, image of the parabolic set. We denote by $M_0^{\pm} = \mathcal{N}^{-1}(S_0) \cap M^{\pm}$, the set formed by the inverse images of the regions of S_0 on M^{\pm} .

If $d(\mathcal{N}) > 0$, then for each region R_i^- contained in M_0^- , exists a copy of R_i^- contained in M_0^+ , that we denoted by R_i^+ . We have $\chi(R_i^+) = \chi(R_i^-)$ and $M^- = \cup_i R_i^-$.

The union of the regions $M^+ \setminus \cup_i R_i^+$ on \mathbb{S}^2 , forms d regions that cover \mathbb{S}^2 . Then $\chi(M^+ \setminus \cup_i R_i^+) = 2d$. In this case, $d = 1$ and $M = \mathbb{S}^2$.

If $d(\mathcal{N}) \leq 0$, for each region R_i^+ contained in M_0^+ , there is a copy of R_i^+ contained in M_0^- , that we denoted by R_i^- and $M^+ = \cup_i R_i^+$. Then $\chi(M^- \setminus \cup_i R_i^-) = -2d$.

In all cases, we have $\chi(M^+) - \chi(M^-) = 2d(\mathcal{N})$ and the result follows from the Definition 2, $\theta_{\chi}(\mathcal{N}) = \chi(M^+) - \chi(M^-)$. \square

Lemma 2. Let $\mathcal{G}(V, E, W)$ be a graph of a stable Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$. Then $\theta_{\chi}(\mathcal{N}) = 2(\theta_V(\mathcal{N}) - \theta_W(\mathcal{N}))$.

Proof. The Euler number of M_i^{\pm} is given by $\chi(M_i^{\pm}) = 2 - 2w_i^{\pm} - E_i^{\pm}$, where w_i^{\pm} and E_i^{\pm} denote (respectively) the genus and the number of boundary components for each surface M_i^{\pm} . Thus, $\chi(M^{\pm}) = \sum_{i=1}^{V^{\pm}} \chi(M_i^{\pm}) = 2V^{\pm} - 2W^{\pm} - E$. Then, it follows from Definition 2, $\theta_{\chi}(\mathcal{N}) = \chi(M^+) - \chi(M^-) = 2(\theta_V - \theta_W)$. \square

The next result provides the necessary condition for a given graph to be associated to a fold Gauss map from a closed and orientable surface.

Theorem 2. Let $\mathcal{G}(V, E, W)$ be a graph of a fold Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$. Then $d(\mathcal{N}) = \theta_V - \theta_W$.

Proof. It follows immediately from Lemmas 1 and 2. \square

Corollary 2. *If \mathcal{N} is a Gauss fold map, then $1 - g(M) = \theta_V - \theta_W$. Consequently, $2(V^- - W^-) = E$.*

3. Beaks and Lips Transitions

Our purpose in this section is to study codimension one transitions and their effects on the topology of the regular set and the singular set from a global point of view.

Definition 4. *A cusp point $x \in \Sigma\mathcal{N}$ is called positive (respectively negative) if its local mapping degree, in a neighbourhood U_x of x is $+1$ (-1 , respectively) with a fixed orientation.*

Definition 5. *Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ be a stable Gauss map. Consider the next global invariants of \mathcal{N}*

- I_E : *is the number of parabolic curves.*
- I_{C^+} : *is the number of cusps with positive sign.*
- I_{C^-} : *is the number of cusps with negative sign.*
- I_{V^+} : *is the number of elliptic regions.*
- I_{V^-} : *is the number of hyperbolic regions.*
- I_{W^+} : *is the total sum of genus of elliptic regions.*
- I_{W^-} : *is the total sum of genus of hyperbolic regions.*

