Graphs and closed surfaces associated to pairing of edges of regular polygons.

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Abstract

In this paper, we define the concept of graph extension, embedded on a closed and orientable surfaces, associated to pairing of edges of regular polygons in order to show that the K-regular pairing of edges graphs can be obtained by the canonical extension of graphs (graphs with a single vertex). We will present examples of K-regular graphs associated to surfaces with genus $g \leq 3$.

Key words: Keywords: trivalent graphs, closed surfaces, pairing of edges, surgeries.
Mathematical subject classification: 14J80, 57M15, 57N05.

1 Introduction

Given a polygon P with $2E$ sides, always is possible to obtain a closed orientable surface M_g by edge pairing (quotient map), where the image of the border of P corresponds to a graph $\mathcal{G}$ with E edges embedded on M_g (see Figure 1). Some authors have searched graphs that can be associated with a edge pairing and the posible edge pairings linked to each of them graphs. In [3], Jorgensen and Naatanen showed that for $E = 9$, there are eight trivalent pairings (all vertices have degree 3) for surfaces with genus $g = 2$. These pairings are associated to five graphs non isomorphic (see Figure 12). For $g = 3$, Nakamura [4] presents a table of 65 graphs (it can be shown in Figure 15) that are associated with trivalent pairings and asserts that there are 927 trivalent pairings associated with these graphs. In [2], was introduced two surgeries of pairing that involves connected sum of graphs, of surfaces and polygons, with the goal of determining families of trivalent graphs for $g \geq 3$, associated to some pairing edges.

In this paper, we will introduce an extension and contraction of graphs on surfaces, for determining families of pairing graphs on closed orientable surfaces with genus g. The

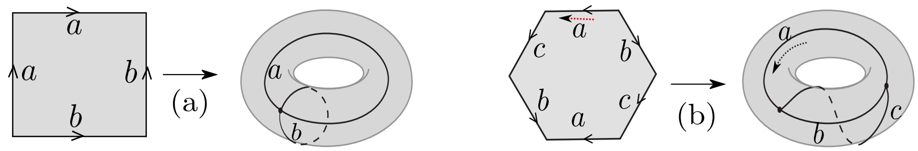

Figure 1: Edge pairing of polygon: (a) $E = 2$ and (b) $E = 3$.

article has a dual purpose: firstly to use the technique of extension of graphs on a surface M_g, to show that all graph can be obtained by some extension of pairing graph with a unique vertex and $E = 2g$ associated to canonical identification, where the regular polygon P has $4g$ sides (see Figure 9); afterwards show the constructions of these K-regular graphs for surfaces with genus $g \leq 3$, by extension of graphs. We have to remark that the objective of this paper is to give theoretical conditions to obtain the pairing graphs but we are not interested in the number of graphs (non isomorphic) that can be computed depending on the genus of the surface.

The organisation of the paper is: in Section 2 we present the definition of extension of graphs and some consequences. In Section 4, (graph of identification) pairing graph and its extensions. In Section 5, we will introduce the K-regular identification graph. Finally we present the construction of k regular graphs for $g \leq 3$.

2 Extension and contraction of graphs

Let G be a graph with vertices set $V(G) = \{v_1, \ldots, v_n\}$. The pair (V, A) will denote the number of vertices and edges of the graph. For the convenience of the reader we review the following concepts thus making our exposition self-contained: A *path* of length n in a graph G is an alternate sequence of vertices and edges of the form $C = \{v_0, z_1, v_1, \ldots, v_{n-1}, z_n, v_n\}$, where $z_i = \{v_{i-1}, v_i\}$ is an edge that connects the vertices v_{i-1} and v_i. A *cycle* of G with length n is a closed path, where the vertices $v_1, \ldots, v_n$ are all different. The number of free cycles of the graph G is given by $\beta(G) = 1 - V + A$. The Euler characteristic of the graph G is given by $\chi(G) = V - A$.

Definition 2.1. *The degree of a vertex v is the number of edges incident to v, with a loop counting two towards the degree of the vertex to which it is incident. The degree of v is denoted by $deg(v)$*

A graph is said K-regular if all its vertices have degree equal to K.

Notation: The graph with a unique vertice and β loops, it will be denoted by I_β.This graph will be 2β-regular.

Example 2.2. Figure 2 illustrates examples of K-regular graphs with four free cycles. From (a) to (i) represents the possible 3-regulars graphs (without isomorphisms) with $(v, a) = (6, 9)$ edges. The graph (j) represents a I_4 and it is 8-regular type $(1, 4)$. The graphs from (k) to (m) are 5-regular type $(2, 5)$ and from (n) to (q) are 4-regular type $(3, 6)$.

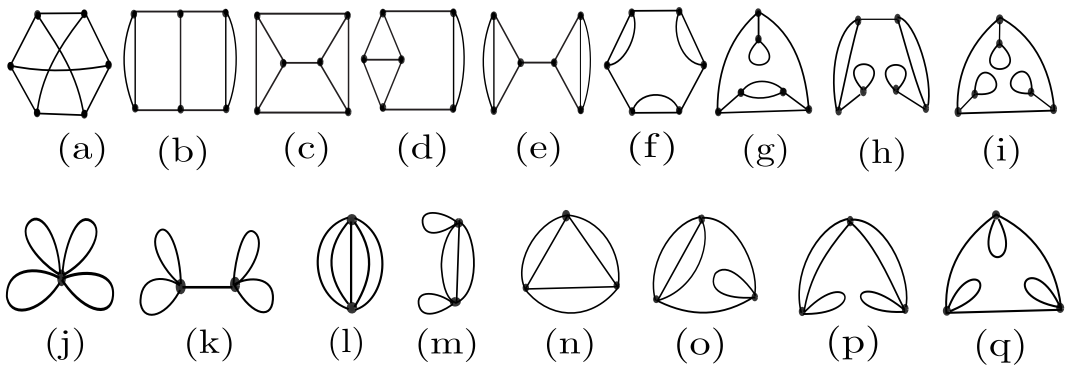

Figure 2: Examples of K-regular graphs.

Definition 2.3. An edge $uv \in G$ it is said *vertex extension* $w \in G_1$ if the vertices $u, v \in G$ and the edge uv can be obtained by "stretching" the vertex w. In this case, we say that the graph G is an *extension* of the graph G_1 or G_1 is a *contraction* of the graph G.