Let I be an invariant of stable Gauss maps. If $\mathcal{N}, \mathcal{N}_1 : M \rightarrow \mathbb{S}^2$ are two stable Gauss maps. Then \mathcal{N} and \mathcal{N}_1 can be joined by a generic homotopy, in the sense that it only passes through codimension one transitions, because the two maps have the same degree that depends on $g(M)$. The difference $I(\mathcal{N}) - I(\mathcal{N}_1)$ is called increment of I and denoted by

$$\theta_I(\mathcal{N}) = I(\mathcal{N}) - I(\mathcal{N}_1).$$

Then, we can compute $\theta_I(\mathcal{N})$ starting from $\theta_I(\mathcal{N}_1)$ and add the increment ΔI along the path from \mathcal{N}_1 to \mathcal{N} . The total sum of all the increment of I along a path between two stable Gauss maps that only passes through codimension one transitions is denoted by $\Delta I = \sum_i \Delta I(\eta_i)$, where $\eta_1 \dots \eta_k$ denotes a sequence of co-dimension one transitions from \mathcal{N}_∞ to \mathcal{N} . These transitions can change the numbers of cusps, double points, singular curves, and regular regions.

Definition 6. *Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ be a stable Gauss map of a closed orientable surface M immersed in \mathbb{R}^3 . Let us denote by C^+ (C^- , respectively) the number of positive cusp points (negative cusp points, respectively) and by $C = C^+ + C^-$ the total number of cusps of \mathcal{N} . The difference between the number of cusp points positive and negative will be denoted by $\theta_C(\mathcal{N}) = C^+ - C^-$.*

Let M be a closed orientable surface immerse in \mathbb{R}^3 and \mathcal{N} its corresponding Gauss map.

(i) **Lips transitions:** The generic transitions in one-parameter families of height functions (defined by generic one-parameter families of embeddings) together with their effects on the corresponding Gauss maps have been described in [9] both in the local and multi-local situation (see [6] for more information). The lips transition corresponds to a Morse transition of maximum or minimum type in the parabolic curve.

It may be done in a region with a positive curvature (or negative curvature) X of M , giving rise to a new region with a negative (positive) curvature of Z . Their common boundary is a connected component of the parabolic set, whose image given by the Gauss map is a closed curve with two cusp points on \mathbb{S}^2 .

Following the notation of [6,8], we denote the lips transition by $L_{v\pm}^\gamma$, where γ corresponds to the sign (“+” if positive and “-” if negative) of the pair of cusps that are born with the lips transition (see Figure 3). We introduce the subindex $v\pm$ to indicate if the number of vertices are being changed V^\pm . The transition $L_{v\mp}^\pm$ occurs in a region $X \subset M^\pm$ and always increases the number of singular components and the number of regular components of M^\mp . The effect of this transition $L_{v\mp}^\pm$ on the graph of \mathcal{N} is to add

a new edge in E and a new vertex in V^\mp , which corresponds with the initial region, now renamed X_1^\mp .



Figure 3. Lips transitions.

(ii) **The beaks transition**, corresponds to a Morse transition of saddle type in the parabolic set. Such a transition occurs as we approach two arcs of the parabolic set until they meet at a common point (*beaks point*) and break, giving a new pair of arcs. As a result, a couple of cusp points are introduced in the branch set. This procedure, from the global point of view, increases the cusp points.

Following the notation of [6,8], a subindex \pm was added to indicate if the number of vertices V^\pm or the weight W^\pm are being changed.

Beak transitions (locally) can be categorised into eight distinct cases, as shown in Figure 4. Table 1 displays the increments of edges, numbers of cusps, vertices, and weight according to lip transitions and beak transitions.

1. $B_{v\pm}^{+\pm}$: the transition increases by 1 the number of vertices in V^\mp and the number of edges E .
2. $B_{v\mp}^{-\pm}$: the transition decreases by 1 the number of vertices in V^\pm and the number of edges E .
3. $B_{w\pm}^{+\pm}$: the transition increases by 1 the weight in W^\pm and decreases by 1 the number of edges E .
4. $B_{w\mp}^{-\pm}$: the transition decreases by 1 the weight in W^\mp and increases by 1 the number of edges E .

Let us remark that the genus of the surface is a constant, $(V^+ - W^+) + (V^- - W^-) = 1 - g(M) + E$. As all transitions change E , so the left side will always change.