Note that the degree of w satisfies $deg(w) = deg(u) + deg(v) - 2$,

Example 2.4. The Figure 3, displays different extensions of the graph I_4 for 3-regular graphs. Note that the graphs are immersed in $\mathbb{R}^3$, locally is not important the position of the edges incident to a vertex, this allows to have different stretches. This does not occur if the size of the space is smaller.

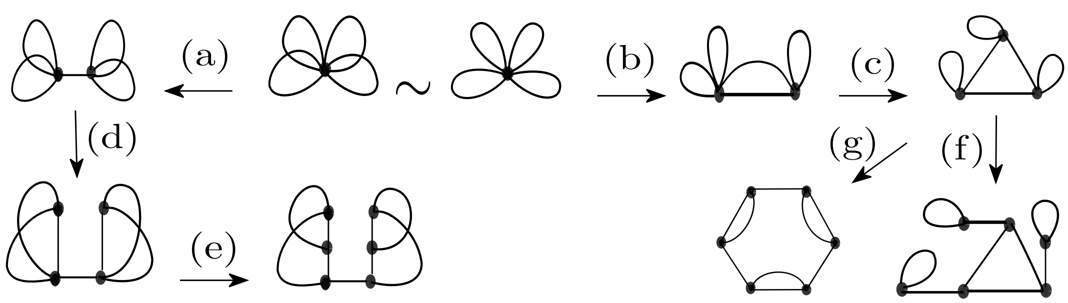

Figure 3: Examples of graph extension.

Proposition 2.5. *Extension and contraction of graphs preserves the number of cycles.*

Proof. The number of cycles in the graph is given by $1 - V + A$. In the stretching of each vertex, the number of edges and vertices always changes by one unit at the same time. Therefore the stretching of each vertex does not change the cycle number. Consequently extension and contraction of graphs do not change the number of cycles. □

Corollary 2.6. *Every connected graph with β free cycles can be contracted in the I_β graph. Consequently, every tree can be contracted at one vertex.*

Proof. If the graph G is connected with β free cycles, then any two vertices u and v in G are connected by a path that can be contracted at a vertex. Consequently, any cycle of G can be contracted in a loop. By contracting all paths and cycles, there will remain only one vertices with β loops. □

3 Embedding graphs on a surface M_g

We will denote by M_g the closed and oriented surface with genus g. The genus of a connected graph G is the smallest number g for which there is an embedding $\iota : G \longrightarrow M_g$. This embedding exists provided that the genus of G is less than g.

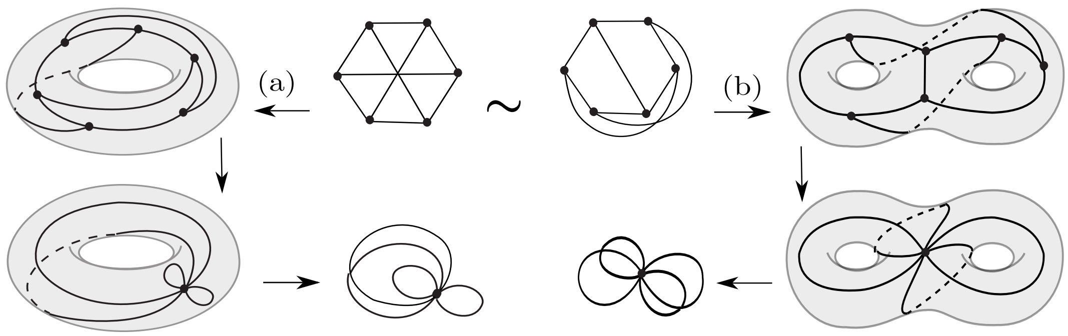

Figure 4: Graphs inmersed in M_1 and M_2.

Example 3.1. Figure 4 illustrates two graphs on the torous and two on the bitorous. The two graphs with six vertices are isomorphic and have genus one. Let us observe that in Figure 4 (a) the complements of the two graphs on the torus have three components that are simply connected and (b) the complements of the two graphs on the torus are simply connected.

We will denote by $\mathcal{G} = \iota(G)$ the graph on M_g. The number of connected components of the complement of $M_g \setminus \mathcal{G}$ we will denote it by F.

Proposition 3.2. If $M_g \setminus \mathcal{G}$ has F simply connected regions (homeomorphic to the disk), then $\chi(G) = 2 - 2g - F$.

Proof. The Euler characteristic of the surface M_g is given by $\chi(M_g) = V - A + F = \chi(G) + F$, when all of F regions of $M_g \setminus \mathcal{G}$ are simply connected. On the other hand, the Euler Characteristic of a closed and oriented surface with genus g it is given by $\chi(M_g) = 2 - 2g$. From these two equalities we have: $\chi(G) = 2 - 2g - F$.

□

Note that F is always less than the number of cycles of $\mathcal{G}$.

If $\mathcal{G}_1$ is a graph embedded on the surface M_g and w is a vertex of $\mathcal{G}_1$, then we can construct a new graph $\mathcal{G}$, embedding the surface M_g, stretching the vertex w on the surface M_g, the two vertices stretching on the torous, as displays Figure 5.

Definition 3.3. If $\mathcal{G}$ is a graph obtained from the graph $\mathcal{G}_1$ by stretching of vertices on the surface M_g, then we will say that $\mathcal{G}$ is a *extension on the surface* M_g of $\mathcal{G}_1$.

Proposition 3.4. *The extension and contraction of graphs on a surface does not change the number of connected components of the graph complement.*

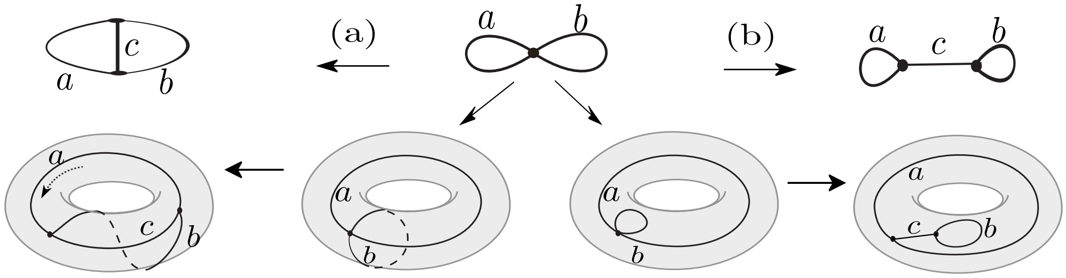

Figure 5: Extension of graphics on the torus.

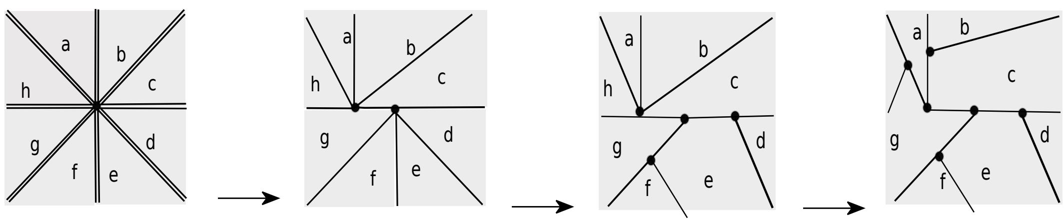

Figure 6: Local example of vertex stretching on a surface.

Proof. The stretching of a vertex in the graph on the surface changes by a unit the number of vertices and the number of edges, which occurs only at the edges of the regions of the complement of the graph, without changing the number of these connected components, as can be seen example in Figure 6 and 11. □

Note that if all components of the complement of M_g are simply connected, by Proposition 3.4 becomes very clear by Proposition 3.2, since $\chi(M)$ is constant and the number of vertices and number of edges that it adds (or decreases) to the contraction of the graph are the same, where we can conclude that the same is true of the number of components of the graph complement.

Theorem 3.5. *Every connected graph embedded on a surface M_g, with β free cycles, can be contracted in the graph type I_β.*

Proof. Let $\mathcal{G}$ be a connected graph, with β free cycles, embedded on a surface M_g. As M_g is connected by path, we can stretch one of the edges of each free of $\mathcal{G}$ on a surface and contract the other edges and vertices into a single vertex. By Corollary 2.6, this contraction does not change the number of cycles in the graph. Thus the graph with only vertex thus obtained is of the type I_β (see Figure 4). □

4 Pairing graphs on a surface

The complement of a graph on a surface M_g may have different numbers of connected components, as you can see in Figures 4 and 5. A natural question is: *which graphs can be embedded on a surface M_g so that the complement of the graph has F simply connected components?.* Here we are interested when $F = 1$. The next definition gives us a motivation to study these graphs.

Definition 4.1. A graph $\mathcal{G}$ is said *pairing graph on the surface* M_g, if the complement $M_g \setminus \mathcal{G}$ is simply connected (homeomorphic to a disk).

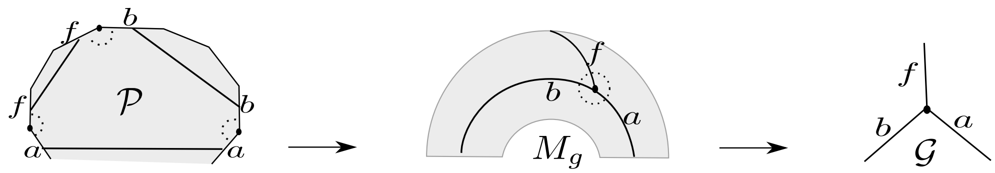

Figure 7: Local example of pairing graphs on a surface.

The pairing graphs $\mathcal{G}$ on a surface M_g, corresponds to the image of the edge of some regular polygon $\mathcal{P}$, by the quotient map $q : \mathcal{P} \longrightarrow M_g$, which takes a couple of edges of $\mathcal{P}$ on a curve arc a_i in M_g and it takes a set of k_i vertices of $\mathcal{P}$ on a point v_i de M_g. The arcs of curves a_i and the points v_i corresponds respectively to the edges and vertices of $\mathcal{G}$, and the number k_i corresponds to the degree of v_i. The polygon will be $\mathcal{P}$ connected with a segment of lines each pair of edges identified by q (see Figures 7 and 8).

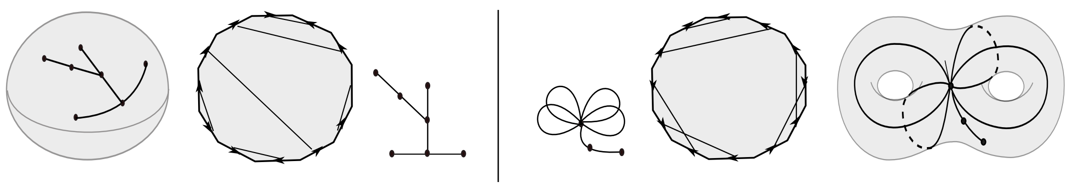

Figure 8: Examples of surface pairing graphs.

Definition 4.2. The set of segments of lines that point the pairs of edges in $\mathcal{P}$ identified on M_g it will be called *pairing diagram*, denoted by $\mathcal{D}$.

Definition 4.3. The pairing of the polygon $\mathcal{P}$, with $4g$ sides, on the surface M_g we will call it *canonical pairing* (see Figure 9).

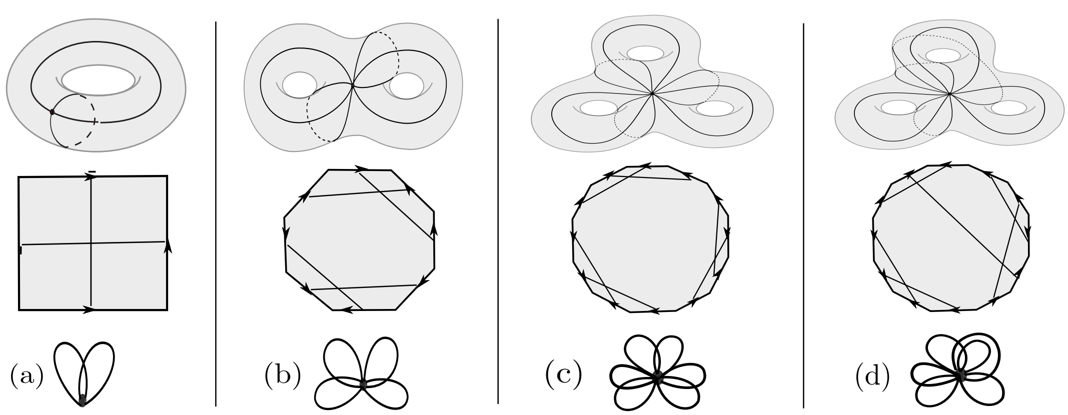

Figure 9: Examples of canonical pairing on M_g, para $1 \leq g \leq 3$.