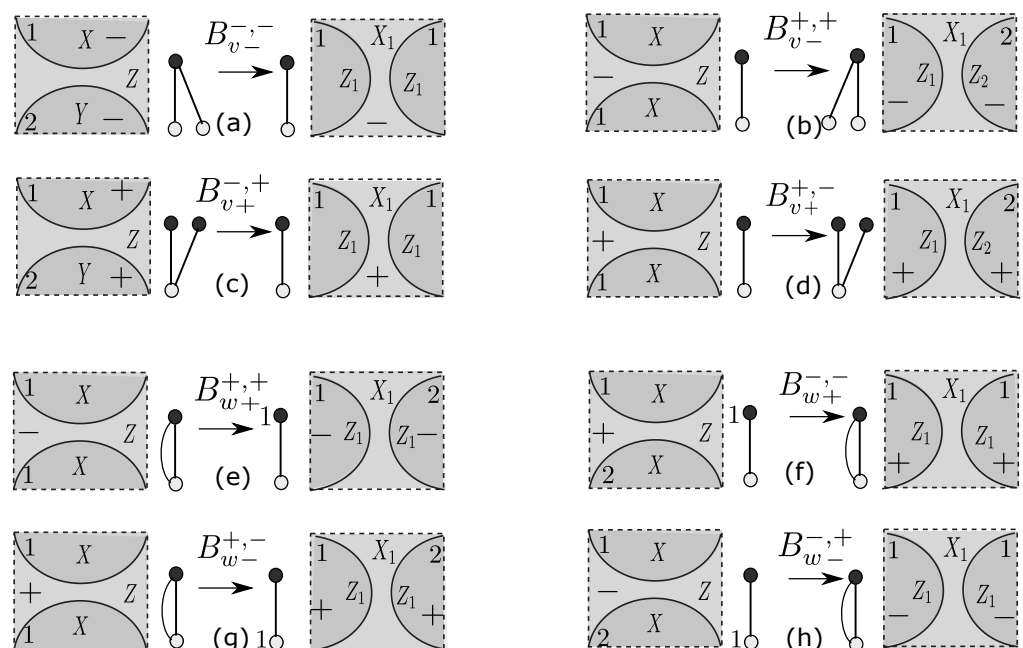


Figure 4. Beaks transitions in M . (a–h) Regions X , X_1 , Y , Z , Z_1 and Z_2 separated by the curves arcs 1 and 2 denote (locally) the regular regions where the transitions hold. The numbers 1 and 2 represents the number of singular curves. If the two arcs have the same number, it shows that the arcs belong to the same singular curve.

Table 1. Increments of lips and beaks transitions.

Trans	Invar	L_{v-}^+	L_{v+}^-	$B_{v-}^{-,+}$	$B_{v-}^{+,-}$	$B_{v+}^{-,+}$	$B_{v+}^{+,-}$	$B_{w+}^{-,+}$	$B_{w+}^{+,-}$	$B_{w-}^{-,+}$	$B_{w-}^{+,-}$
	ΔE		1	1	-1	1	-1	1	-1	1	-1
ΔC^+		2	0	0	2	2	0	2	0	0	2
ΔC^-		0	2	2	0	0	2	0	2	2	0
ΔV^+		0	1	0	0	-1	1	0	0	0	0
ΔV^-		1	0	-1	1	0	0	0	0	0	0
ΔW^+		0	0	0	0	0	0	1	-1	0	0
ΔW^-		0	0	0	0	0	0	0	0	1	-1

A consequence of Table 1 is the following result:

Proposition 1. Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ a stable Gauss map, then:

- (i) $\theta_C(\mathcal{N}) + \theta_V(\mathcal{N}) + \theta_W(\mathcal{N}) = 1 \pmod 2$.
- (ii) $I_E + I_{C^+} + I_{C^-} + I_{V^+} + I_{V^-} + I_{W^+} + I_{W^-} = 0 \pmod 2$.

Figure 5a–d shows examples about the effects of beaks and lips transitions. Let us remark, the map (d) is obtained only passing from L^+ to $-B_{v+}^{-,+}$. Figure 5e–h illustrates beaks transitions in a sequence of stable Gauss maps on the torus, where the first and the last map are Gauss fold maps. The first with 4 singular curves, separating two elliptic areas of two hyperbolas, and it finishes with two singular curves that separate two areas homeomorphic to the cylinder (see [6]).

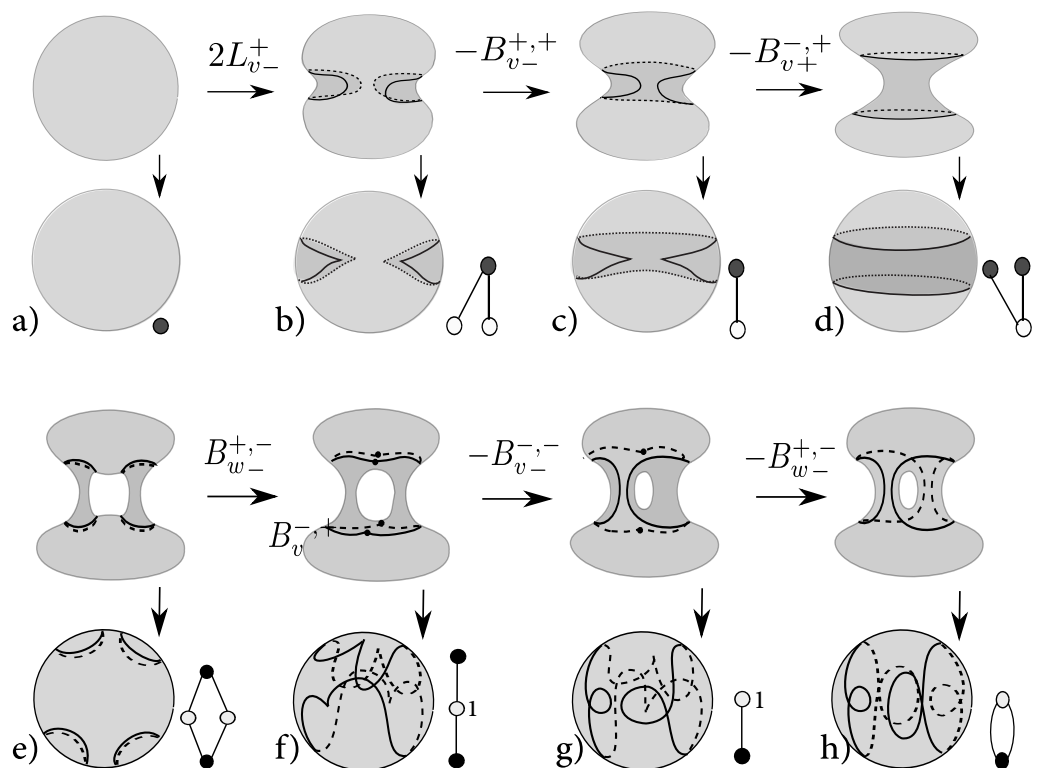


Figure 5. Example of beaks and lips transitions.

Let $\zeta \in \{L_{v\mp}^{\pm}, B_{v\pm}^{\pm,\pm}, B_{w\pm}^{\pm,\pm}\}$ a codimension one transition. We denote by ζ the increment corresponding to ζ along the path that connects two stable Gauss maps.

Analogous to the case of maps between surfaces (see [8]), we have the following results for Gauss maps.

Lemma 3. *If \mathcal{N}_1 and \mathcal{N}_2 are two stable Gauss maps, then the increments of the number of cusps, of the number of parabolic curves, elliptic and hyperbolic regions, of the total genus of the regions, along with a path between two stable Gauss maps, are given by*

$$\begin{aligned} \Delta C^\pm &= 2(|B_{v\mp}^{+\pm}| + |B_{v\pm}^{-\pm}| + |B_{w\pm}^{+\pm}| + |B_{w\mp}^{-\pm}| + |L_{v\mp}^\pm|), \\ \Delta E &= (|L_{v-}^+| + |L_{v+}^-| + |B_{v-}^{+,+}| + |B_{v+}^{+,-}| + |B_{w+}^{-,-}| + |B_{w-}^{-,+}|) - (|B_{v+}^{-,+}| + |B_{v-}^{-,-}| \\ &\quad + |B_{w+}^{+,+}| + |B_{w-}^{+,-}|), \\ \Delta V^\pm &= |L_{v\pm}^\mp| + |B_{v\pm}^{+, \mp}| - |B_{v\pm}^{-, \mp}|, \\ \Delta W^\pm &= |B_{w\pm}^{+, \mp}| - |B_{w\pm}^{-, \mp}|. \end{aligned}$$