Notation: We will denote by I_{2g} the graph of canonical pairing on the surface M_g.

Proposition 4.4. *If $\mathcal{G}$ is a pairing graph on M_g, then $V - A = 1 - 2g$.*

Proof. If $\mathcal{G}$ is a pairing graph on M_g, the complement of $\mathcal{G}$ on M_g is simply connected, therefore $F = 1$. By Proposition 3.2, we have $A - V = 2g - 2 + F = 2g - 1$. □

Corollary 4.5. The pairing graph $\mathcal{G}$ is a tree if, and only if, $M_g = S^2$.

Proposition 4.6. *Let $\mathcal{G}$ be an pairing graph with V vertices and A edges, on a surface M_g. The graph $\mathcal{G}$ has V_i vertices with degree K_i, for $i = 1, \cdots, r$, then*

$$2A = \sum_{i=1}^{r} K_i V_i \quad \textit{and} \quad \sum_{i=1}^{r}(K_i - 2)V_i = 2(2g - 1).$$

Proof. If $\mathcal{G}$ has V_i vertices with degree K_i,for $i = 1, \cdots, r$, the total number of vertices is given by $V = \sum_{i=1}^{r} V_i$ and the total number of edges is given by $A = \dfrac{1}{2}\sum_{i=1}^{r} V_i K_i$,because each edge connects two vertices. So $2(A-V) = \sum_{i=1}^{r} V_i(K_i - 2)$. According to Proposition 4.7 it follows $2(A - V) = 2(2g - 1)$. From this two equalities it follows the statement. □

Proposition 4.7. *All regular polygons P_{2A}, with $A > 1$, are polygons of pairing for some pair $(M_g, \mathcal{G})$, where $A \geq 2g$ and the graph $\mathcal{G}$has A edges and $V = A + 1 - 2g$ vertices.*

Proof. Given the polygon P_{2A}, to obtain the sphere, $g = 0$, we take the quotient map $q : P_{4g} \longrightarrow M_g$, in way that the pairing diagram does not have points of intersection. For $A = 4g > 0$, the canonical pairing guarantees the existence of the pair (M_g, I_{2g}). For $0 < 2g < A$, we can identify, as in the previous case, the $4g$ first consecutive edges, forming a type graph I_{2g} with a single vertex v. The $m = 2A - 4g$ remaining sides of the polygon can be identified the consecutive pairs or pairs so that the corresponding lines of the pairing diagram do not have intersection points. In the graph, this pairing forms a tree-like branch connected to v (see Figure 8). □

Lemma 4.8. *Let $\mathcal{G}$ be an pairing graph on the surface M_g. If $\mathcal{G}_1$ is an extension or contraction on M_g of the graph $\mathcal{G}$, then $\mathcal{G}_1$ is also an pairing graph on M_g.*

Proof. The extension of a vertex of the pairing graph on M_g increases two sides in the pairing polygon because the changed edges always remain on the edge of the polygon, as you can see in Figure 10. By Proposition 3.4, the extension (contraction) of graphs on M_g does not change the number of the connected component of the graph complement. Since $\mathcal{G}_1$ is an extension (or contraction) of $\mathcal{G}$ and $M_g \setminus \mathcal{G}$ is simply connected, then $M_g \setminus \mathcal{G}_1$is also simply connected. Then $\mathcal{G}_1$ is an pairing graph on M_g, by Definition 4.1. □

Example 4.9. Figure 11 illustrates 16 pairing graphs with four vertices on M_3. These graphs may be obtained by a sequence of extensions de graphs of type I_6 (see (c) and (d) Figure 9 .

Note that these graphs are not K-regular and that the vertex stretches can be made within a region homeomorphic to disk.

Remark 4.10. An extension or contraction of any pairing graph in the space may lead to a non-pairing graph. We can guarantee that the resulting graph of an extension or contraction is a pairing graph if (the extension or contraction) is made on the surface.

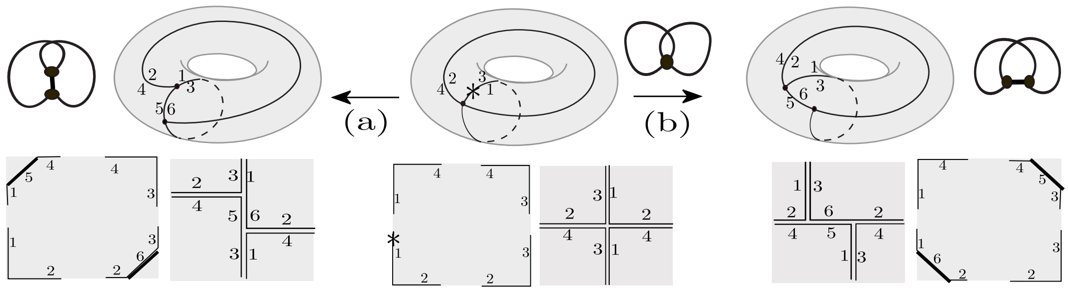

Figure 10: Examples of graph extension on the torus.