Proof. Equalities can be deduced from Table 1. \square

Lemma 4. *If \mathcal{N}_1 and \mathcal{N}_2 are two stable Gauss, then the increments of θ_C , θ_V , and θ_W are given by*

$$\begin{aligned} \Delta\theta_C(\mathcal{N}) &= 2(|L_{v-}^+| + |B_{v-}^{+,+}| + |B_{v+}^{-,-}| + |B_{w+}^{+,+}| + |B_{w-}^{-,-}|) - 2(|L_{v+}^-| + |B_{v+}^{+,-}| + |B_{v-}^{-,-}| \\ &\quad + |B_{w-}^{+,+}| + |B_{w+}^{-,-}|), \\ \Delta\theta_V(\mathcal{N}) &= (|L_{v+}^-| + |B_{v+}^{+,-}| + |B_{v-}^{-,-}|) - (|L_{v-}^+| + |B_{v-}^{+,+}| + |B_{v+}^{-,-}|), \\ \Delta\theta_W(\mathcal{N}) &= (|B_{w+}^{+,+}| + |B_{w-}^{-,-}|) - (|B_{w-}^{+,+}| + |B_{w+}^{-,-}|), \\ \Delta\theta_\chi(\mathcal{N}) &= 2[\Delta\theta_V(\mathcal{N}) - \Delta\theta_W(\mathcal{N})] = -\Delta\theta_C(\mathcal{N}). \end{aligned}$$

Proof. Let us remark $\Delta\theta_C = \Delta C^+ - \Delta C^-$, $\Delta\theta_V = \Delta V^+ - \Delta V^-$ e $\Delta\theta_W = \Delta W^+ - \Delta W^-$). The result is directly deduced from Lemmas 2 and 3. \square

Theorem 3. *Let $\mathcal{N}_1, \mathcal{N} : M \rightarrow \mathbb{S}^2$ be two stable Gauss maps. If \mathcal{N}_1 is a fold Gauss then*

$$\theta_\chi(\mathcal{N}) + \theta_C(\mathcal{N}) = \theta_\chi(\mathcal{N}_1).$$

Proof. If \mathcal{N}_1 is a fold Gauss map then $\theta_C(\mathcal{N}_1) = 0$ and $\theta_C(\mathcal{N}) = \Delta\theta_C(\mathcal{N})$. The equality holds $\theta_\chi(\mathcal{N}) = \theta_\chi(\mathcal{N}_1) + \Delta\theta_\chi(\mathcal{N})$ and Lemma 4. \square

Corollary 3. *If \mathcal{N}_1 and \mathcal{N} are two fold maps $\theta_C(\mathcal{N}) = 0$ then $\theta_\chi(\mathcal{N}) = \theta_\chi(\mathcal{N}_1)$.*

4. Global Invariant of Stable Maps

We now see the relation between θ_χ , θ_C and d for a stable Gauss map from closed orientable surfaces in the 3-space.

Theorem 4. *Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ be stable Gauss map of a closed orientable surface M . If \mathcal{N} has degree $d(\mathcal{N})$, then*

$$\theta_\chi(\mathcal{N}) + \theta_C(\mathcal{N}) = 2d(\mathcal{N}).$$

Proof. According to Lemma 1, for a fold Gauss map \mathcal{N}_1 we have $\theta_\chi(\mathcal{N}_1) = 2d(\mathcal{N}_1)$.

If \mathcal{N}_1 is a Gauss fold map, by Theorem 3, $\theta_\chi(\mathcal{N}) + \theta_C(\mathcal{N}) = \theta_\chi(\mathcal{N}_1) = 2d(\mathcal{N}_1)$.

Then the result follows from the relation between the degrees of \mathcal{N} and \mathcal{N}_1 , namely $d(\mathcal{N}) = d(\mathcal{N}_1)$. Note that the result does not depend on the choice of \mathcal{N}_1 . \square

Note that for a stable Gauss map $\mathcal{N} : M \rightarrow \mathbb{S}^2$, we have

$$\theta_\chi(\mathcal{N}) = \chi(M) - 2\chi(M_{\mathcal{N}}^-)$$

and $\theta_C(\mathcal{N})$ is the total sum of the signs of the cusps of \mathcal{N} . Thus, we can see that the Theorem 4 is equivalent to the Quine’s Theorem (for $N = \mathbb{S}^2$), which was proved in [13]. Here we give a new proof, for this particular case, based on the previous results.

Theorem 5. *Let $\mathcal{G}(V, E, W)$ be the graph of a stable Gauss map \mathcal{N} from closed orientable surfaces M . If d is a degree of \mathcal{N} then $2(\theta_V - \theta_W) = 2d - \theta_C$.*

Proof. If $\mathcal{G}(V, E, W)$ is a graph of a stable Gauss map \mathcal{N} , according to Theorem 4, we have $\theta_\chi(\mathcal{N}) = 2d(\mathcal{N}) - \theta_C(\mathcal{N})$ and by Theorem 2, we have $\theta_\chi(\mathcal{N}) = 2(\theta_V - \theta_W)$. Then, the result it follows from these two equalities. \square

Corollary 4. *If $\mathcal{G}(2, 1, 0)$ is a graph of a stable Gauss map \mathcal{N} with degree d , then $2\theta_W = \theta_C - 2d$.*

The next result establishes that the limit for θ_C depends on the genus of M and the degree of \mathcal{N} . We present a brief proof of the result proved in [14].

Definition 7. *An apparent contour of a stable map $f \in C^\infty(M, N)$ is said to be irreducible if Σf has only one connected component.*

Theorem 6 ([14]). *Let $\mathcal{N} : M \rightarrow \mathbb{S}^2$ be a stable Gauss map with degree d . If \mathcal{N} has an irreducible contour, then*

$$-2(g(M) - d) \leq (\theta_C) \leq 2(g(M) + d).$$

Proof. If \mathcal{N} has irreducible contour, then $E = 1$. We apply Theorem 1 to get $g(M) = W^+ + W^-$. Let us remark $\theta_W = W^+ - W^- = g(M) - 2W^-$. According to Corollary 4, we have $2g(M) = 4W^- + \theta_C - 2d$.

For $d > 0$, as $W^- \geq 0$, then $2g(M) \geq \theta_C - 2d$, consequently,

$$\theta_C \leq 2(g(M) + d). \tag{1}$$

On the other hand, if $d < 0$, as $g(M) - W^+ = W^-$, we have $2g(M) = 4W^+ - \theta_C + 2d$. Since $W^+ \geq 0$, we have $2g(M) \geq -(\theta_C - 2d)$. Then

$$-2(g(M) - d) \leq (\theta_C). \tag{2}$$

Thus, the result follows from the Equations (1) and (2). \square

The sphere has $g(M) = 0$ and $d = 1$. If the singular set is empty, then $(\theta_C) = 0 \leq 2(g(M) + d)$. Going through a transition L^+ , then $(\theta_C) = 2$ and the equality holds.

This inequality can be seen in Figure 6, which illustrates other examples of surfaces with some beaks associated to weighted graphs (V, E, W) :