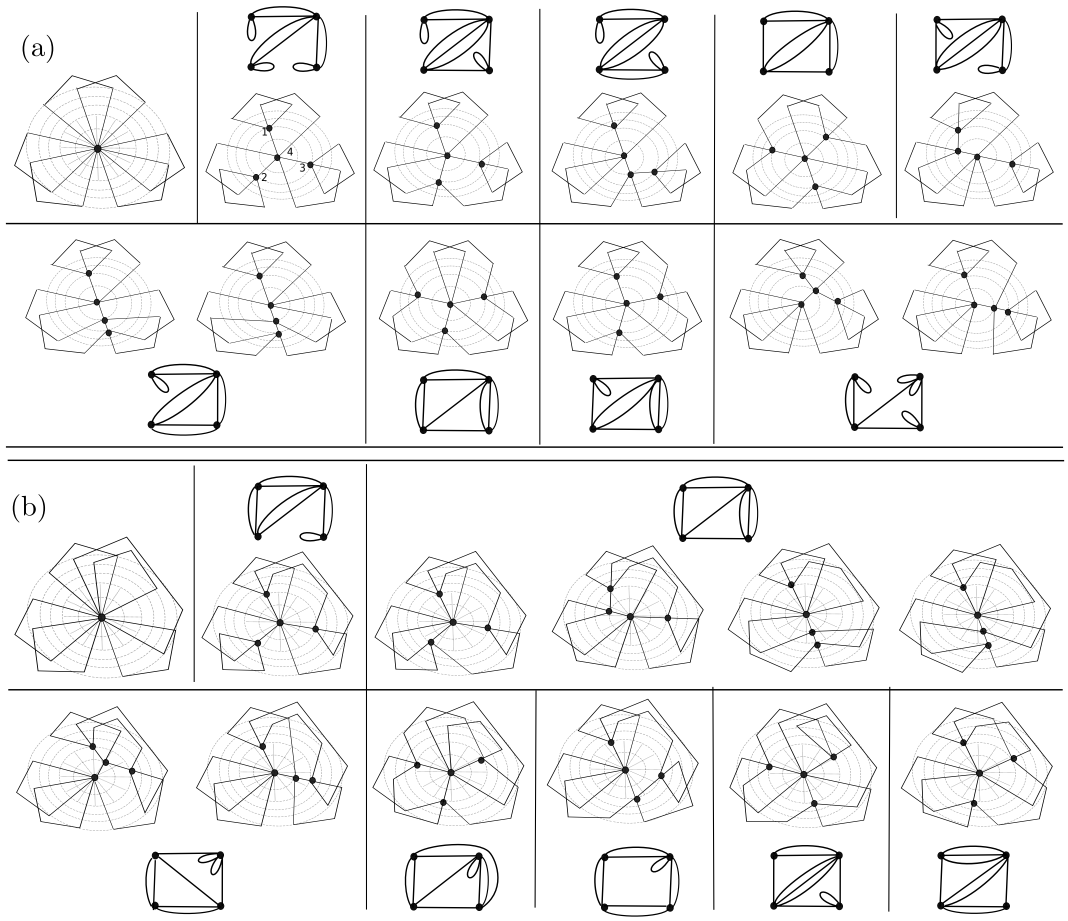

Figure 11: Examples of graphs with four vertices on M_3.

Theorem 4.11. *All pairing graph $\mathcal{G}$ on M_g, with $g > 0$, can be obtained by extension of the pairing canonical graph I_{2g}.*

Proof. If $\mathcal{G}$ is a pairing graph with β cycles, by Proposition 4.4, $A - V = 2g - 1$ e $\beta = 1 - V + A = 2g$. Then by Theorem 3.5, $\mathcal{G}$ can be contracted in the graph type I_{2g} on a surface M_g. Since $\mathcal{G}$ is pairing graph, then I_{2g} is a pairing graph on M_g, by Lemma 4.8. Doing the inverse path, by stretching vertices on M_g, we obtain the pairing graph $\mathcal{G}$ from the canonical pairing graph I_{2g}. □

As consequence of Theorem3.5, we have the following result:

Corollary 4.12. *All pairing graphs K-regular, with $K > 1$, can be obtained by some extension of the graph I_{2g}.*

In particular, all pairing graph on M_g with $(V, A) \leq (4g - 2, 6g - 3)$ can be obtained by some extension on M_g of the pairing graph I_{2g}.

5 Pairing K-regular graphs

From now on we will give special attention to pairing k-regular graphs. Note that the only 1-regular graph is a tree and can not be associated with a regular polygon. The 2-regular graphs are homeomorphic to the circle and also can not be pairing graphs, since it does not have a simply connected complement on a surface.

5.1 Possible values for V and A

Let see the possible values for V and A for pairing K-regular graphs on surfaces M_g as a function of g and K. On the torous exists a unique 3-regular graph (non-isomorphic) and single 4-regular graph that is I_2 (see Figure 10). Thus, we can state the following,

Definition 5.1. A regular polygon $\mathcal{P}$ it will be said *pairing K-regular polygon* if exists a pair $(M_g, \mathcal{G})$, where $\mathcal{G}$is a pairing K-regular graph on a surface M_g.

Proposition 5.2. *Let $\mathcal{G}$ be a pairing graph on a surface M_g. If $\mathcal{G}$ is K-regular and $g > 0$, then $K \geq 3$. Consequently, if P_{2A} is a K-regular polygon then $A \geq 2$.*

Corollary 5.3. *Let $\mathcal{G}$ be a K-regular pairing graph on M_g, with V vertices. If $V = 2V_1$, then there are $2(K-1)$-regular pairing graphs on M_g, with V_1 vertices.*

Consequently, $V = 2mV_m$, with $m, V_m \in I\!N$, then there are $2m(K-1)$-regula pairing graphs on M_g, with V_m vertices.

Proposition 5.4. *Let $\mathcal{P}$ be a pairing K-regular polygon associated with the pair $(M_g, \mathcal{G})$. Then*

$$(V, A) = (\frac{2(2g-1)}{K-2}, \frac{K(2g-1)}{K-2}). \tag{1}$$

Consequently, the pairing K-regular polygon has $n = 2K(2g-1)/(K-2)$ sides.

Proof. If $\mathcal{G}$ is a pairing K-regular graph, the quotient map $q : \mathcal{P} \longrightarrow M_g$, it takes a couple of edges $\mathcal{P}$ on an edge of $\mathcal{G}$.

Besides, q tales K vertices of $\mathcal{P}$ on a vertex of $\mathcal{G}$. Since the number of vertices and edges of the polygon $\mathcal{P}$ are equal, Then $2A = KV$. From Proposition 4.4, we have $\mathcal{G}$ satisfies$A - V = 2g - 1$. From these two equality it follows $A = K(2g-1)/(K-2)$ and $V = 2(2g-1)/(K-2)$. □

Corollary 5.5. *If $\mathcal{G}$ is a pairing K-regular graph on M_g, then $g = \frac{1}{2K}[(K-2)A + K]$.*

g	V	A	K	n
1	1	2	4	4
	2	3	3	6
2	1	4	8	8
	2	5	5	10
	3	6	4	12
	6	9	3	18
3	1	6	12	12
	2	7	7	14
	5	10	4	20
	10	15	3	30

Table 1: Values for the pairing k-regular graphs.