- (a) $g(M) = 2$, fold Gauss map with $d = 1 - g(M) = -1$ and $(V, E, W) = (5, 4, 2)$. Two of the curves are borders of simply connected elliptical regions and also borders of cylindrical hyperbolic regions (homeomorphic to the cylinder). The other two curves are boundaries of the bitorous with two edge curves (bitorus minus two disks).
- (b) A graph with $(V, E, W) = (3, 2, 2)$ is realized by a stable Gauss map obtained of (a) by $2(B_{v_+}^{-,+})$, creating four positive cusps and joining the three elliptic regions.
- (c) A graph of $(3, 2, 2)$ type is realized by a fold Gauss map obtained of (b), removing the cusps and joining the two elliptic regions, by $-B_{v_-}^{+,+}$ followed by $-B_{v_+}^{-,+}$, which separates the elliptic region into two, one with genus 2 with a hole and another homeomorphic to the disc. The hyperbolic region is homeomorphic to the cylinder.
- (d) A graph with $(3, 2, 2)$ is equivalent to the map (c).
- (e) $g(M) = 3$, fold Gauss map with $d = 1 - g(M) = -2$ and $(V, E, W) = (4, 4, 2)$. The singular curves separate two cylindrical hyperbolic regions from elliptical regions, one of this region is cylindrical and the other is a bitorous with two boundary curves.
- (f) A graph with $(V, E, W) = (3, 2, 3)$ is realized by a stable Gauss map obtained from (e) by $B_{v_+}^{-,+}$ which creates two cusps joining the two elliptic regions and by $B_{w_+}^{+,+}$, creating two new cusps, joining two parabolic curves that separate the elliptic region of genus three from two hyperbolic regions each homeomorphic to the disc.
- (g) A graph of $(2, 1, 3)$ type is realized by a stable Gauss map obtained from (f) by $-B_{v_-}^{+,+}$ which eliminate a pair of cusps and joins the two hyperbolic regions.

- (h) A graph with $(V, E, W) = (3, 2, 3)$ is realized by a fold Gauss map obtained from (g) by $-B_{v+}^{-,+}$ removing the two cusps, and thus separating the parabolic curve into two which separate a hyperbolic region homeomorphic to the cylinder and the two elliptic regions, one with genus three and another homeomorphic to the disc.

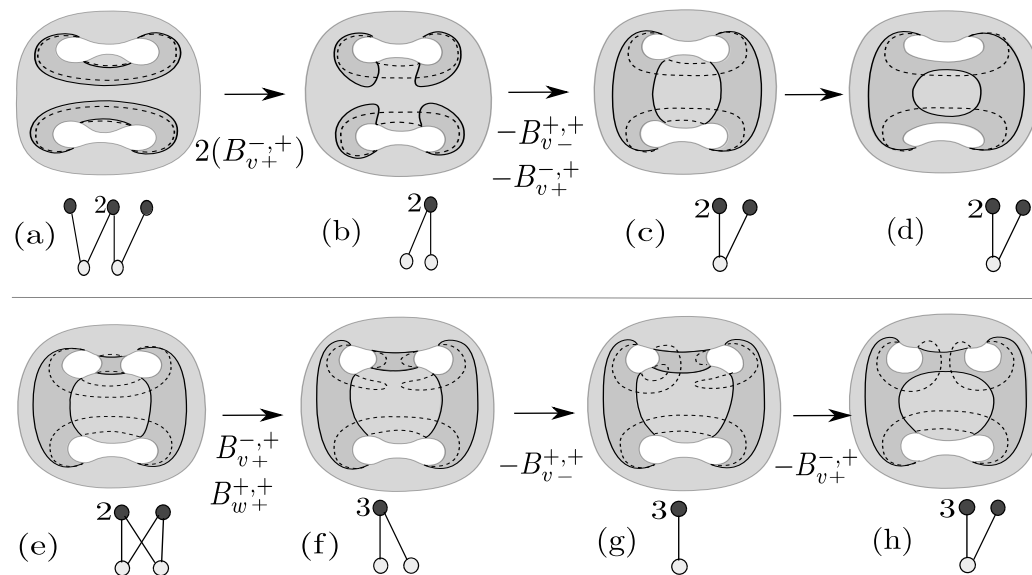


Figure 6. Example of beaks on bitorous and tritorous.

Author Contributions: Formal analysis, C.M.d.J. and P.D.R.; Investigation, C.M.d.J. and P.D.R.; Methodology, C.M.d.J.; Resources, P.D.R.; Supervision, P.D.R.; Writing—original draft, C.M.d.J. and P.D.R.; Writing—review & editing, C.M.d.J. and P.D.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the grant number INDI21/15 from Universidad CEU Cardenal Herrera and by ESI International Chair@CEU-UCH.

Conflicts of Interest: The authors declare no conflict of interest.

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