Remark 5.6. Fixed $0 < g \leq 4$, from Proposition 5.4 whe have the possible values$K > 2$ for pairing K-regular graphs on M_g:

1. For $g = 1$, we have $(V, A) = (\dfrac{2}{(K-2)}, \dfrac{K}{(K-2)})$. Then A and V it will be positive integers only for $K = 3$, with $(V, A) = (2, 3)$ and for $K = 4$, with $(V, A) = (1, 2)$. In both cases, there is only one graph (see Figure10).

2. For $g = 2$, $(V, A) = (\dfrac{6}{(K-2)}, \dfrac{3K}{(K-2)})$. it will be positive integers for A and V for $K = 3$, with $(V, A) = (6, 9)$, for $K = 4$, with $(V, A) = (3, 6)$, for $K = 5$, with $(V, A) = (2, 5)$ and for $K = 8$, with $(V, A) = (1, 4)$.

3. For $g = 3$, $(V, A) = (\dfrac{10}{(K-2)}, \dfrac{5K}{(K-2)})$. The numbers A and V it will be positive integers for $K = 3$, with $(V, A) = (10, 15)$, for $K = 4$, with $(V, A) = (5, 10)$, for $K = 7$, with $(V, A) = (2, 7)$ and for $K = 12$, with $(V, A) = (1, 6)$.

4. For $g = 4$, $(V, A) = (\dfrac{14}{(K-2)}, \dfrac{7K}{(K-2)})$. The numbers A and V it will be positive integers for $K = 3$, with $(V, A) = (14, 21)$, for $K = 4$, with $(V, A) = (7, 14)$, for $K = 9$, with $(V, A) = (2, 9)$ and for $K = 16$, with $(V, A) = (1, 8)$.

Table 1 shows a summary of possible values V, A, K e $n = 2A$ associated to pairing k-regular graph. Note $n = KV$.

5.2 Pairing graphs for $g \leq 3$

Remark 5.6 describes the possible K for pairing k-regular graphs associated to M_g, with $g < 4$.

We are going to present the possible diagrams associated with the pairing graphs on M_1 e M_2. For M_3, we are going to depict examples because we are limited by the huge number of pairings (see [4], for $V = 10$).

(i) For M_1: the possible values for K are: 3 and 4. Figure 10 illustrates the unique pairing 3-regular graph obtained by extension of graph type I_2.

(ii) For M_2: the possible values for K are: 3, 4, 5 e 8. We will now see the possible pairing graphs for $V = 1, 2, 3$ and 6 (see Table 1) by extentions of the graphs type I_4. Figure 12 iillustrates the possibles pairing K-regular graphs (non isomorphic) on M_2 ($K = 3, 4, 5, 8$). The reader can verify any other K- regular diagram on M_2 is equivalent to one of this fours. For

> $V = 1$: (a) and (b) illustrates the two possible pairing graphs type I_4 with their respective diagrams on the polygon with 8 sides.
>
> $V = 2$: (c), (d) and (e) illustrates the three possible pairing 5-regular graphs (non isomorphic) on M_2, with their respective diagrams on the polygon with 10 sides. Note that the graph (d) is associated with two pairings.
>
> $V = 3$: (f), (g) e (h) illustrates the three pairing 4-regular graphs (non isomorphic) on M_2, with their respective diagrams on the polygon with 12 sides. Note that the graph (g) is associated with two pairings.
>
> $V = 6$: (i), (j), (k), (l) and (m), illustrates the five pairing 3- regular graphs (non isomorphic) on M_2, with their respective diagrams on the polygon with 18 sides. These graphs can be obtained by different extensions of 5-regular graphs or $4-$regular graphs. Note that the graphs (i) and (k), has an unique associated pairing. In [3], Jorgensen and Naatanen shows all possible pairing 3-regular graphs which corresponds to eight different pairing 3-regular diagram on M_2.

(iii) For M_3: the possible values for K are: 3, 4, 7 e 12 (see Table 1). Figures 13, 14, 15 illustrates the pairing K-regular graphs (non isomorphic) on M_3.

We will now see some examples of pairing graphs for $V = 1, 2, 5$ and 10, by extension beginning with graph type I_6.

> $V = 1$: Figure 13-(a) illustrates eight examples of pairing diagrams associated to graphs type I_6, on the polygon with 12 sides. The reader can check that there are other pairing diagrams on the polygon with 12 sides, besides these eight.

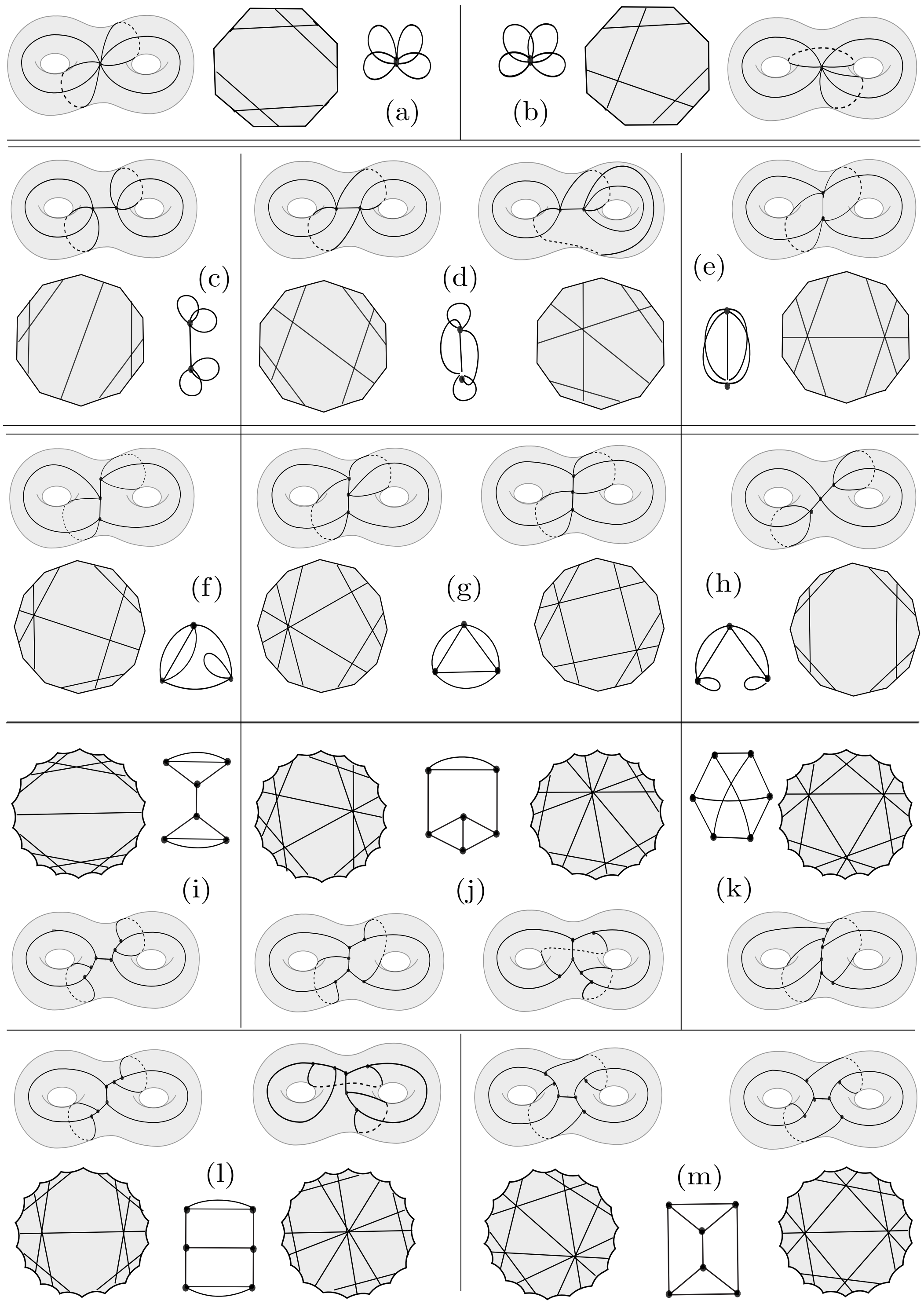

Figure 12: Examples of extension of pairing graphs K-regular on M_2.

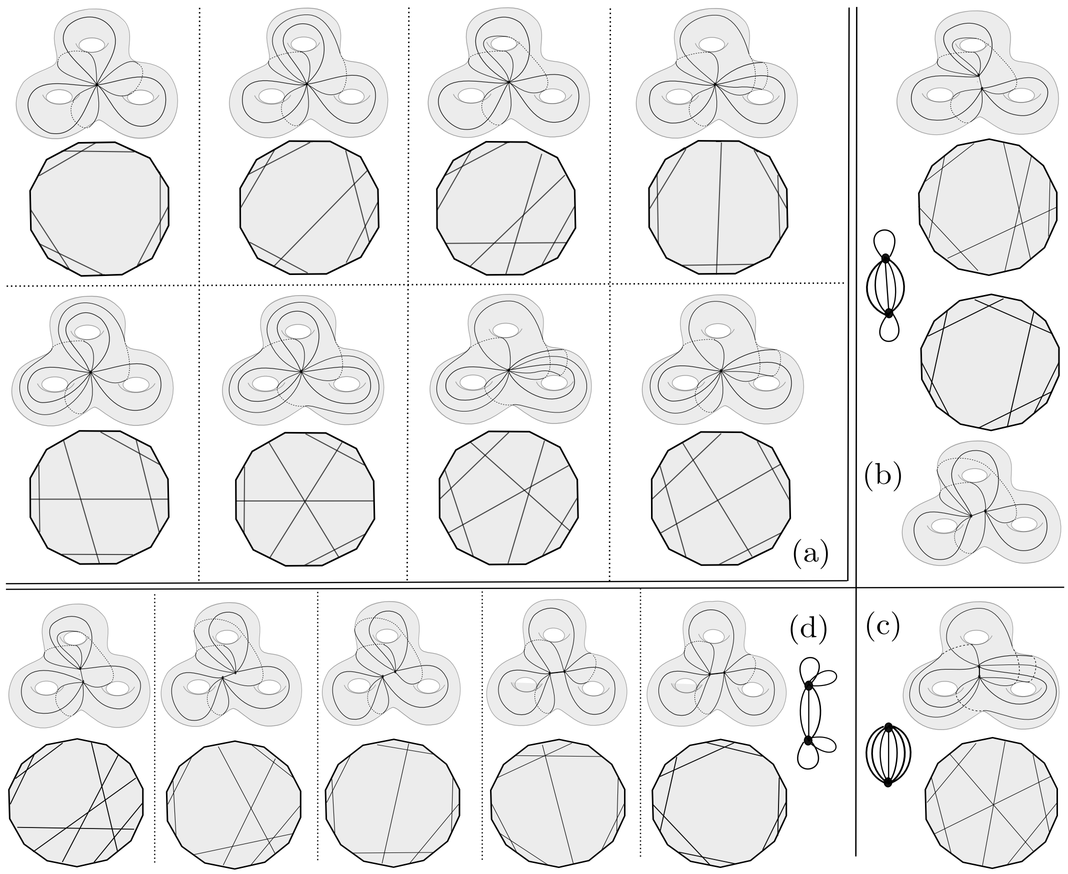

Figure 13: Extension for graphs of 3-regular pairings on M_2 .

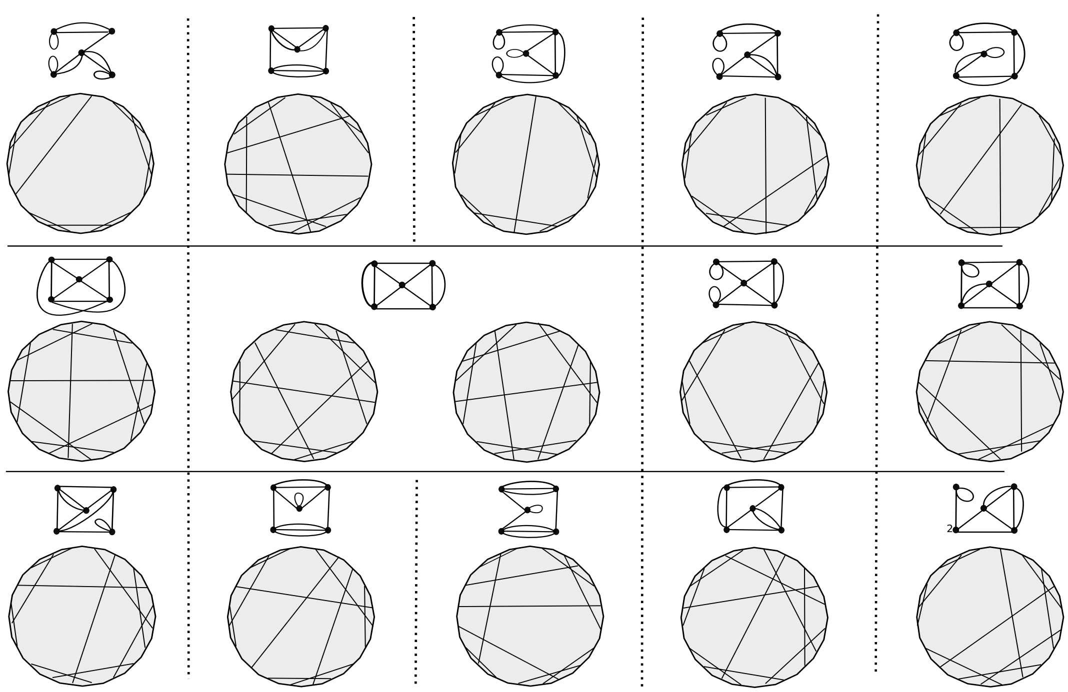

Figure 14: Examples of 4-regular diagrams and graphs of graphs on M_3.

$V = 2$: Figure-13(b), (c) and (d), illustrates the eight examples of pairing 7-regular, associated to three 7-regular graphs, with their respective pairing diagrams. The reader can verify that exists other pairing diagrams on the polygon with 14 sides. These graphs can be obtained by different extension of pairing graphs type I_6 (Figure 13).

$V = 5$: Figure-14, displays some examples of pairing 4-regular graphs on M_3 (non isomorphic), with some of their pairing diagrams. Note that each graph can be associated to a various pairings. The reader can verify that any pairing 4-regular graph on M_3 can be obtained by extension of some pairing graph with 4 vertices, see Figure 11.

$V = 10$: in [4], Nakamura shows that exists 65 pairing 3-regular graphs (see Figure 15) that are non isomorphic on M_3 and the authors show the 927 pairing diagrams associated with these graphs, on the polygon with 30 sides.

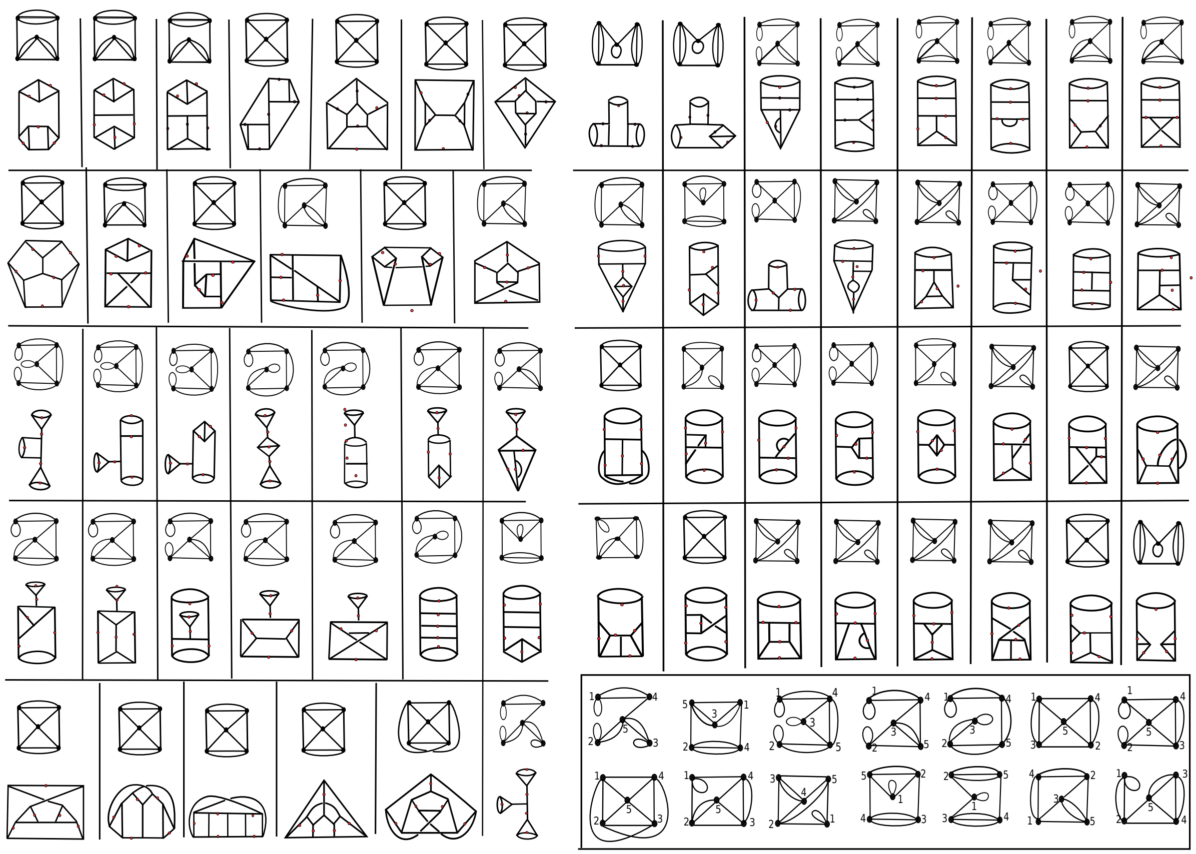

Figure 15: Possible 3-regular pairing graphs on M_3.

Figure 15 illustrates the 65 pairing 3-regular graphs given by Nakamura, in [4]. These graphs can be obtained by extension of the graphs displayed in Figure 14 (shown in the right-hand rectangle below). These extensions are not unique, because one pairing 3-regular graph can be obtained from different extensions of graphs 4−regular.

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